Reconstruction and deceleration-acceleration transitions in modified gravity

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We discuss the cosmological reconstruction in modified Gauss-Bonnet and $F(R)$ gravities. Two alternative representations of the action (with and without auxiliary scalar) are considered. The approximate description of deceleration-acceleration transition cosmologies is reconstructed. It is shown that cosmological solution containing Big Bang and Big Rip singularities may be reconstructed only using the representation with the auxiliary field. The analytical description of the deceleration-acceleration transition cosmology in modified Gauss-Bonnet gravity is demonstrated to be impossible at sufficiently general conditions.

I. INTRODUCTION

The modified gravity approach (for general review, see [1]) became the essential element of the modern cosmology. It is quite remarkable that some change of the classical gravitational action may resolve the number of cosmological problems, including inflationary paradigm, dark energy and dark matter. It turns out that it is not necessary to introduce the extra ingredients (usually, scalar or fluid) as all these phenomena could be understood as gravitational manifestations. For instance, the unification of the early-time inflation and late-time acceleration may be achieved in $F(R)$ gravity (for first realistic model of that sort, see [2]) without the need to introduce the inflaton and (scalar) dark energy by hands. Several models of modified gravity may successfully describe dark matter as gravitational effect (for a recent review, see [3]). The coincidence problem effectively disappears in the modified gravity approach because dark matter and dark energy are caused by the universe expansion governed by specific theory. It is expected that modified gravity may be helpful also in high-energy physics (for instance, for hierarchy problem).

Unfortunately, the realistic modified gravity has usually highly non-linear structure in terms of geometric invariants (curvature, Gauss-Bonnet invariant, etc.). As the result, its background evolution is very hard to describe analytically unlike to the case of General Relativity where number of viable analytic solutions are available. In turn, with only approximate FRW solutions of modified gravity it is extremely difficult to study the cosmological perturbations. At best, such cosmological perturbations are studied in further approximation neglecting the higher-derivatives non-linearities which is definitely not sufficient. In order to study the background evolution of the alternative gravities, so-called reconstruction method has been developed (for the introduction, see [4]). Within the reconstruction method, given FRW cosmology may be used to reconstruct the modified gravity where such cosmology is the solution of the equations of motion.

In the present paper we develop the reconstruction method for modified Gauss-Bonnet gravity [5]. It is demonstrated how to reconstruct the theory which admits the deceleration-acceleration transition (the transition to ΛCDM epoch). Such background evolution turns out to be the very complicated and approximate one. For quite general class of $F(G)$-functions we show that there is no analytical description of deceleration-acceleration transition. It turns out that it is very difficult (if possible at all) to construct such a model which admits such transition analytically (the co-existence of matter dominance and accelerating solutions [6]). The comparison with $F(R) = R + f(R)$ theory is done. The alternative presentation for $F(R)$ and $F(G)$ modified gravity using the auxiliary scalar is considered. It is shown that reconstruction using such representation leads to wider class of cosmological solutions, including the deceleration-acceleration transition ones. It is demonstrated that cosmological solution containing the Big Bang as well as Big Rip singularity may be reconstructed from $F(R)$ gravity.

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II. ANALYTICAL APPROACH TO DECELERATION-ACCELERATION TRANSITION IN $f(G)$-GRAVITY

Let us study modified gravity with the following action [5]:

$$S_{F(G)} = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} + F(G) + \mathcal{L}_{\text{matter}} \right). \tag{1}$$

Here G is the Gauss-Bonnet invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\text{matter}}$ is the Lagrangian of matter. It is convenient to put $2\kappa^2 = 1$ in this section. We will discuss only the FRW background:

$$g_{\mu\nu} = diag(-n^2, a^2, a^2, a^2). \tag{2}$$

The variation of the action (1) with respect to lapse-function n gives the modified Friedman equation:

$$6H^2 + F - F'G + 24F''H^3\dot{G} = \rho, \tag{3}$$

and the variation with respect to scale factor a gives more complicated equation:

$$4\dot{H} + 6H^2 + F - F'G + 8F'''H^2\dot{G}^2 + \frac{2G}{3H}F''\dot{G} + 8H^2F''\ddot{G} = -p. \tag{4}$$

Here ρ and p is the matter energy-density and pressure, respectively, which arises from $\mathcal{L}_{\text{matter}}$. We also note that prime $'$ denotes partial differentiation of function F with respect to its argument. Using analogy with the FRW equations in the Einstein gravity one may define $\rho_G \equiv -F + F'G - 24F''H^3\dot{G}$ and $p_G \equiv F - F'G + 8F'''H^2\dot{G}^2 + \frac{2G}{3H}F''\dot{G} + 8H^2F''\ddot{G}$, so the equations (3) and (4) take the following form:

$$6H^2 = \rho_{\text{tot}}, \tag{5}$$

$$4\dot{H} + 6H^2 = -p_{\text{tot}}, \tag{6}$$

where $\rho_{\text{tot}} = \rho + \rho_G$ and $p_{\text{tot}} = p + p_G$. Note that different cosmological solutions for above theory have been discussed in refs.[7].

The barotropic equation of matter state $p = w\rho$ is considered below. To close our system one adds conservation energy law which takes the following form:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{7}$$

Note also that equation (3) is just the first integral of the system (4)-(7). Now let us consider the possibility of occurrence of late-time universe acceleration due to function $F(G)$ analytically. In other words, one searches some function $F(G)$, which plays non-essential role during the standard dust stage($w = 0$), but gives leading contribution at late times. Our purpose is the analytical description of such deceleration-acceleration transition. We discuss functions with $F(0) = 0$, because otherwise we will have some analogue of cosmological constant. (Of course, permitting the effective cosmological constant may qualitatively change the results obtained below). The effective equation of state parameter may be easily found by using the expressions (5)-(6): $w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}$.

Now let us suppose that there exists some function $F(G)$ which leads to late-time acceleration and has the following properties:

$$F(0) = 0, \ F'(0) = 0, \ F''(0) = 0, \ F'''(0) = 0. \tag{8}$$

From another side it is known that at the deceleration-acceleration transition point $G = 24\frac{\dot{a}^2\ddot{a}}{a^3} = 0$. This point is reached when $w_{\text{tot}} = -1/3$. By calculating the values of ρ_G and p_G at the transition point, we find that they vanish. This means that there is no any effective matter besides the usual matter ρ at this moment, so $\rho_{\text{tot}} = \rho$ and $p_{\text{tot}} = p$, hence w_{tot} must be equal to some value of w which is bigger than $-1/3$. This logical contradiction proves that any function satisfying the conditions (8) cannot reach the deceleration-acceleration transition point. Note also that the condition $F'(0) = 0$ may be removed from (8) because its contribution to ρ_G and p_G contains G as a factor. This result is complimentary to the one of ref.[6] where it has been shown that some class of $F(G)$-theories which allow an exact power-law solution can not explain transition from deceleration to acceleration. Actually, the following general form of function which allows *exact* power-law decelerating solution (see (15) in [6]) is: $F(G) = AG^{0.5} + BG^k$ where

$k < \frac{3}{2}$ and this function does not satisfy to our condition (8). From another side, one may easily find functions which satisfy the conditions (8) but do not allow *exact* power-law solution. For example,

$$F = \sum_{N=4}^{\infty} a_N G^N, \quad F = \frac{G^N}{c_1 G^N + c_2}. \tag{9}$$

The situation is the following: there is some function F, which does not allow exact power-law decelerating solution, but allows it approximately with very good accuracy. This solution may be unstable and leading to acceleration. The examples of such approximate deceleration-acceleration transition will be discussed below.

III. COMPARISON WITH $F(R)$-GRAVITY

It is interesting to compare results of the previous section with $F(R)$-gravity. Its action has the following form:

$$S_{f(R)} = \int d^4 x \sqrt{-g} \left(\frac{1}{2\kappa^2} [R + f(R)] + \mathcal{L}_{\text{matter}} \right). \tag{10}$$

FRW equations of motion are (here again $2\kappa^2 = 1$):

$$6H^2 + f - f'R + 6H^2 f' + 6H f'' \dot{R} = \rho, \tag{11}$$

and

$$4\dot{H} + 6H^2 + f - f'R + 6H^2 f' + 2f''' \dot{R}^2 + 4H f'' \dot{R} + 2f'' \ddot{R} = -p. \tag{12}$$

The latter equation is a consequence of (11) and (7). It is well known that $R = 0$ identically for the regime $a \sim t^{1/2}$, which corresponds to $w_{\text{eff}} = \frac{1}{3}$. So using developed analysis one may try to investigate the possibility to reach $a \sim t^{1/2}$ regime. We will study only functions satisfying the conditions

$$f(0) = 0, \ f'(0) = 0, \ f''(0) = 0, \ f'''(0) = 0. \tag{13}$$

Let us consider the theory with fixed EoS w_m matter besides the relativistic matter ($w_m = \frac{1}{3}$). Following previous section we define $\rho_{f(R)} \equiv -f + f'R - 6H^2 f' - 6H f'' \dot{R}$ and $p_{f(R)} \equiv f - f'R + 6H^2 f' + 2f''' \dot{R}^2 + 4H f'' \dot{R} + 2f'' \ddot{R}$ to rewrite equations (11)-(12) in the canonical form (5)-(6). In this case we have the logical contradiction: from the one side it must be $w_{\text{tot}} = \frac{1}{3}$ on $a \sim t^{1/2}$ regime, but from another side we have $w_{\text{tot}} = w_m \neq \frac{1}{3}$ because there is no any contribution to ρ_{tot} and p_{tot} from $f(R)$-terms at this regime ($\rho_{f(R)} = 0$, $p_{f(R)} = 0$ due to (13)).

So we have the following result. The analytical description of transition to regime $a \sim t^{1/2}$ in the universe with any perfect fluid except $w_m = \frac{1}{3}$ in $f(R)$-gravity with (13) is very hard to realize (compare with [8] where similar conclusion is made). Of course, other classes of functions $f(R)$ or account of the effective cosmological constant may change this conclusion.

Note also that there is no any problem with deceleration-acceleration transition in $f(R)$ gravity. A number of such theories admitting the transition is well known. For example, most general function which leads from matter dominated era to the ΛCDM cosmology was constructed in [9, 10] by using reconstruction method. Below we try to use this method to solve the problem described in previous section.

IV. RECONSTRUCTION AND THE DECELERATION-ACCELERATION TRANSITION IN $F(G)$-GRAVITY

Let us investigate possibility to find the theories (1) which allow transition from deceleration to acceleration phase by using reconstruction method. This method developed in ref.[9] may be easily adopted to our Gauss-Bonnet modified gravity (1). We start from the equation (3). First of all we would like to use a new variable N instead of the cosmological time t, defined by $N = \ln \frac{a}{a_0}$. Here a_0 is the value of the scale factor $a(t)$ in (2) at a fixed time. This variable is related with the redshift z by $e^{-N} = 1 + z$. Since $\frac{d}{dt} = H \frac{d}{dN}$ and $\frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$, one can rewrite (3) as

$$6H^2 + F(G) - 24H^3 (H' + H)F'(G) + 24^2 F''(G)H^6 (HH'' + 3H'^2 + 4HH') = \rho. \tag{14}$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$, but $F' = dF/dG$ like above. Here we have used also $G = 24H^2(\dot{H} + H^2) = 24H^3(H' + H)$. If the matter energy density ρ is given by a sum of the fluid densities with constant EoS parameter w_i, we find

$$\rho = \sum \rho_{i0} a^{-3(1+w_i)} = \sum \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N} . \tag{15}$$

Here ρ_{i0} is a constant. Let the Hubble rate is given in terms of N via some function $k(N)$ as

$$H = k(N) = k(-\ln(1+z)) . \tag{16}$$

Note now that the expression $G = 24k(N)^3 k'(N) + 24k(N)^4$ may be solved with respect to N as $N = N(G)$. Then by using (15) and (16), one can rewrite (14) as

$$6\left(k\left(N\left(G\right)\right)\right)^2 + F(G) - 24\left(k\left(N\left(G\right)\right)\right)^3 F'(G)\left[k'\left(N\left(G\right)\right) + k\left(N\left(G\right)\right)\right]$$
$$+24^2 F''(G)\left(k\left(N\left(G\right)\right)\right)^6 \left[k\left(N\left(G\right)\right) k''\left(N\left(G\right)\right) + 3\left(k'\left(N\left(G\right)\right)\right)^2 + 4k\left(N\left(G\right)\right) k'\left(N\left(G\right)\right)\right]$$
$$= \sum \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N} . \tag{17}$$

This equation is differential equation for $F(G)$ and may be simplified by introducing $h(N) \equiv (k(N))^2 = H^2$:

$$6h(N(G)) + F(G) - 12\frac{dF(G)}{dG}\left[h(N(G))h'(N(G)) + 2\left(h\left(N\left(G\right)\right)\right)^2\right]$$
$$+24^2 \frac{d^2 F(G)}{dG^2} h\left(N\left(G\right)\right)^3 \left[\frac{1}{2}h''\left(N\left(G\right)\right) + 2h'\left(N\left(G\right)\right) + \frac{h'\left(N\left(G\right)\right)^2}{h\left(N\left(G\right)\right)}\right]$$
$$= \sum \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)N} . \tag{18}$$

Note that the Gauss-Bonnet invariant is given by $G = 24h(N)^2 + 12h(N)h'(N)$. Hence, when we find $F(G)$ satisfying the differential equation (18), such $F(G)$ theory admits the solution (16) and therefore such gravity realizes above cosmological solution. This is essentially the cosmological reconstruction.

Now let us discuss the simplest example which is related with previous discussion and which reproduces the ΛCDM-era. In the Einstein gravity the FRW equation for the ΛCDM cosmology is given by

$$6H^2 = 6H_0^2 + \rho_0 a^{-3} = 6H_0^2 + \rho_0 a_0^{-3} e^{-3N} . \tag{19}$$

Here H_0 and ρ_0 are constants. This equation reproduces the universe with dust matter which enters to ΛCDM-era at late time (for sufficiently small H_0). Therefore, it reaches the point $\ddot{a} = 0$ at some moment. So we have

$$h(N) = H_0^2 + \frac{1}{6}\rho_0 a_0^{-3} e^{-3N} . \tag{20}$$

Substituting this relation into expression for G we find:

$$G(N) = 24H_0^4 + 2H_0^2 \rho_0 a_0^{-3} e^{-3N} - \frac{1}{3}\rho_0^2 a_0^{-6} e^{-6N} , \tag{21}$$

which may be solved to find $N(G)$. It is convenient to introduce $x = \rho_0 a_0^{-3} e^{-3N}$, so finally we have

$$24H_0^4 + 2H_0^2 x - \frac{1}{3}x^2 = G . \tag{22}$$

It is interesting to note that $x > 0$ at any moment. Moreover, one may easily calculate the transition point from deceleration to acceleration which corresponds to $G = 0$: $x(G = 0) = 12H_0^2$. The solution of (22) is:

$$x_{1,2} = 3H_0^2 \pm \sqrt{9^2 H_0^4 - 3G} . \tag{23}$$

Note that sign "$-$" must be excluded because it corresponds to non-physical negative values of x for negative G, which corresponds to accelerated regimes. Now we can see that only values $G < 27H_0^4$ are resolved. Now it is necessary to use the function (20) in order to find the theory $F(G)$ as the solution of the differential equation (18). It turns out that this differential equation is extremely complicated and the corresponding solution may be found only numerically for different asymptotics (near to transition point). In principle, it is easier to construct such solutions in the alternative representation of $F(G)$ theory with auxiliary scalar. The corresponding examples are found in third and fourth papers from ref.[7]. That is why we will no go further to technical details of the solution of eq.(18). Hence, in principle it is possible to construct $F(G)$ which allows the transition from deceleration to acceleration era (of course if we can solve the corresponding differential equation).

V. ALTERNATIVE REPRESENTATIONS OF $F(R)$-GRAVITY AND $F(G)$-GRAVITY AND THE RECONSTRUCTION

Let us discuss the alternative representation for $F(R)$-gravity and $F(G)$-gravity with the actions given by (10) and (1), respectively.

In addition to the problems mentioned in the previous sections, there appear other problems in $F(R)$- and $F(G)$-gravities. First problem is easy to understand in terms of $F(R)$ gravity. The Einstein gravity coupled with perfect fluid with constant equation of state (EoS) parameter w can be reproduced by the following $F(R)$ theory

$$F(R) \propto R^m, \quad m = \frac{9w + 7 \pm \sqrt{45w^2 + 126w + 53}}{6\,(w+1)} \text{ or } w = -1 - \frac{2(m-2)}{3(m-1)(2m-1)}. \tag{24}$$

We may investigate the modified gravity with Big Bang singularity with $w_{\mathrm{BB}} > 0$ and Big Rip singularity [11] with $w = w_{\mathrm{BR}} < 0$. In both of the Big Bang singularity and Big Rip singularity, the scalar curvature R diverges. This shows that if we construct a model describing both of Big Bang singularity and Big Rip singularity, the corresponding $F(R)$ must be double valued function of R.

Similarly for $F(G)$-gravity, if we try to construct a realistic model, where there is a transition from decelerating phase to the accelerating phase, $F(G)$ may become a double valued function or it may become purely imaginary function.

In the following, we consider how the above problem could be solved. At least locally we can rewrite the actions (10) and (1) by introducing the auxiliary scalar field ϕ as follows

$$\tilde{S}_{F(R)} = \int dx^4 \sqrt{-g} \left(\frac{P(\phi) + Q(\phi)R}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right), \tag{25}$$

and

$$\tilde{S}_{F(G)} = \int dx^4 \sqrt{-g} \left(\frac{R}{2\kappa^2} - V(\phi) + f(\phi)G + \mathcal{L}_{\mathrm{matter}} \right). \tag{26}$$

We should note, however, the actions (25) and (26) express more wide class of theories than the actions (10) and (1) (for related discussion, see also [12]). For example, we may consider the following model corresponding to $F(R)$ gravity:

$$P(\phi) = \frac{1}{3}\phi^3 + \beta\phi^2, \quad Q(\phi) = \gamma\phi. \tag{27}$$

Here β and γ are constants. (The following arguments do apply even for $F(G)$ gravity.) Then by the variation of ϕ, one finds

$$0 = \phi^2 + 2\beta\phi + \gamma R, \tag{28}$$

which can be solved with respect ϕ as

$$\phi = -\beta \pm \sqrt{\beta^2 - \gamma R}, \tag{29}$$

which gives

$$\tilde{S}_{F(R)} = \int dx^4 \sqrt{-g} \left(\frac{F_\pm(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right), \quad F_\pm(R) \equiv \left(-\frac{2\beta^2}{3} + \frac{\gamma R}{3} \right) \left(-\beta \pm \sqrt{\beta^2 - \gamma R} \right) \tag{30}$$

The action (30) is double-valued function and furthermore the value of R is restricted to be $\gamma R < \beta^2$ in order to have the real $\tilde{S}_{F(R)}$. Hence, the action (10) describes the theory corresponding to one of the branches of double-valued function and R is restricted to be $\gamma R < \beta^2$. We should note, however, that we need not to start from the action (10) but from the action (25). The action (25) may describe the scalar field theory with potential $-\frac{P(\phi)}{2\kappa^2}$ and the Brans-Dicke non-minimal coupling $\frac{Q(\phi)}{2\kappa^2}$ but without the kinetic term. If we start with the action (25) instead of (10) from the very beginning, even if we consider the model (27), we may obtain the theory with transition between $F_+(R)$ and $F_-(R)$. Note that, in the model corresponding to (25) with (27), the value of R can be, in general, in the region $\gamma R < \beta^2$, which is forbidden for the action (10).

Let us clarify it in more detail. For simplicity, we neglect the contribution from matter by omitting $\mathcal{L}_{\mathrm{matter}}$. First we consider the following model corresponding to (25):

$$P(\phi) = e^{\tilde{g}(\phi)/2}\tilde{p}(\phi) , \quad \tilde{g}(\phi) = -10\ln\left[\left(\frac{\phi}{t_0}\right)^{-\gamma} - C\left(\frac{\phi}{t_0}\right)^{\gamma+1}\right] ,$$

$$\tilde{p}(\phi) = \tilde{p}_+\phi^{\beta_+} + \tilde{p}_-\phi^{\beta_-} , \quad \beta_\pm \equiv \frac{1 \pm \sqrt{1 + 100\gamma(\gamma+1)}}{2} ,$$

$$Q(\phi) = -6\left[\frac{d\tilde{g}(\phi)}{d\phi}\right]^2 P(\phi) - 6\frac{d\tilde{g}(\phi)}{d\phi}\frac{dP(\phi)}{d\phi} , \tag{31}$$

Here t_0, C, and $\tilde{p}_\pm$ are constants. Now $P(\phi)$ and $Q(\phi)$ are smooth functions of ϕ as long as

$$0 < \phi < t_s \equiv t_0 C^{-1/(2\gamma+1)} . \tag{32}$$

The exact solution of the FRW equation is

$$H(t) = \left(\frac{10}{t_0}\right)\left[\frac{\gamma\left(\frac{t}{t_0}\right)^{-\gamma-1} + (\gamma+1)C\left(\frac{t}{t_0}\right)^{\gamma}}{\left(\frac{t}{t_0}\right)^{-\gamma} - C\left(\frac{t}{t_0}\right)^{\gamma+1}}\right] , \tag{33}$$

When $t \to 0$, i.e., $t \ll t_s$, $H(t)$ behaves as

$$H(t) \sim \frac{10\gamma}{t} , \tag{34}$$

which corresponds to the Big Bang singularity at $t = 0$. On the other hand, when $t \to t_s$, we find

$$H(t) \sim \frac{10}{t_s - t} . \tag{35}$$

which corresponds to the Big Rip singularity. Then in the form (25) of the action, one can obtain the cosmological model describing both of the Big Bang and Big Rip singularities. In this alternative presentation with auxiliary scalar, it is easy also to construct the deceleration-acceleration transition solutions. Such reconstruction has been presented already for $F(R)$ and $F(G)$ theories in refs.[4, 10] and third and fourth papers from ref.[7]. That is why we do not give the details of such cosmologically-viable theories here.

We should also note that the modified gravity which exhibits the transition from deceleration epoch to acceleration epoch can be obtained without introducing the auxiliary field ϕ (compare with reconstruction in refs.[9, 10, 13]). For example, we consider the following form of $F(R)$:

$$F(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) . \tag{36}$$

Here A and B are constants, $F(\alpha, \beta, \gamma; x)$ is Gauss' hypergeometric function, x is defined by $x = \frac{R}{3H_0^2} - 3$, and

$$\gamma = -\frac{1}{2} , \alpha + \beta = -\frac{1}{6} , \quad \alpha\beta = -\frac{1}{6} . \tag{37}$$

The action has an exact solution which reproduces, without real matter, the ΛCDM era whose FRW equation is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0 a^{-3} . \tag{38}$$

Next we consider the following $F(G)$ gravity model corresponding to the action (26):

$$V(\phi) = \frac{3}{\phi_0\kappa^2}\left(1 + g_1\frac{\phi_0}{\phi}\right)^2 - \frac{6g_1}{\phi_0^2\kappa^2}\left(1 + g_1\frac{\phi_0}{\phi}\right)\left(\frac{\phi}{\phi_0}\right)^{g_1} W\left(-g_1 - 1, \frac{\phi}{\phi_0}\right) ,$$

$$f(\phi) = \frac{\phi_0^2 g_1}{4\kappa^2}\int^{\frac{\phi}{\phi_0}} dx \frac{e^x x^{g_1}}{(1 + g_1 x)^2} W\left(-g_1 - 1, x\right) . \tag{39}$$

Here g_1 and ϕ_0 and positive constants and $W(\alpha, x)$ is given by the incomplete gamma function:

$$W(\alpha, x) = \int^x dy \mathrm{e}^{-y} y^{\alpha-1} \, . \tag{40}$$

Note that the functions $V(\phi)$ and $f(\phi)$ are smooth functions as long as $\phi > 0$. An exact solution of the model is given by

$$H(t) = \frac{1}{\phi_0} + \frac{g_1}{t} \, . \tag{41}$$

When t is small $H(t)$ describes the Big Bang singularity where the expansion of the universe is decelerating if $g_1 < 1$. On the other hand, when t is large H goes to a constant: $H \to \frac{1}{\phi_0}$, which corresponds to the de Sitter universe and the universe is expanding with the acceleration.. Hence, starting from the theory with the action (26), one can explicitly construct a model which admits the approximate transition from decelerating phase to the accelerating phase,

VI. DISCUSSION

In summary, we discussed the cosmological reconstruction method for modified Gauss-Bonnet and $F(R)$ gravities. Two alternative representations for the action is used: with and without the auxiliary scalar field. It turns out that the cosmological solutions in the representation with the auxiliary scalar follow from the wider class of theories. Moreover, it is easier to reconstruct modified gravity in such representation. For instance, the cosmological solution which contains the Big Bang and Big Rip singularities may be reconstructed in such formulation with the auxiliary scalar but not in the original formulation. Special attention is paid to the cosmologies admitting the deceleration-acceleration transitions. It is shown that such cosmological solutions may be reconstructed in both representations of modified gravity but only approximately. The analytical deceleration-acceleration transition cosmology in modified Gauss-Bonnet gravity satisfying to some reasonable conditions is shown to be impossible. It is extremely hard (if possible at all) to find such analytical solutions in modified Gauss-Bonnet gravity.

The detailed understanding of the background evolution of modified gravity is the necessary step in the development of the cosmological perturbations. Hence, even the approximate background evolution realized via the reconstruction method may serve for this purpose in order to select the most realistic theories confronting them with the observational data.

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