Semi-classical treatment of proton-neutron monopole interaction

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Abstract

We apply a time dependent variational method to a many-body Hamiltonian consisting of a spherical shell model term, a proton-proton and neutron-neutron pairing interaction and a monopole particle-hole and particle-particle proton-neutron interaction. The variational state is a generalized BCS state where all $T = 1$ Cooper pairs with $T_z = 0, \pm 1$ are included. Stationary solutions correspond to generalized BCS equations and define the static ground state. The linearized equations of motion are of RPA type and describe small oscillations of the nuclear system around the static ground state. Numerical application is made for a one level case. In contrast to previous treatments, the proton-neutron particle-particle interaction is included first in the mean field equations, defining the quasiparticle approximation, And then the residual interaction is taken into account by the RPA approach. In this way one obtains a non-collapsing RPA ground state.

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I. INTRODUCTION

Pairing correlations of nucleons of the same charge have been extensively studied in various physical contexts [1–6]. Many properties of atomic nuclei like gap parameters, moments of inertia in deformed nuclei, spectroscopic factors are nicely explained by considering the pairing interaction among alike nucleons. The standard way to treat the pairing correlations in nuclei is the BCS approach. In such a formalism symmetries like gauge and isospin invariance are broken. Since there are physical observables which are sensitive to the departure from these symmetries, several groups worked on restoring these symmetries [7–10].

Though not so extensively, the proton-neutron Cooper pairs have been also investigated [11–26]. Including the proton neutron pairing, various phenomena like stability of the ground state against the particle particle proton-neutron interaction, the structure of nuclei lying close to the proton drip line, the structure of the $N = Z$ nuclei and the structure of superdense nuclear matter might be realistically approached. The proton-neutron pairing has the peculiarity, comparing it to the proton-proton and neutron-neutron pairing, that besides the $T = 1$ pairs there might exist also $T = 0$ pairing and the corresponding pairs may play an important role in some phenomena like the structure of the $N = Z$ nuclei [16,20] or the Gamow Teller beta transitions [21,22]. The competition of the $T = 1$ and $T = 0$ pairing has been considered by several authors [17,20,23,24] particularly in the context of high spin states.

The BCS formalism for alike nucleons has the salient feature that the ground state is described by a coherent state for the SU(2) group associated to the quasi-spin algebra. This provides not only some beauty to the mathematical framework but also a great technical simplifications. In a time dependent variational treatment, it is relatively easy to find out a set of canonically conjugate coordinates in the classical phase space [27]. Such a property is lost when besides this pairing interaction the proton-neutron pairing is switched on. Guided by previous experience, the ansatz for the variational state describing the ground state of a system with pp, pn and pn pairing is chosen to be a vacuum for the quasiparticle

operators defined by a generalized Bogoliubov-Valatin (BV) transformation appropriate for the particular problem under consideration.

The aim of the present paper is to study a many body Hamiltonian, commonly used to study the beta transitions of Fermi type [26,28,29], which involves the proton-neutron monopole particle-hole and particle-particle two body interaction. In the remaining of the text we refer to these interactions as to the *ph* and *pp* interactions, respectively. Since the *pp* interaction used in the model Hamiltonian is the $T = 1$ proton-neutron pairing, it is worthwhile to treat it on an equal footing with the proton-proton and neutron-neutron pairing. This allows to investigate the point whether one may cure the long standing problem of the ground state instability reached at a certain critical value of the *pp* interaction strength.

We state from the beginning that there is no attempt here to discuss generalized BV transformations that are relevant for the study of other problems like deformed nuclei and high spin states. Indeed, many achievements on this line have been reported, in the literature. Our work here is motivated by a well known problem found in the description of the nuclei undergoing a double beta decay. Since 1983, when Cha [30] noticed that the β^+ transition rates are very sensitive to the strength of the *pp* interaction, all studies of double beta decay based on pnQRPA formalism, or related formalisms, where including the *pp* interaction, although in the standard RPA calculations such terms of the two body interaction are ignored. Such studies show that as a function of the *pp* interaction strength, the transition amplitude has a plateau and then decreases very fast to zero (see for instance [31]). In this interval of fast decrease, the transition amplitude reaches values compatible with experimental data, soon the amplitude vanishes and immediately after the pnQRPA breaks down. A lot of work has been devoted to cure the instability of the ground state around the critical value of the strength of the *pp* interaction. Here we propose a solution to cure this ill-behavior of the QRPA ground state. Our point here is that the proton-neutron interaction is contributing first to the mean field and therefore it is taken into account in the minimization procedure used to find the static ground state. Furthermore the residual interaction is treated by the usual QRPA procedure.

The ground state is the static solution of a time dependent variational principle (TDVP) set of equations and includes correlations due to both the proton-proton, neutron-neutron pairing and the proton-neutron *ph* and *pp* interaction. The variational state is taken as a product of three coherent states describing the proton-proton, neutron-neutron and proton-neutron T=1 pairing. We determine not only the ground state properties by solving the generalized pairing equations but also the small oscillations around the static ground state. This project is achieved in several steps according to the following plan. In Section 2 we define our model Hamiltonian, formulate the time dependent formalism and derive the classical equations of motion. The static solution is discussed in Section 3. The equations of motion are linearized around the point where the energy is minimum and RPA like equations are derived in Section 4. Besides the energy there are another two constants of motion, therefore there are two spurious states of vanishing solutions which are completely decoupled from the physical solutions. This property is also discussed in Section 4. Some final conclusions are analytically obtained for the case of a single j in Section 5. Numerical application refers to a single j and is analyzed in Section 6. The final conclusions are drawn in Section 7.

II. THE MODEL HAMILTONIAN AND THE CLASSICAL EQUATIONS OF MOTION

In the present paper we shall deal with a heterogeneous system of nucleons which move in a spherical shell model mean field and interact among themselves in the following manner. Nucleons of similar charge interact through monopole pairing forces while protons and neutrons interact by a monopole particle-hole and a monopole particle-particle two body term. The many body Hamiltonian describing such a system reads:

$$H = \sum_{\tau,j,m} (\epsilon_\tau - \lambda_\tau) c^\dagger_{\tau jm} c_{\tau jm} - \frac{G_p}{4} \sum_{j,m;j',m'} c^\dagger_{pjm} c^\dagger_{\widetilde{pjm}} c_{\widetilde{pj'm'}} c_{pj'm'} - \frac{G_n}{4} \sum_{j,m;j',m'} c^\dagger_{njm} c^\dagger_{\widetilde{njm}} c_{\widetilde{nj'm'}} c_{nj'm'}$$
$$+ 2\chi (c^\dagger_{pj} c_{nj})_0 (c^\dagger_{nj'} c_{pj'})_0 - 2\chi_1 (c^\dagger_{pj} c^\dagger_{nj})_0 (c_{nj'} c_{pj'})_0 . \qquad (2.1)$$

$c^{\dagger}_{\tau jm}(c_{\tau jm})$ denotes the creation (annihilation) operator of one particle of $\tau(=p,n)$ type in the spherical shell model state $|\tau;nljm\rangle \equiv |\tau jm\rangle$. The time reversed state corresponding to $|\tau;nljm\rangle$ is denoted by $|\tau;\widetilde{jm}\rangle = (-)^{j-m}|\tau;j-m\rangle$. This Hamiltonian is very often used in the literature to describe single and double beta Fermi transitions within a pnQRPA formalism [26,28,29].

The model Hamiltonian will be treated within a time dependent variational formalism. Thus we are looking for the solutions of the following variational equation [27,32,33]:

$$\delta \int_0^t \langle \Psi | H - i\frac{\partial}{\partial t'} | \Psi \rangle dt' = 0. \tag{2.2}$$

If the state $|\Psi\rangle$ spans the whole Hilbert space for the many nucleons system, solving the equation (2.2) is equivalent to solving the time dependent Schroedinger equation. That would mean that we replaced one very difficult problem with one with similar degree of difficulty. Since we are interested to study not only the properties of the static ground state but also the small oscillations around it we choose a variational state of the following form:

$$|\Psi\rangle = \Psi(z_p, z_p^*; z_n, z_n^*; z_{pn}, z_{pn}^*) = e^{T_{pn}} e^{T_p} e^{T_n} |0\rangle. \tag{2.3}$$

where the transformations specified by the operators T are given by:

$$T_{pn} = \sum_{jm} (z_{pnj} c^{\dagger}_{pjm} c^{\dagger}_{\widetilde{njm}} - z^*_{pnj} c_{\widetilde{njm}} c_{pjm}), \tag{2.4}$$

$$T_{p} = \sum_{jm} (z_{pj} c^{\dagger}_{pjm} c^{\dagger}_{\widetilde{pjm}} - z^*_{pj} c_{\widetilde{pjm}} c_{pjm}), \tag{2.5}$$

$$T_{n} = \sum_{jm} (z_{nj} c^{\dagger}_{njm} c^{\dagger}_{\widetilde{njm}} - z^*_{nj} c_{\widetilde{njm}} c_{njm}). \tag{2.6}$$

and $|0\rangle$ stands for the particle vacuum state. These transformations depend on the parameters z which are complex functions of time. The corresponding complex conjugate functions are denoted by z^*. The parameters (z, z^*) play the role of classical coordinates and conjugate momenta, respectively. We recognize immediately the Bogoliubov-Valatin transformations for proton-proton, neutron-neutron and proton-neutron pairing interactions, respectively:

$$b^{\dagger}_{pjm} \equiv e^{T_p} c^{\dagger}_{pjm} e^{-T_p} = U_{pj} c^{\dagger}_{pjm} - V_{pj} c_{\widetilde{pjm}}, \tag{2.7}$$

$$b^{\dagger}_{njm} \equiv e^{T_n} c^{\dagger}_{njm} e^{-T_n} = U_{nj} c^{\dagger}_{njm} - V_{nj} c_{\widetilde{njm}}, \tag{2.8}$$

$$a^{\dagger}_{pjm} \equiv e^{T_{pn}} c^{\dagger}_{pjm} e^{-T_{pn}} = U_j c^{\dagger}_{pjm} - V_j \widetilde{c_{njm}}, \tag{2.9a}$$

$$a^{\dagger}_{njm} \equiv e^{T_{pn}} c^{\dagger}_{njm} e^{-T_{pn}} = U_j c^{\dagger}_{njm} - V_j \widetilde{c_{pjm}} \tag{2.9b}$$

The coefficients U, V entering the above equations are related with the parameters z, defining the variational state $|\Psi\rangle$, by the equations:

$$\begin{aligned} z_{pj} &= \rho_{pj} e^{i\varphi_{pj}},\ U_{pj} = \cos 2\rho_{pj},\ V_{pj} = e^{-i\varphi_{pj}} \sin 2\rho_{pj}, \\ z_{nj} &= \rho_{nj} e^{i\varphi_{nj}},\ U_{nj} = \cos 2\rho_{nj},\ V_{nj} = e^{-i\varphi_{nj}} \sin 2\rho_{nj}, \\ z_{pnj} &= \rho_{pnj} e^{i\varphi_{pnj}},\ U_j = \cos \rho_{pnj},\ V_j = e^{-i\varphi_{pnj}} \sin \rho_{pnj}. \end{aligned} \tag{2.10}$$

It is worthwhile to remark on a very important feature of the state Ψ. Using the Cambel-Hausdorf factorization [35] for the first two exponential operators and then bringing the operator $\exp(T_{pn})$ near the particle vacuum state one obtains:

$$\begin{aligned} |\Psi\rangle &= N_p N_n e^{T_{pn}} e^{\sum_{jm} A_{pj} c^{\dagger}_{pjm} \widetilde{c^{\dagger}_{pjm}}} e^{\sum_{jm} A_{nj} c^{\dagger}_{njm} \widetilde{c^{\dagger}_{njm}}} |0\rangle \\ &= N_p N_n e^{\sum_{jm} A_{pj} a^{\dagger}_{pjm} \widetilde{a^{\dagger}_{pjm}}} e^{\sum_{jm} A_{nj} a^{\dagger}_{njm} \widetilde{a^{\dagger}_{njm}}} e^{T_{pn}} |0\rangle = e^{T'_p} e^{T'_n} |BCS\rangle_{pn}. \end{aligned} \tag{2.11}$$

with the notations:

$$N_\tau = \prod_j (U_{\tau j})^{\Omega_j},\ A_{\tau j} = \frac{V_{\tau j}{}^*}{U_{\tau j}},\ \tau = p, n. \tag{2.12}$$

The notation $|BCS\rangle_{pn}$ stands for the vacuum state of the quasiparticle operators $a_{\tau jm}$ defined by eqs. (2.9), i.e. the BCS state for the proton-neutron pairing interaction alone. The operators T'_p, T'_n are obtained from the corresponding operators T_p, T_n defined by eqs. (2.5, 2.6) by replacing the particle operators $c^{\dagger}, c$ with the corresponding quasiparticle operators $a^{\dagger}, a$. This new form of the trial function shows that it is more general than the variational state used in ref. [26]. The expression (2.11) shows that our variational state can be written as a product of two BCS-like operators of quasiparticles acting on a standard proton-neutron BCS state. From this function one can immediately obtain an equivalent form:

$$|\Psi\rangle = \prod_{j,m\rangle 0} \left[f_0 + f_p c^\dagger_{pjm} c^\dagger_{\widetilde{pjm}} + f_n c^\dagger_{njm} c^\dagger_{\widetilde{njm}} + f_{pn}(c^\dagger_{pjm} c^\dagger_{\widetilde{njm}} + c^\dagger_{njm} c^\dagger_{\widetilde{pjm}}) \right] |0\rangle, \tag{2.13}$$

where the amplitudes f have simple expressions in terms of the U and V coefficients defined above. Equation (2.13) has the form of the general BCS wave function for $T = 1$ ($J = 0$) pairing. Denoting by $\hat{U}$ the unitary transformation (2.3) and by α the images of the fermion operators through $\hat{U}$

$$\begin{pmatrix} \alpha^\dagger_{1jm} \\ \alpha^\dagger_{2jm} \\ \alpha_{\widetilde{1jm}} \\ \alpha_{\widetilde{2jm}} \end{pmatrix} = \hat{U} \begin{pmatrix} c^\dagger_{pjm} \\ c^\dagger_{njm} \\ c_{\widetilde{pjm}} \\ c_{\widetilde{njm}} \end{pmatrix} \hat{U}^\dagger, \tag{2.14}$$

and taking into account the BV equations (2.7-9) we obtain the total quasiparticle transformation:

$$\begin{pmatrix} \alpha^\dagger_{1jm} \\ \alpha^\dagger_{2jm} \\ \alpha_{\widetilde{1jm}} \\ \alpha_{\widetilde{2jm}} \end{pmatrix} = \begin{pmatrix} U_{pj}U_j & -V_{pj}V^*_j & -V_{pj}U_j & -U_{pj}V_j \\ -V_{nj}V^*_j & U_{nj}U_j & -U_{nj}V_j & -V_{nj}U_j \\ V^*_{pj}U_j & U_{pj}V^*_j & U_{pj}U_j & -V^*_{pj}V_j \\ U_{nj}V^*_j & V^*_{nj}U_j & -V^*_{nj}V_j & U_{nj}U_j \end{pmatrix} \begin{pmatrix} c^\dagger_{pjm} \\ c^\dagger_{njm} \\ c_{\widetilde{pjm}} \\ c_{\widetilde{njm}} \end{pmatrix}. \tag{2.15}$$

This is our BV matrix for generalized monopole pairing. It can be written as a product of three matrices describing one BV transformation and two Hartree Fock transformations in the isospin space [25]. The transformation coefficients are defined as functions of the U and V coefficients given by the equations (2.10). Thus, although the factorised form involves three distinct BV transformations, this can be written in an alternative form which agrees with the Bloch-Messiah theorem [25]. The transformation (2.14) does not mix states of different angular momentum since our model Hamiltonian involves only monopole pairing interactions.

In order to write the equations of motion provided by the variational principle (2.2), one needs to know the expression for the expectation values of the model Hamiltonian and the classical action. By a direct calculation one finds for the expectation value of H the expression given in the Appendix A. In the present paper we shall neglect the re-

normalization of the single particle energies due to the two body terms, in the spirit of the standard BCS approach. Therefore the classical energy will be taken as:

$$\mathcal{H} = \sum_j (\epsilon_{pj} - \lambda_p)(2j+1)V^2_{\text{eff},pj} + \sum_j (\epsilon_{nj} - \lambda_n)(2j+1)V^2_{\text{eff},nj} \\ -\frac{|\Delta_p|^2}{G_p} - \frac{|\Delta_n|^2}{G_n} + \frac{2|\beta_-|^2}{\chi} - \frac{2|\Delta_{pn}|^2}{\chi_1}, \tag{2.16}$$

where the following notations have been used:

$$V^2_{\text{eff},pj} = (U_j^2|V_{pj}|^2 + |V_j|^2 U_{nj}^2),$$

$$V^2_{\text{eff},nj} = (U_j^2|V_{nj}|^2 + |V_j|^2 U_{pj}^2),$$

$$\Delta_p \equiv \frac{G_p}{2}\langle\Psi|\sum_{j,m} c^\dagger_{pjm} c^\dagger_{\widetilde{pjm}}|\Psi\rangle = \frac{G_p}{2}\sum_j (U_j^2 U_{pj}V_{pj} - V_j^2 U_{nj}{V_{nj}}^*),$$

$$\Delta_n \equiv \frac{G_n}{2}\langle\Psi|\sum_{j,m} c^\dagger_{njm} c^\dagger_{\widetilde{njm}}|\Psi\rangle = \frac{G_n}{2}\sum_j (U_j^2 U_{nj}V_{nj} - V_j^2 U_{pj}{V_{pj}}^*),$$

$$\beta_- \equiv \chi\langle\Psi|\sum_j (c^\dagger_{pj}c_{nj})_0|\Psi\rangle = \chi\sum_j \hat{j}(U_j V_j^* U_{pj}V_{pj} + U_j V_j U_{nj}{V_{nj}}^*),$$

$$\beta_+ \equiv (\beta_-)^* = \chi\langle\Psi|\sum_j (c^\dagger_{nj}c_{pj})_0|\Psi\rangle,$$

$$\Delta_{pn} \equiv \chi_1\langle\Psi|\sum_j (c^\dagger_{pj}c^\dagger_{nj})_0|\Psi\rangle = \chi_1\sum_j \hat{j}U_jV_j(U^2_{pj} - |V_{nj}|^2),\ \hat{j} = \sqrt{2j+1}. \tag{2.17}$$

The Fermi level energies λ_p and λ_n are determined so that the average number of protons and neutrons are equal to Z and N, respectively:

$$Z = \sum_j (2j+1)V^2_{\text{eff},pj}, \\ N = \sum_j (2j+1)V^2_{\text{eff},nj}. \tag{2.18}$$

The classical action has the expression:

$$\langle\Psi| - i\frac{\partial}{\partial t}|\Psi\rangle = \sum_j \Omega_j[V^2_{pj}\ \dot{\varphi}_{pj} + V^2_{nj}\ \dot{\varphi}_{nj} + 2V_j^2(1 - V^2_{pj} - V^2_{nj})\ \dot{\varphi}_{pnj}], \tag{2.19}$$

where $\bullet$ stands for the time derivative.

Having in mind the aim of quantizing the classical trajectories it is useful to have the equations of motion in the canonical Hamilton form. To this aim we change the classical generalized coordinates by the following transformation:

$$
\begin{aligned}
r_{0j} &= \Omega_j(2\sin^2\rho_{pnj}-1)(1-\sin^2 2\rho_{pj}-\sin^2 2\rho_{nj}), \quad \varphi_{0j}=\varphi_{pnj};\ \Omega_j=\frac{2j+1}{2},\\
r_{-j} &= \frac{1}{2}\Omega_j(\sin^2 2\rho_{pj}-\sin^2 2\rho_{nj}), \qquad \varphi_{-j}=\varphi_{pj}-\varphi_{nj},\\
r_{+j} &= \frac{1}{2}\Omega_j(\sin^2 2\rho_{pj}+\sin^2 2\rho_{nj}), \qquad \varphi_{+j}=\varphi_{pj}+\varphi_{nj}-2\varphi_{pnj},
\end{aligned}
\tag{2.20}
$$

In terms of the new coordinates, the equations of motion provided by the variational principle (2.2) have the canonical form:

$$
\begin{aligned}
\frac{\partial\mathcal{H}}{\partial r_{0j}} &= -\dot{\varphi}_{0j}, \qquad \frac{\partial\mathcal{H}}{\partial\varphi_{0j}} = \dot{r}_{0j},\\
\frac{\partial\mathcal{H}}{\partial r_{-j}} &= -\dot{\varphi}_{-j}, \qquad \frac{\partial\mathcal{H}}{\partial\varphi_{-j}} = \dot{r}_{-j},\\
\frac{\partial\mathcal{H}}{\partial r_{+j}} &= -\dot{\varphi}_{+j}, \qquad \frac{\partial\mathcal{H}}{\partial\varphi_{+j}} = \dot{r}_{+j}\ .
\end{aligned}
\tag{2.21}
$$

These equations suggest that the variables r_{0j}, r_{-j}, r_{+j} play the role of classical coordinates while $\varphi_{0j}, \varphi_{-j}, \varphi_{+j}$ are their corresponding conjugate momenta. Two of the chosen coordinates have a nice physical meaning. Indeed, denoting by $\hat{N}_{\tau j}$ the τ-particle number operator for the shell j:

$$
\hat{N}_{\tau j} = \sum_m c^\dagger_{\tau jm} c_{\tau jm},
\tag{2.22}
$$

one easily checks that the following equations hold:

$$
\begin{aligned}
r_{0j} &= \langle\Psi|-\Omega_j+\frac{\hat{N}_{pj}+\hat{N}_{nj}}{2}|\Psi\rangle \equiv \langle\Psi|\hat{M}_{zj}|\Psi\rangle,\\
2r_{-j} &= \langle\Psi|\frac{\hat{N}_{pj}-\hat{N}_{nj}}{2}|\Psi\rangle \equiv \langle\Psi|\hat{T}_{zj}|\Psi\rangle.
\end{aligned}
\tag{2.23}
$$

These equations indicate that $r_{0j}, 2r_{-j}$ are classical variables associated to the z-components of the *pn* quasi-spin ($\hat{M}_{zj}$) and isospin ($\hat{T}_{zj}$) , for each shell j, respectively. In terms of the new coordinates the effective occupation probabilities are:

$$
\begin{aligned}
V^2_{\text{eff},pj} &= \frac{1}{2}+\frac{2r_{-j}+r_{0j}}{2\Omega_j},\\
V^2_{\text{eff},nj} &= \frac{1}{2}+\frac{-2r_{-j}+r_{0j}}{2\Omega_j}.
\end{aligned}
\tag{2.24}
$$

The equations of motion are written explicitly in Appendix B. From the results presented there one finds:

$$\sum_j \dot{r}_{0j} = 0,$$
$$\sum_j \dot{r}_{-j} = 0. \tag{2.25}$$

which results in having two constants of motion:

$$\mathcal{M}_z = \sum_j \langle \Psi | \hat{M}_{zj} | \Psi \rangle,$$
$$\mathcal{T}_z = \sum_j \langle \Psi | \hat{T}_{zj} | \Psi \rangle. \tag{2.26}$$

This is a remarkable result. Indeed, although the trial function breaks gauge and isospin symmetries, the classical trajectories conserve these symmetries. This is a reminiscence of the fact that the quantum mechanical operator H commutes with $\sum_j \hat{M}_{zj}$ and $\sum_j \hat{T}_{zj}$. As we shall see later on, the nice consequence of this property is that the spurious solutions of the RPA equations are fully separated from the physical ones. Of course there exists a third constant of motion which is the classical energy. Indeed, using the equations of motion, one easily checks that

$$\dot{\mathcal{H}} = 0. \tag{2.27}$$

This feature is specific to any system governed by equations of motion derived from a variational principle. Therefore the classical trajectories are lying on the surface

$$\mathcal{H} = const, \tag{2.28}$$

which is conventionally called the energy surface. Note that the minima of this surface correspond to the potential energy of the system, since there the system has vanishing kinetic energy.

III. THE STATIC GROUND STATE

The equations of motion (2.18) are highly non-linear and therefore can not be solved analytically. However a good deal of information about the local behavior of the solution

of the non-linear equations can be drawn from the analysis of the solution of the linearized equations. Such a solution describes small oscillations around a static ground state. Therefore we have first to search for the stationary points, where the time derivatives of the generalized coordinates vanish. The equations (2.18) show that they are also stationary points for the energy surface. We consider the classical system at rest which implies that $\varphi_{0j} = \varphi_{-j} = \varphi_{+j} = 0$. Consequently $\Delta_p, \Delta_n, \Delta_{pn}, \beta_-$ are real quantities and the equations corresponding to the time derivatives of the coordinates, are automatically obeyed. For a given j-shell, the remaining equations are:

$$\epsilon_{pj} - \lambda_p + \epsilon_{nj} - \lambda_n - \frac{U_{pj}V_{pj} + U_{nj}V_{nj}}{V_{pj}^2 - U_{nj}^2}(\Delta_p + \Delta_n)$$
$$-\frac{2}{\hat{j}}\frac{U_j^2 - V_j^2}{U_jV_j}\frac{U_{pj}V_{pj} + U_{nj}V_{nj}}{V_{pj}^2 - U_{nj}^2}\beta_- - \frac{2}{\hat{j}}\frac{U_j^2 - V_j^2}{U_jV_j}\Delta_{pn} = 0, \tag{3.1a}$$

$$\epsilon_{pj} - \lambda_p - (\epsilon_{nj} - \lambda_n) - \frac{U_{pj}^2 - V_{pj}^2}{2U_{pj}V_{pj}}\left(U_j^2\Delta_p - V_j^2\Delta_n - \frac{4}{\hat{j}}U_jV_j\beta_-\right)$$
$$-\frac{U_{nj}^2 - V_{nj}^2}{2U_{nj}V_{nj}}\left(V_j^2\Delta_p - U_j^2\Delta_n + \frac{4}{\hat{j}}U_jV_j\beta_-\right) = 0, \tag{3.1b}$$

$$\frac{U_{pj}^2 - V_{pj}^2}{2U_{pj}V_{pj}}\left(U_j^2\Delta_p - V_j^2\Delta_n - \frac{4}{\hat{j}}U_jV_j\beta_-\right) - \frac{U_{nj}^2 - V_{nj}^2}{2U_{nj}V_{nj}}\left(V_j^2\Delta_p - U_j^2\Delta_n + \frac{4}{\hat{j}}U_jV_j\beta_-\right)$$
$$+\frac{V_j^2 - U_j^2}{U_jV_j}\frac{U_{pj}V_{pj} + U_{nj}V_{nj}}{V_{pj}^2 - U_{nj}^2}\left[(\Delta_p + \Delta_n)U_jV_j + \frac{2}{\hat{j}}(U_j^2 - V_j^2)\beta_-\right] - \frac{2}{\hat{j}}\frac{1}{U_jV_j}\Delta_{pn} = 0. \tag{3.1c}$$

From (3.1) one can easily express the quantities U^2, V^2 in terms of $\Delta_p, \Delta_n, \Delta_{pn}, \lambda_p, \lambda_n, \beta_-$:

$$\begin{pmatrix} V_{pj}^2 \\ U_{pj}^2 \end{pmatrix} = \frac{1}{2}\left(1 \mp \frac{S_p|U_j^2(\epsilon_{pj} - \lambda_p) - V_j^2(\epsilon_{nj} - \lambda_n) + \frac{4}{\hat{j}}U_jV_j\Delta_{pn}|}{\sqrt{[U_j^2(\epsilon_{pj} - \lambda_p) - V_j^2(\epsilon_{nj} - \lambda_n) + \frac{4}{\hat{j}}U_jV_j\Delta_{pn}]^2 + (U_j^2\Delta_p - V_j^2\Delta_n - \frac{4}{\hat{j}}U_jV_j\beta_-)^2}}\right), \tag{3.2}$$

$$\begin{pmatrix} V_{nj}^2 \\ U_{nj}^2 \end{pmatrix} = \frac{1}{2}\left(1 \mp \frac{S_n|U_j^2(\epsilon_{nj} - \lambda_n) - V_j^2(\epsilon_{pj} - \lambda_p) + \frac{4}{\hat{j}}U_jV_j\Delta_{pn}|}{\sqrt{[U_j^2(\epsilon_{nj} - \lambda_n) - V_j^2(\epsilon_{pj} - \lambda_p) + \frac{4}{\hat{j}}U_jV_j\Delta_{pn}]^2 + (U_j^2\Delta_n - V_j^2\Delta_p - \frac{4}{\hat{j}}U_jV_j\beta_-)^2}}\right), \tag{3.3}$$

$$\left(\begin{array}{c} V_j^2 \\ U_j^2 \end{array}\right) = \frac{1}{2}\left(1 \mp \frac{S|\epsilon_{pj} - \lambda_p + \epsilon_{nj} - \lambda_n - \frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}(\Delta_p + \Delta_n)|}{\sqrt{[\epsilon_{pj} - \lambda_p + \epsilon_{nj} - \lambda_n - \frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}(\Delta_p + \Delta_n)]^2 + \frac{16}{2j+1}(\Delta_{pn} + \frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}\beta_-)^2}}\right). \tag{3.4}$$

The factors S_p, S_n, S are phases determined in the following way. The three equations from (3.1) can be written in the alternative form:

$$\begin{aligned} \frac{\mathcal{A}_j}{\mathcal{C}_j} &= \frac{U_j^2 - V_j^2}{2U_jV_j}, \\ \frac{\mathcal{A}_{\tau j}}{\mathcal{C}_{\tau j}} &= \frac{U_{\tau j}^2 - V_{\tau j}^2}{2U_{\tau j}V_{\tau j}}, \tau = p, n. \end{aligned} \tag{3.5}$$

with obvious notations for the terms from the left hand sides. The phases entering the equations defining the U and V coefficients are given by :

$$S = sign\left(\frac{\mathcal{A}_j}{\mathcal{C}_j}\right),\ S_\tau = sign\left(\frac{\mathcal{A}_{\tau j}}{\mathcal{C}_{\tau j}}\right),\ \tau = p, n. \tag{3.6}$$

Note that the equations defining U and V coefficients are coupled with each other. However in the single j case, equation (3.4) is very much simplified due to the following equality :

$$\frac{U_pV_p + U_nV_n}{V_p^2 - U_n^2} = -\frac{\chi_1\beta_-}{\chi\Delta_{pn}}. \tag{3.7}$$

In this way the coefficients U, V depend exclusively on gaps, Fermi energies and β_-. Inserting their expression into the eqs (3.2) and (3.3) one finds that the above statement is also true for $V_{pj}, U_{pj}, V_{nj}, U_{nj}$.

The three square root quantities involved in the equations defining the U and V coefficients define three quasiparticle energies. None of them is of pure neutron or pure proton type. For each of them both the proton and neutron from the given shell participate. While in two of these energies the single particle energies enter through the difference of the weighted single particle proton and neutron energies in the remaining case the sum of the proton and neutron energies appears. The first two cases differ from the standard quasiparticles appearing in homogeneous systems where the single particle energies are normalized by the

two body interaction only by additive terms. Indeed, here also a contraction, due to the multiplicative U_j^2 and V_j^2 factors, appears.

Denoting by N_s the number of the single particle levels taken into consideration, and assuming the same single particle space for protons and neutrons, there are $3N_s+2$ equations for $3N_s+2$ unknowns $(V_{pj}, V_{nj}, V_j, \lambda_p, \lambda_n)$: $3N_s$ static equations (3.1) and two constraints providing equations for the number of protons and the number of neutrons. The solutions of these equations are known once we solve the six equations defining the gaps, β_-, the number of protons and the number of neutrons. Indeed these equations, after replacing the U and V coefficients with their expressions in terms of gaps, β_- and λ_p, λ_n, become 6 nonlinear equations for the unknowns: $\Delta_p, \Delta_n, \Delta_{pn}, \beta_-, \lambda_p, \lambda_n$. Solving these equations and calculating the coefficients for the BV transformation one obtains the values for the canonical coordinates. It is worth mentioning that the stationary equations are necessary but not sufficient conditions for minima of $\mathcal{H}$. A rigorous test for minima stems from the properties of the RPA equations. Indeed, if the stationary point, found in the way described above, is a minimum then all solutions of the equations linearized around that point are real. We address this problem in the next Section.

IV. THE RPA EQUATIONS

Let us denote by (q,p) the deviations of the current variables from their stationary values, denoted by $\mathring{r}_{0j}, \mathring{\varphi}_{0j}, \mathring{r}_{-j}, \mathring{\varphi}_{-j}, \mathring{r}_{+j}, \mathring{\varphi}_{+j}$, respectively:

$$\begin{aligned} q_{1j} &= r_{0j} - \mathring{r}_{0j}, \quad p_{1j} = \varphi_{0j} - \mathring{\varphi}_{0j}, \\ q_{2j} &= r_{-j} - \mathring{r}_{-j}, \quad p_{2j} = \varphi_{-j} - \mathring{\varphi}_{-j}, \\ q_{3j} &= r_{+j} - \mathring{r}_{+j}, \quad p_{3j} = \varphi_{+j} - \mathring{\varphi}_{+j} \ . \end{aligned} \tag{4.1}$$

Expanding the right hand side of equations (2.18) around the minimum point and keeping only the linear terms in deviations one obtains:

$$\dot{p}_{kj} = \sum_{k'j'} A_{kj,k'j'} q_{k'j'},$$

$$\dot{q}_{kj} = \sum_{k'j'} B_{kj,k'j'} p_{k'j'}. \tag{4.2}$$

To save space we don't give the explicit expressions for the matrices $A_{kj,k'j'}$ and $B_{kj,k'j'}$. For the special case of a single level we give in Appendix C only two coefficients, A_{33} and B_{33}, which are needed to calculate the frequency of the harmonic mode. Note that the coefficients of the expansion in eqs. (4.2) are just the second order derivatives of the classical Hamiltonian taken in the considered stationary point. Therefore, to the linear terms in the equations of motion correspond the quadratic terms of the classical energy expansion. Note that in the first equation (4.2) linear terms in p do not appear while in the second equation the linear terms in q are missing. Their presence would violate the time reversal invariance of $\mathcal{H}$. The condition for the existence of the minimum value for $\mathcal{H}$ is that the associated Hessian be positive, which results in having a positive definite quadratic form for $\mathcal{H}$. As a matter of fact this assures that frequencies for the classical trajectories are all real numbers. It is worth mentioning that the linearized equations of motion can be written in the Hamilton canonical form (2.18) with $\mathcal{H}$ expanded up to the second order. Therefore the canonical form is not altered by the linearization process. Moreover the equations admit as constants of motion:

$$\begin{aligned} \mathcal{M}_z &= \sum_j q_{1j}, \\ \mathcal{T}_z &= \sum_j q_{2j}. \end{aligned} \tag{4.3}$$

This is implied by the following property of the matrix $B_{kj,k'j'}$:

$$\sum_j B_{kj,k'j'} = 0, \ k = 1,2; \ k' = 1,2,3. \tag{4.4}$$

A. Equations of motion in the RPA form

In what follows we write the equations of motion in a standard RPA form. To this aim we look for the transformation to a new set of generalized coordinates:

$$\begin{pmatrix} Q_i \\ P_i \end{pmatrix} = \begin{pmatrix} X_i & Y_i \\ Z_i & W_i \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{4.5}$$

chosen so that the following equations are fulfilled:

$$\dot{Q}_i = \omega P_i, \quad \dot{P}_i = -\omega_i Q_i. \tag{4.6}$$

The index "i" labels the solutions of the equations written above, if they exist. The equations (4.5) and (4.6) provide four sets of equations relating the matrix elements X, Y, W, Z. Only two of them are independent, the other two being obtained from the first ones by replacements:

$$Z_i = -X_i, \; W_i = Y_i. \tag{4.7}$$

The independent equations read:

$$B^T X = \omega Y, \; A^T Y = -\omega X. \tag{4.8}$$

It is easy to check that the new canonical coordinates satisfy the equations:

$$\{Q_k, \mathcal{H}\} = \dot{Q}_k, \; \{P_k, \mathcal{H}\} = \dot{P}_k, \tag{4.9}$$

where $\{,\}$ denotes the Poisson bracket defined in the standard way. To any two functions f and g, defined in the classical phase space, spanned by the conjugate coordinates (q, p), one associates the Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}. \tag{4.10}$$

The matrix elements X, Y defining the transformation (4.5) are related to the "forward" and "backward" phonon amplitudes in the quantum mechanical picture. To prove this we quantize [27,34] the classical motion. This can be achieved by the following algebra mapping $\{q_{kj}, p_{kj}, \{,\}\} \to \{\hat{q}, \hat{p}, \frac{1}{i}[,]\}$, where the inner multiplication operation is the Poisson bracket in the classical algebra while in the quantal algebra it is the commutator. By this mapping to

the canonical conjugate variables (q_{kj}, p_{kj}) correspond the coordinate and linear momentum operators obeying the commutation relations:

$$[\hat{q}, \hat{p}] = i. \tag{4.11}$$

Similarly one defines the operators $\hat{Q}, \hat{P}$ by quantizing the classical conjugate coordinates, Q, P. To these operators one associates the boson operators,

$$\begin{aligned} \left[C_k, C_{k'}^\dagger\right] &= \delta_{k,k'}, \\ \left[c_{kj}, c_{k'j'}^\dagger\right] &= \delta_{k,k'}\delta_{j,j'}. \end{aligned} \tag{4.12}$$

by the transformation:

$$\begin{aligned} \hat{q}_{kj} &= \frac{1}{\sqrt{2}}(c_{kj}^\dagger + c_{kj}),\ \hat{p}_{kj} = \frac{1}{i\sqrt{2}}(-c_{kj}^\dagger + c_{kj}), \\ \hat{Q}_k &= \frac{1}{\sqrt{2}}(C_k^\dagger + C_k),\ \hat{P}_k = \frac{1}{i\sqrt{2}}(-C_k^\dagger + C_k). \end{aligned} \tag{4.13}$$

Quantizing the equation (4.5) relating the (q, p) coordinates to the (Q, P) ones and then using the equations (4.13) one obtains an equation relating the bosons $C, C^\dagger$ and $c, c^\dagger$:

$$C_k^\dagger = \frac{1+i}{\sqrt{2}} \sum \left[\frac{X_{k,lj} + Y_{k,lj}}{\sqrt{2}} c_{lj}^\dagger - \frac{-X_{k,lj} + Y_{k,lj}}{\sqrt{2}} c_{lj} \right]. \tag{4.14}$$

From this equation one obtains the relation between the forward $(\widetilde{X})$ and backward $(\widetilde{Y})$ amplitudes and the amplitudes defining the transformation (4.5) in the phase space:

$$\widetilde{X}_{k,lj} = \frac{X_{k,lj} + Y_{k,lj}}{\sqrt{2}},\ \widetilde{Y}_{k,lj} = \frac{-X_{k,lj} + Y_{k,lj}}{\sqrt{2}}. \tag{4.15}$$

Replacing X and Y by their expressions in terms of $\widetilde{X}$ and $\widetilde{Y}$, using the inverse transformation (4.15), the equations (4.8) can be written in an alternative form:

$$\begin{pmatrix} -\frac{A^T - B^T}{2} & -\frac{A^T + B^T}{2} \\ \frac{A^T + B^T}{2} & \frac{A^T - B^T}{2} \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \omega \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}. \tag{4.16}$$

This equation is nothing else but the RPA equation for the generalized pairing vibration with the standard normalization for the amplitudes $\tilde{X}$ and $\tilde{Y}$:

$$\sum_{lj} \left[\left|\widetilde{X}_{k,lj}\right|^2 - \left|\widetilde{Y}_{k,lj}\right|^2 \right] = 1. \tag{4.17}$$

B. Spurious states

As we have seen the system admits three constants of motion and two of them reflect the invariance of the model Hamiltonian against the rotations around z axis in the spaces of the *pn* quasi-spin and isospin, respectively. These two symmetries are consequences of the equation (4.4). This equation determines important features for the solutions of the equations (4.8), i.e, the RPA equations. Indeed, there are two independent solutions of vanishing energies:

$$\begin{aligned} &X_{1,1j} = \mathcal{A}_1,\ \forall j;\ \ X_{1,kj} = 0, k = 2,3, \forall j, \\ &Y_{1,kj} = 0,\ \forall (k,j), \\ &X_{2,2j} = \mathcal{A}_2,\ \forall j;\ \ X_{2,kj} = 0, k = 1,3, \forall j, \\ &Y_{2,kj} = 0,\ \forall (k,j). \end{aligned} \tag{4.18}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants with respect to the index j. In both cases, $\widetilde{X} = \widetilde{Y}$ and the RPA equations become:

$$B^T \widetilde{X} = 0 \tag{4.19}$$

Since for the $\omega_k = 0, (k = 1,2)$ states $Y = 0$, the equation (4.5) yields:

$$P = -Q. \tag{4.20}$$

and therefore $\{Q, P\} = 0$. On the contrary for physical solutions the Poisson bracket is a non-vanishing quantity

$$\{Q_k, P_k\} = 2 \sum_{k'j'} X_{k,k'j'} Y_{k,k'j'} = 1, \tag{4.21}$$

which results in having the corresponding boson normalized to unity.

V. THE SINGLE J CASE

In this case many simplifications appear and several finite analytical results can be obtained. In what follows we shall omit the index j specifying the single particle orbit. Even

if we are in a simple situation, it is convenient to consider some extreme cases which will define some reference features with respect to which the more general cases may be studied. From the particle number equations one easily derives very simple equations:

$$V_p^2 - V_n^2 = \frac{Z-N}{2\Omega},$$
$$V_p^2 + V_n^2 = \frac{1}{1-2V^2}\left(\frac{Z+N}{2\Omega} - 2V^2\right), \tag{5.1}$$

which allows one to determine V_p and V_n in terms of V

$$\begin{pmatrix} V_p^2 \\ V_n^2 \end{pmatrix} = \frac{1}{2}\left(\frac{Z-N}{2\Omega} \pm \frac{1}{1-2V^2}(\frac{Z+N}{2\Omega} - 2V^2)\right). \tag{5.2}$$

Therefore, in order to solve the static equations one has to use the equation for V^2 and then the equations (5.2) which express V_p^2 and V_n^2 in terms of V^2, in connection with the gaps, β_- and particle number equations. In this way one obtains a set of 6 equations for the six unknowns $\Delta_p, \Delta_n, \Delta_{pn}, \beta_-, \lambda_p, \lambda_n$.

Let us now consider some particular cases:

I) $\chi = \chi_1 = 0$. This is the standard case, when only the nucleons of similar charge are paired. From the formulae presented in the previous sections by considering vanishing values for χ and χ_1, we obtain decoupled pairing equations for protons and neutrons.

$$\Delta_\tau = G_\tau \Omega U_\tau V_\tau,$$
$$N_\tau \equiv Z\delta_{\tau,p} + N\delta_{\tau,n} = 2\Omega V_\tau^2,\ \tau = p, n. \tag{5.3}$$

From here the standard equations for the gap and the Fermi level are analytically obtained:

$$\Delta_\tau = G_\tau \left(\frac{1}{2}N_\tau\Omega(1 - \frac{N_\tau}{2\Omega})\right)^{\frac{1}{2}},$$
$$|\epsilon_\tau - \lambda_\tau| = G_\tau\Omega\left(\frac{1}{4} - \frac{N_\tau}{2\Omega}(1 - \frac{N_\tau}{2\Omega})\right)^{\frac{1}{2}}. \tag{5.4}$$

Note that if the input data are so that $Z = N = \Omega$ then

$$\epsilon_p - \lambda_p = \epsilon_n - \lambda_n = 0,$$
$$\Delta_\tau = \frac{1}{2}G_\tau\Omega. \tag{5.5}$$

while for empty or filled shell one automatically gets $\Delta_\tau = 0$.

II) We consider now the same strength for the pp and nn pairing interactions and non-vanishing χ, χ_1:

$$G_p = G_n \equiv G,$$
$$\chi, \chi_1 \neq 0. \tag{5.6}$$

In what follows we shall discuss the static equations in terms of two parameters q and P defined as follows:

$$q \equiv \frac{Z-N}{2\Omega} = V_p^2 - V_n^2,$$
$$P \equiv 1 - \frac{Z+N}{2\Omega} = (1-2V^2)(1-V_n^2-V_p^2). \tag{5.7}$$

These quantities are related to the constants of motion r_0 and r_- by simple relations:

$$P = -\frac{r_0}{\Omega},$$
$$q = 2\frac{r_-}{\Omega}. \tag{5.8}$$

From the gap's equations one derives a useful relation between all three gap's.

$$\frac{\Delta_{pn}(\Delta_p + \Delta_n)}{\beta_-} = \frac{G\chi_1}{\chi}\Omega P. \tag{5.9}$$

Let us consider the case q=0. Concerning P, the following extreme cases are interesting:

a) $P = 0$, which corresponds to a half filled shell,

b) $P = -1$, the shell is completely filled,

c) $P = 1$, empty shell and,

d) $P = \frac{1}{2}$, one quarter filled shell.

If $q = 0$ and $\epsilon_p = \epsilon_n \equiv e_p + \lambda_p$, then several simplifications in the equations for the U and V coefficients appear. Since from $q = 0$ we get $V_p^2 = V_n^2$ and with the convention that U's and V's are positive we get that the signs in the eqs (3.2-3.4) are related by $S_p = S_n$, the gaps are given by:

$$\Delta_n = \Delta_p = G\Omega(1-2V^2)U_pV_p,$$
$$\Delta_{pn} = \chi_1\sqrt{2\Omega}(1-2V_p^2)UV,$$
$$\beta_- = 2\frac{\chi}{\chi_1}\frac{\Delta_p\Delta_{pn}}{\Omega PG}. \tag{5.10}$$

The equation for $U's$ and $V's$ can be written in a very symmetric form:

$$U_p^2 - V_p^2 = S_p\frac{|e_p + \frac{2}{\Omega\chi_1}\frac{\Delta_{pn}^2}{P}|}{\sqrt{[e_p + \frac{2}{\Omega\chi_1}\frac{\Delta_{pn}^2}{P}]^2 + [\Delta_p - \frac{\Delta_p\Delta_{pn}^2}{P^2}\frac{\chi}{G}(\frac{2}{\Omega\chi_1})^2]^2}},$$
$$U^2 - V^2 = S\frac{|e_p + \frac{2}{\Omega G}\frac{\Delta_p^2}{P}|}{\sqrt{[e_p + \frac{2}{\Omega G}\frac{\Delta_p^2}{P}]^2 + \frac{2}{\Omega}[\Delta_{pn} - \frac{\Delta_p^2\Delta_{pn}}{P^2}\frac{\chi}{\chi_1}(\frac{2}{\Omega G})^2]^2}} \tag{5.11}$$

Denoting the quantities under the square root signs in equations (5.11) by A and B respectively, and multiplying the two equations (5.11) side by side one obtains:

$$P[AB]^{\frac{1}{2}} = S_pS\left|\left(e_p + \frac{2\Delta_{pn}^2}{P\Omega\chi_1}\right)\left(e_p + \frac{2\Delta_p^2}{P\Omega G}\right)\right| \tag{5.12}$$

From the above equation and the definition of P(see eq. (5.7)) one may conclude that

$$S_p = S,\ for\ 0 < P < 1 \text{ (the shell is less than half filled)},$$
$$S_p = -S,\ for\ -1 < P < 0 \text{ (the shell is more than half filled)}. \tag{5.13}$$

The equation $P = 0$ is fulfilled when either $V_p^2 = \frac{1}{2}$ and/or $V^2 = \frac{1}{2}$. From the gap's equation one obtains $\Delta_{pn} = 0$ for the first case while for the second solution the proton gap is vanishing. For what follows it is convenient to write the equations (5.11) in the equivalent form:

$$\left[\left(e_pP + \frac{2\Delta_{pn}^2}{\Omega\chi_1}\right)^2 + \left(\Delta_pP - \frac{2\Delta_{pn}\beta_-}{\Omega\chi_1}\right)^2\right]^{\frac{1}{2}} = \left|(1-2V^2)e_p + \frac{4}{\sqrt{2\Omega}}UV\Delta_{pn}\right|,$$
$$\left[\left(e_pP + \frac{2\Delta_p^2}{\Omega G}\right)^2 + \frac{2}{\Omega}\left(\Delta_{pn}P - \frac{2\Delta_p\beta_-}{\Omega G}\right)^2\right]^{\frac{1}{2}} = |(1-2V_p^2)e_p + 2G\Omega(1-2V^2)U_p^2V_p^2|. \tag{5.14}$$

For the first solution mentioned above, i.e. $V_p^2 = \frac{1}{2}$, we notice that the above equations are satisfied provided $V^2 = \frac{1}{2}$, which results in having $\Delta_p = 0$. Concluding for $q = 0,\ P = 0$ one has the solution $V^2 = V_p^2 = \frac{1}{2}$ and $\Delta_p = \Delta_{pn} = 0$.

For the general case of $P \neq 0$, eqs. (5.11) could be written, after some algebraic manipulations, in a suitable form which allows to *express analytically the gaps* Δ_p *and* Δ_{pn} *in terms of* P:

$$e_p P + \frac{2}{\chi_1 \Omega}\Delta_{pn}^2 = \pm\frac{P^2 G\Omega}{2}\left|1 - \left(\frac{2}{\chi_1\Omega}\right)^2 \frac{\chi}{G}\left(\frac{\Delta_{pn}}{P}\right)^2\right|$$
$$e_p P + \frac{2}{G\Omega}\Delta_p^2 = \pm P^2\chi_1\left|1 - \left(\frac{2}{G\Omega}\right)^2 \frac{\chi}{\chi_1}\left(\frac{\Delta_p}{P}\right)^2\right| \tag{5.15}$$

When the space of single particle states is restricted to one level, the RPA equations are very simple and moreover analytically solvable. Indeed, since we have two constants of motion the equations for q_3, p_3 are decoupled. Moreover they can be easily integrated with the result of a harmonic motion with the frequency

$$\omega = \sqrt{-A_{33}B_{33}}. \tag{5.16}$$

where the involved matrix elements are given in Appendix C.

VI. NUMERICAL APPLICATION

For illustration, the formalism developed in the previous sections is applied here for the case of a single $j = \frac{19}{2}$ shell of energies $\epsilon_p = \epsilon_n = 3$ MeV. First, we consider a system of 4 protons and 12 neutrons. The strengths of the two body interactions are as follows:

$$G_p = 0.25\text{MeV},\ G_n = 0.12\text{MeV}, \chi = 0.2\text{MeV}. \tag{6.1}$$

The strength for the *pp* interaction χ_1 is considered as a free parameter and varied in the interval from 0 to 5 MeV. We solve first the generalized pairing equations and determine the gaps and the Fermi levels. We note that having r_0, r_- as constants of motion, the numbers of protons and neutrons are conserved along a given classical trajectory. There is no need for additional constraints for particle number conservation. However we keep these constraints on the Fermi level energies, in order to fix the values of the two constants of motion. The results are collected in Figs. 1 and 2. From Fig. 1a one sees that the β_- values are almost

insensitive to χ_1 and the neutron gap is slightly changing at the variation of χ_1. The proton gap is however increasing rapidly with χ_1. The reason is that in the decoupling regime (small χ_1) the occupation probability for protons is small, which results in a small proton gap parameter. This causes a relatively larger effect of the perturbation produced by the *pp* interaction. For large values of χ_1, the proton and neutron gaps are similar. The dependence of proton and neutron gaps on χ_1, as well as the independence of β_-, are related to the fact that while the proton-proton, neutron-neutron and proton-neutron interaction of strength χ_1, are all of particle-particle type, β_- characterizes the particle-hole interaction. The phase space coordinates corresponding to the solutions of the pairing equations define stationary points of the energy surface. The corresponding energies are plotted by solid line in Fig. 2. These points are minima points since the RPA equations have a real positive root, as shown in Fig. 1b.

It is worth commenting on the behavior of ω as a function of χ_1. Indeed it is increasing with χ_1 in the same way as Δ_{pn}. This behavior is in contrast to the standard one where the RPA energy decreases with increasing strength of the attractive force. *The reason is that in the present work the attractive force modifies the mean field for the quasiparticle motion and therefore the RPA ground state does not collapse with increasing* χ_1.

In addition to the χ_1 dependent BCS solution shown in Fig. 1, there is another one that is independent of χ_1 and has the following values

$$\Delta_p = 1.0\text{MeV}, \Delta_n = 0.588\text{MeV}, \beta_- = \Delta_{pn} = 0.0 \text{ MeV}, \mathcal{H} = -5.32\text{MeV}. \tag{6.2}$$

The energy corresponding to this solution is shown in Fig. 2 by dotted line. This solution is of a different nature. Since for this case one has $U = 1$, $V = 0$, r_0 and r_+ are, up to an additive constant, identical. Therefore the state described by r_+ is spurious and corresponds to $\omega = 0$. In this case the ground state is degenerate. The full and dotted energy lines cannot be compared with each other since they correspond to different phases. The Goldstone mode is a bridge between two modes of different nature, or in other words, it separates two distinct nuclear phases. In the present study the phase with all gaps different

from zero is well defined while the second phase is not. The RPA mode associated to the latter phase is collapsing to a Goldstone mode in the single shell case. In order to reach another stable phase one has to include more shells and to change also the strength of the other two body terms.

Next we consider the case $N = Z$. As we said before, for N=Z the $T = 0$ pn pairing is expected to be important. Such a pairing interaction affects mainly the Gamow-Teller beta transitions and not the Fermi ones. Therefore the isoscalar pairing is ignored in the present work, due to the specific structure of the model Hamiltonian. It is remarkable that even in the absence of the $T = 0$ pairing, the probability to have a proton paired with a neutron comes out to be comparable to or larger than the probability for having it paired with another proton depending on the χ_1 value (likewise for neutrons). This case is analyzed using the following input data:

$$Z = N = 8, G_p = G_n = 0.125\text{MeV},\ \chi = 0.20\text{MeV}. \tag{6.3}$$

The results are shown in Figs. 3 and 4. Since the strengths of pairing for alike nucleons are equal and are close to the monopole particle-hole strength, the gaps $\Delta_p = \Delta_n$ and β_- are not very different. The solutions of the static equations are minima points for the classical energy. Indeed, as shown in Fig 3b the RPA equation has positive roots. The minimal energies are plotted in Fig. 4 by full line. For each value of χ_1 there are another two solutions of the static equations. One is characterized by $\beta_- = \Delta_p = \Delta_n = 0$ and $\Delta_{pn} \neq 0$. The non-vanishing values for Δ_{pn} and $\mathcal{H}$ are shown in Fig. 3b and 4, respectively, by dashed lines. The other solution consists of

$$\beta_- = \Delta_{pn} = 0,\ \Delta_p = \Delta_n = 0.612\ \text{MeV},\ \mathcal{H} = -4.0\ \text{MeV}. \tag{6.4}$$

and does not depend on χ_1. The constant energy is presented, for the sake of completeness in Fig. 4. The RPA modes on these two static solutions are spurious and consequently have ω values equal to zero. Indeed, in each of these two cases the coordinate r_+ becomes equal to one of the coordinates which are constants of motion. This will certainly not happen in

the multilevel situation but they will define distinct phases, which will persist for a manifold in the strength parameters space.

The advantage of using the single j case is the possibility of testing the semi-classical approach by comparing the predictions with the corresponding exact result. Therefore we diagonalize the model Hamiltonian in the non-orthogonal basis:

$$|n_1, n_2, n_3\rangle = \mathcal{N}_{n_1,n_2,n_3}(A^\dagger_{pp})^{n_1}(A^\dagger_{nn})^{n_2}(A^\dagger_{pn})^{n_3}|0\rangle, \tag{6.5}$$

where $|0\rangle$ denotes the vacuum and

$$\begin{aligned} A^\dagger_{\tau\tau} &= \sum_m c^\dagger_{\tau jm} c^\dagger_{\widetilde{\tau jm}},\ \tau = p, n, \\ A^\dagger_{pn} &= \sum_m c^\dagger_{pjm} c^\dagger_{\widetilde{njm}}. \end{aligned} \tag{6.6}$$

The integers n_1, n_2, n_3 are subject to the constraints:

$$2n_1 + n_3 = Z,\ 2n_2 + n_3 = N \tag{6.7}$$

As in the numerical application described before we consider the case $(Z, N) = (4, 12)$ in the $j = \frac{19}{2}$ shell. The equations (6.7) have three solutions which determine the following basis states:

$$|1\rangle \equiv |2, 6, 0\rangle,\ |2\rangle \equiv |1, 5, 2\rangle,\ |3\rangle \equiv |0, 4, 4\rangle. \tag{6.8}$$

We note that in our basis states all nucleons are paired. Moreover these states are components of the trial function (see eq. 2.13). Inclusion of some broken pairs is straightforward but it is not necessary since the states with broken pairs are not linked to the basis states (6.5) by the model Hamiltonian. With some efforts the matrix elements of the model Hamiltonian, in the above basis, can be derived analytically. Although it is apparently simple we give, for the sake of completeness, few details about the diagonalization procedure. Basically we aim at solving the eigenvalue equations

$$H|\Phi\rangle = E|\Phi\rangle, \tag{6.9}$$

with the ansatz:

$$|\Phi\rangle = \sum_k C_k |k\rangle, \tag{6.10}$$

where the non-orthogonal states $|k\rangle$ are defined by the equation (6.5). Denoting by C the column vector with the components C_k and by O the overlap matrix, the equation (6.9) can be written as:

$$\bar{H}C = EOC, \tag{6.11}$$

with $\bar{H}$ standing for the matrix of H in the basis (6.5). Consider now the eigenvalue equation associated to O

$$Ov_i = a_i v_i, i = 1, 2, 3, \tag{6.12}$$

and denote the eigenvectors matrix by

$$W = (v1, v2, v3). \tag{6.13}$$

The overlap matrix is positive definite and therefore the following diagonal matrix can be defined

$$M = \begin{pmatrix} \sqrt{a_1} & 0 & 0 \\ 0 & \sqrt{a_2} & 0 \\ 0 & 0 & \sqrt{a_3} \end{pmatrix}. \tag{6.14}$$

The overlap matrix O may be written in the factorised form

$$O = UU^T, \tag{6.15}$$

with

$$U = WM, \tag{6.16}$$

and T standing for the transposition operation. Transforming the vector C into

$$X = U^T C, \tag{6.17}$$

equation (6.11) is transformed into an ordinary eigenvalue equation for a symmetric matrix:

$$\widetilde{H}X = EX, \tag{6.18}$$

where

$$\widetilde{H} = U^{-1}\bar{H}(U^{-1})^T. \tag{6.19}$$

The results of the exact diagonalization are shown and compared to the results of the semi-classical approach in Figures 5,6,7. In FIG. 5, the energy of the ground state obtained by diagonalization is compared to the predictions of the approach used in the present paper. We note that although, as expected, the exact ground state energy is lower than the one obtained with the semi-classical method, the energy variations with respect to χ_1 follow the same trend. In Fig. 6 we plot the energies of the first excited state yielded by the diagonalization procedure and by the semi-classical procedure(eq. 5.16) by dashed and solid lines, respectively.

Usually the model Hamiltonian (2.1) is treated as follows: First it is written in terms of quasiparticles, using independent BV transformations for protons and neutrons. Then, the resulting Hamiltonian is treated by the QRPA approach. As a result three independent modes are obtained: a proton-proton, a neutron-neutron and a proton-neutron QRPA mode. In Fig. 6 we also plot, for comparison, the energies of the proton-proton and proton-neutron modes as functions of χ_1 by dash-dotted and dotted lines, respectively. The energy of the proton-proton mode is not collapsing since the proton-proton pairing is included in the mean field. However this is not the case for the proton-neutron mode that, as already mentioned, decreases fast and collapses with increasing χ_1. The standard $pnQRPA$ mode, represented by dotted line in Fig. 6, describes oscillations around dotted line ground state in Fig. 2. From the latter figure it is obvious that starting from $\chi_1 \approx 0.2$ the true static ground state corresponds to the situation where $\Delta_{pn} \neq 0$.

At low χ_1, the first exact excited state is higher in energy. For $\chi_1 = 0$ its energy is close to the 4 quasiparticle energies $2E_p + 2E_n$ with E_p,E_n standing for the proton and neutron

quasiparticle energies respectively. This can be understood better from Fig. 7. Indeed the structure of the states given by diagonalization can be seen either by looking at the components of the corresponding eigenvectors or comparing the diagonal terms of the model Hamiltonian. Thus one notes that for $\chi_1 < 1.9$, the lowest state is $|1\rangle$. To excite the system to the state $|2\rangle$ one has to break one proton-proton pair and one neutron-neutron pair and create instead 2 proton-neutron pairs. The excitation energy shown in Fig. 6 should be close to the energy spacing between the lowest two lines shown in Fig. 7. This spacing is decreasing up to $\chi_1 \approx 1.9$ and then it is increasing. This explains the minimum value of the excitation energy at about $\chi_1 = 1.6$, from Fig.6. This reflects indeed a transition from the phase where the dominant ground state component has only like nucleons paired to a new phase with a dominant ground state component having no proton-proton pairs, all protons participating in proton-neutron pairs. For the above mentioned value of χ_1 the three states are quasi-degenerate and apparently the system might be excited at an almost vanishing cost. As a matter of fact that happens in the standard pnQRPA treatment (see the dotted line in Fig. 6). However in the exact calculation the minimum excitation energy caused by the off-diagonal matrix elements and shown in Fig. 6 is very close to twice the proton-neutron gap shown in Fig.1 at $\chi_1 \approx 1.6$. This is nothing else but a beautiful confirmation provided by the exact calculation for the mechanism which prevents the ground state to collapse. It is interesting to see that there are two values for χ_1 where the prediction of the present approach coincides with the exact result. This is an indication that for these values the components of alpha-like (two proton-neutron pairs) excitations prevail.

Of course there are differences between the curves shown by full and dashed lines in Fig.6 caused mainly by the fact that while the exact eigenstates have definite numbers of protons and neutrons, in the semi-classical approach these numbers are conserved only in average, in the static ground state. Therefore in the present approach the phonon operator can connect the ground state of a (N,Z) nucleus with excited states in the neighboring odd-odd as well as even-even nuclei. Such a transition operator was constructed within a different approach and different arguments in ref. [36]. In order to obtain a detail structure of the phonon operator

obtained by quantizing the classical motion of the independent degrees of freedom additional investigations are necessary. They are under progress and will be published elsewhere.

The oscillating sign of the deviations of the exact excitation energy from the value of ω (5.16) reflects the fact that the semi-classical approach accounts for the average properties of the many body system.

VII. CONCLUSIONS

In the previous Sections we developed a semi-classical formalism to treat a many body Hamiltonian which is often used in the literature to study the single and double beta Fermi transitions. Usually such a Hamiltonian is treated in two steps. First one treats the pairing interaction defining a BCS ground state where alike nucleons are paired to vanishing angular momentum. In the next step the monopole particle-hole and particle-particle proton-neutron interactions are treated by the proton neutron quasiparticle RPA (pnQRPA). In such a treatment the ground state is collapsing for a critical value of the strength of the particle particle interaction(see for instance [28,29]). This feature is troublesome since a) the realistic value of the strength for the *pp* interaction is close to its critical value, and b) the value of the strength that reproduces the experimental value of the double beta transition amplitude is also close to the value where the pnQRPA breaks down. The *pp* interaction is just the proton-neutron T=1 pairing interaction and therefore the instability of the pnQRPA ground state is an indication that there must exist a new static ground state which includes the proton-neutron correlations. If that ground state exists indeed, the residual *pp* interaction does not produce any longer instability of the RPA solutions since the dangerous graphs have been already introduced in the static ground state. These ideas guided us in performing the present work.

The model Hamiltonian has been treated within a semi-classical formalism. The variational state is obtained by applying on the bare vacuum three BV transformations: for proton-proton, neutron-neutron and proton-neutron pairing. We have proved that this s-

tate may be written as two BV transformations applied on the BCS vacuum for the proton-neutron pairing. For monopole pairing this is a general transformation mixing protons and neutrons. Indeed, it depends on 6 real parameters for each j-shell. The classical equations of motion are written in a Hamilton canonical form. The canonical coordinates have very nice physical interpretation. Two of them describe, classically, the rotation around the z axes in the space of the proton-neutron quasi-spin and the space of isospin respectively. Moreover the r_0 variable describes rotations in the gauge space. Three constants of motion have been found. These are the third components of the total quasi-spin and isospin respectively and the classical energy. The static equations are, in fact, the generalized pairing equations whose solutions determine the static ground state. Within this approximation one defines not only the static ground state but also three quasiparticle energies corresponding to the three BV transformations. Although the corresponding quasiparticles are independent modes, their energies are related through the particle-hole two body term -which renormalizes the three gaps- and through a gap dependent re-normalization of the single particle energies.

The proof that the found solutions correspond to the minimum energy is given by the fact that the RPA equations have a real positive solution.

Numerical applications have been made for the case of a single level. In this case the model is integrable since we have three degrees of freedom and three constants of motion. Yet, we use the BCS and RPA approximations to illustrate the improvement gained over the previous attempts for the same Hamiltonian, where the self-consistency between the mean field and the RPA mode was not implemented. *In the numerical cases considered here we found a ground state with all gaps, i. e.* $\Delta_p, \Delta_n, \Delta_{pn}, \beta_-$, *different from zero. To this ground state it corresponds a real positive RPA energy which, contrary to the up to date knowledge, is increasing with the strength of the particle-particle interaction.*

Several situations when a Goldstone mode appears have been found. This is a signature for a phase transition. An additional phase, distinct from the one mentioned above, cannot be found in the one level case since when $\Delta_{pn} = 0$ the r_+ coordinate becomes a constant of motion, and the ground state corresponding to the new phase becomes degenerate.

The results of the present paper are very important since they elucidate a long standing problem in the many body treatment of the Fermi transitions which includes the particle particle interaction in the $pnQRPA$ formalism.

Several salient features of the present approach are pointed out by comparing its predictions with those obtained by an exact treatment of the model Hamiltonian.

VIII. APPENDIX A

The explicit expression of the classical energy is:

$$\begin{aligned}
\mathcal{H} \equiv \langle\Psi|H|\Psi\rangle = \sum_j \Bigg[& (\epsilon_{pj} - \lambda_p) - \frac{G_p}{2}(U_j^2|V_{pj}|^2 + |V_j|^2 {U_{nj}}^2) + \frac{(2\chi + \chi_1)}{2j+1}(U_j^2 {U_{nj}}^2 + |V_j|^2|V_{pj}|^2) \\
& - \frac{\chi_1}{2j+1}(U_j^2|V_{nj}|^2 + |V_j|^2 {U_{pj}}^2)\Bigg] (U_j^2|V_{pj}|^2 + |V_j|^2 {U_{nj}}^2)(2j+1) \\
& + \sum_j \Bigg[(\epsilon_{nj} - \lambda_n) - \frac{G_n}{2}(U_j^2|V_{nj}|^2 + |V_j|^2 {U_{pj}}^2) + \frac{(2\chi + \chi_1)}{2j+1}(U_j^2 {U_{pj}}^2 + |V_j|^2|V_{nj}|^2) \\
& - \frac{\chi_1}{2j+1}(U_j^2|V_{nj}|^2 + |V_j|^2 {U_{pj}}^2)\Bigg] (U_j^2|V_{nj}|^2 + |V_j|^2 {U_{pj}}^2)(2j+1) \\
& + (-\chi + \chi_1)\sum_j (|V_{pj}|^2 {U_{nj}}^2 + |V_{nj}|^2 {U_{pj}}^2) - \chi \sum_j (|V_{nj}|^2 - |V_{pj}|^2) \\
& + 2\chi_1 \sum_j (U_j^2 {V_j^*}^2 U_{pj}|V_{pj}|U_{nj}|V_{nj}| + h.c.) - \\
& - \frac{|\Delta_p|^2}{G_p} - \frac{|\Delta_n|^2}{G_n} + \frac{2|\beta_-|^2}{\chi} - \frac{2|\Delta_{pn}|^2}{\chi_1}. \qquad \text{(A.1)}
\end{aligned}$$

IX. APPENDIX B

Taking into account the analytical expression of the classical energy $\mathcal{H}$ one can easily obtain the equations of motion in an explicit form:

$$\begin{aligned}
\dot{r}_{0j} = & -i\Omega_j[(U_{nj}V_{nj}^* V_j^2 - U_{pj}U_j^2 V_{pj})\Delta_p^* - (U_{nj}V_{nj}{V_j^*}^2 - U_{pj}V_{pj}^* U_j^2)\Delta_p] \\
& -i\Omega_j[(U_{pj}V_{pj}^* V_j^2 - U_{nj}V_{nj}U_j^2)\Delta_n^* - (U_{pj}V_{pj}{V_j^*}^2 - U_{nj}V_{nj}^* U_j^2)\Delta_n] \\
& -2i\hat{j}(U_{pj}^2 - |V_{nj}^2|)(V_j^*\Delta_{pn} - V_j\Delta_{pn}^*)U_j, \qquad \text{(B.1)}
\end{aligned}$$

$$\begin{aligned}
\dot{r}_{-j} &= \frac{i}{2}\Omega_j[(U_{pj}V_{pj}U_j^2 - U_{nj}V_j^2V_{nj}^*)\Delta_p^* - (U_{pj}V_{pj}^*U_j^2 - U_{nj}V_{nj}V_j^{*2})\Delta_p] \\
&+\frac{i}{2}\Omega_j[(U_{pj}V_{pj}^*V_j^2 - U_{nj}V_{nj}U_j^2)\Delta_n^* - (U_{pj}V_{pj}V_j^{*2} - U_{nj}V_{nj}^*U_j^2)\Delta_n] \\
&+i\hat{j}[(U_jV_jU_{pj}V_{pj}^* + U_jV_j^*U_{nj}V_{nj})\beta_- - (U_jV_j^*U_{pj}V_{pj} + U_jV_jU_{nj}V_{nj}^*)\beta_-^*], \qquad \text{(B.2)}
\end{aligned}$$

$$\begin{aligned}
\dot{r}_{+j} &= \frac{i}{2}\Omega_j[(U_{pj}V_{pj}U_j^2 + U_{nj}V_{nj}^*V_j^2)\Delta_p^* - (U_{pj}V_{pj}^*U_j^2 + U_{nj}V_{nj}V_j^{*2})\Delta_p] \\
&+\frac{i}{2}\Omega_j[(U_{pj}V_{pj}^*U_j^2 + U_{nj}V_{nj}U_j^2)\Delta_n^* - (U_{pj}V_{pj}U_j^2 + U_{nj}V_{nj}^*U_j^2)\Delta_n] \\
&+i\hat{j}[(U_jV_jU_{pj}V_{pj}^* - U_jV_j^*U_{nj}V_{nj})\beta_- - (U_jV_j^*U_{pj}V_{pj} - U_jV_jU_{nj}V_{nj}^*)\beta_-^*], \qquad \text{(B.3)}
\end{aligned}$$

$$\begin{aligned}
-\dot{\varphi}_{0j} &= (\epsilon_{pj} - \lambda_p) + (\epsilon_{nj} - \lambda_n) \\
&-\frac{1}{2}\frac{1}{|V_{pj}|^2 - U_{nj}^2}\left[\left(U_{pj}V_{pj} + U_{nj}V_{nj}^*\frac{V_j}{V_j^*}\right)\Delta_p^* + \left(U_{pj}V_{pj}^* + U_{nj}V_{nj}\frac{V_j^*}{V_j}\right)\Delta_p\right] \\
&-\frac{1}{2}\frac{1}{|V_{pj}|^2 - U_{nj}^2}\left[\left(U_{nj}V_{nj} + U_{pj}V_{pj}^*\frac{V_j}{V_j^*}\right)\Delta_n^* + \left(U_{nj}V_{nj}^* + U_pV_p\frac{V_j^*}{V_j}\right)\Delta_n\right] \\
&-\frac{1}{\hat{j}}\frac{U_j^2 - |V_j|^2}{U_j}\left(\frac{1}{V_j^*}\Delta_{pn}^* + \frac{1}{V_j}\Delta_{pn}\right) \\
&+\frac{1}{\hat{j}}\frac{U_j^2 - |V_j|^2}{U_{pj}^2 - |V_{nj}|^2}\frac{1}{U_j|V_j|^2}[(U_{pj}V_{pj}V_j + U_{nj}V_{nj}V_j^*)\beta_-^* + (U_{pj}V_{pj}^*V_j^* + U_{nj}V_{nj}V_j)\beta_-], \qquad \text{(B.4)}
\end{aligned}$$

$$\begin{aligned}
-\dot{\varphi}_{-j} &= 2[(\epsilon_{pj} - \lambda_p) - (\epsilon_{nj} - \lambda_n)] \\
&-\frac{1}{2}\left[\frac{U_j^2(U_{pj}^2 - |V_{pj}|^2)}{U_{pj}}\left(\frac{\Delta_p^*}{V_{pj}^*} + \frac{\Delta_p}{V_{pj}}\right) + \frac{U_{nj}^2 - |V_{nj}|^2}{U_{nj}}\left(\frac{V_j^2}{V_{nj}}\Delta_p^* + \frac{V_j^{*2}}{V_{nj}^*}\Delta_p\right)\right] \\
&+\frac{1}{2}\left[\frac{U_j^2(U_{nj}^2 - |V_{nj}|^2)}{U_{nj}}\left(\frac{\Delta_n^*}{V_{nj}^*} + \frac{\Delta_n}{V_{nj}}\right) + \frac{U_{pj}^2 - |V_{pj}|^2}{U_{pj}}\left(\frac{V_j^2}{V_{pj}}\Delta_n^* + \frac{V_j^{*2}}{V_{pj}^*}\Delta_n\right)\right] \\
&+\frac{2}{\hat{j}}\left[\frac{U_j(U_{pj}^2 - |V_{pj}|^2)}{U_{pj}}\left(\frac{V_j^*}{V_{pj}^*}\beta_-^* + \frac{V_j}{V_{pj}}\beta_-\right) - \frac{U_j(U_{nj}^2 - |V_{nj}|^2)}{U_{nj}}\left(\frac{V_j^*}{V_{nj}^*}\beta_- + \frac{V_j}{V_{nj}}\beta_-^*\right)\right], \qquad \text{(B.5)}
\end{aligned}$$

$$\begin{aligned}
-\dot{\varphi}_{+j} &= -\frac{1}{2}\left[\frac{U_j^2(U_{pj}^2 - |V_{pj}|^2)}{U_{pj}}\left(\frac{\Delta_p^*}{V_{pj}^*} + \frac{\Delta_p}{V_{pj}}\right) - \frac{U_{nj}^2 - |V_{nj}|^2}{U_{nj}}\left(\frac{V_j^2}{V_{nj}}\Delta_p^* + \frac{V_j^{*2}}{V_{nj}^*}\Delta_p\right)\right. \\
&\left.+2\frac{|V_j|^2 - U_j^2}{|V_{pj}|^2 - U_{nj}^2}\left[\left(U_{pj}V_{pj} + U_{nj}V_{nj}^*\frac{V_j}{V_j^*}\right)\Delta_p^* + \left(U_{pj}V_{pj}^* + U_{nj}V_{nj}^*\frac{V_j^*}{V_j}\right)\Delta_p\right]\right] \\
&-\frac{1}{2}\left[\frac{U_j^2(U_{nj}^2 - |V_{nj}|^2)}{U_{nj}}\left(\frac{\Delta_n^*}{V_{nj}^*} + \frac{\Delta_n}{V_{nj}}\right) - \frac{U_{pj}^2 - |V_{pj}|^2}{U_{pj}}\left(\frac{V_j^2}{V_{pj}}\Delta_n^* + \frac{V_j^{*2}}{V_{pj}^*}\Delta_n\right)\right. \\
&\left.+2\frac{|V_j|^2 - U_j^2}{|V_{pj}|^2 - U_{nj}^2}\left[\left(U_{nj}V_{nj} + U_{pj}V_{pj}^*\frac{V_j}{V_j^*}\right)\Delta_n^* + \left(U_{nj}V_{nj}^* + U_{pj}V_{pj}\frac{V_j^*}{V_j}\right)\Delta_n\right]\right]
\end{aligned}$$

$$+\frac{2}{\hat{j}}\left[\frac{U_j(U_{pj}^2-|V_{pj}|^2)}{U_{pj}}\left(\frac{V_j^*}{V_{pj}^*}\beta_-^*+\frac{V_j}{V_{pj}}\beta_-^*\right)+\frac{U_j(U_{nj}^2-|V_{nj}|^2)}{U_{nj}}\left(\frac{V_j^*}{V_{nj}^*}\beta_-+\frac{V_j}{V_{nj}}\beta_-^*\right)\right.$$
$$\left.-\frac{(U_j^2-|V_j|^2)^2}{U_{pj}^2-|V_{nj}|^2}\frac{1}{U_j|V_j|^2}\left[(U_{pj}V_{pj}V_j+U_{nj}^*V_{nj}^*V_j^*)\beta_-^*+(U_{pj}V_{pj}^*V_j^*+U_{nj}V_{nj}V_j)\beta_-\right]\right]$$
$$+\frac{2}{\hat{j}}\frac{1}{U_j}\left(\frac{\Delta_{pn}^*}{V_j^*}+\frac{\Delta_{pn}}{V_j}\right). \tag{B.6}$$

X. APPENDIX C

Here we give the analytical expressions for the coefficients A_{33} and B_{33} involved in the equations of motion characterizing the single j case. These coefficients determine, by means of (5.16), the energy of the harmonic mode.

$$A_{33}=\frac{1}{2\Omega_j}\left[4\frac{V^2-U^2}{V_p^2-U_n^2}\left(\left(\frac{U_p^2-V_p^2}{U_pV_p}-\frac{U_n^2-V_n^2}{U_nV_n}\right)\left(\Delta_p+\Delta_n-\frac{2}{\hat{j}}\frac{V^2-U^2}{UV}\beta_-\right)+\frac{V^2-U^2}{U^3V^3}\frac{\Delta_{pn}}{\hat{j}}\right)\right.$$
$$-16\frac{V^2-U^2}{(V_p^2-U_n^2)^2}\left(U_pV_p+U_nV_n)(\Delta_p+\Delta_n-2\frac{V^2-U^2}{UV}\frac{\beta_-}{\hat{j}}\right)$$
$$-\frac{1}{U_p^3V_p^3}\left(U^2\Delta_p-V^2\Delta_n-\frac{4}{\hat{j}}UV\beta_-\right)-\frac{1}{U_n^3V_n^3}\left(U^2\Delta_n-V^2\Delta_p-\frac{4}{\hat{j}}UV\beta_-\right)$$
$$+G_p\Omega_j\left[\frac{U^2(U_p^2-V_p^2)}{U_pV_p}-\frac{V^2(U_n^2-V_n^2)}{U_nV_n}+2\frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}(V^2-U^2)\right]^2$$
$$+G_n\Omega_j\left[\frac{U^2(U_n^2-V_n^2)}{U_nV_n}-\frac{V^2(U_p^2-V_p^2)}{U_pV_p}+2\frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}(V^2-U^2)\right]^2$$
$$\left.-4\chi U^2V^2\left[\frac{U_p^2-V_p^2}{U_pV_p}+\frac{U_n^2-V_n^2}{U_nV_n}+\frac{U_pV_p+U_nV_n}{V_p^2-U_n^2}\frac{(V^2-U^2)^2}{U^2V^2}\right]^2+4\frac{\chi_1}{U^2V^2}\right]$$
$$B_{33}=\frac{{\Delta_p}^2}{2G_p}+\frac{{\Delta_n}^2}{2G_n}-2\frac{{\beta_-}^2}{\chi}+\frac{\Omega_j}{2}[-G_p\Omega_j(U_pV_pU^2+U_nV_nV^2)^2$$
$$-G_n\Omega_j(U_pV_pV^2+U_nV_nU^2)^2+4\chi U^2V^2(U_pV_p-U_nV_n)^2)]. \tag{C.1}$$

REFERENCES

[1] S. T. Belyaev, Matt. Phys. Medd. **31**(1959) 5-55.

[2] B. F. Bayman, Nucl. Phys. **15**(1960)33.

[3] R. R. Chasman, Phys. Rev. 132 (1963) 343.

[4] L. S. Kisslinger and R. A. Sorensen, Mat. Fys. Medd. Dan. Vid. Selsk. **32**,No 16 (1960), pp. 5-82.

[5] R. W. Richardson and N. Sherman, Nucl. Phys. **52** (1964) 221,*ibid* 253.

[6] N. Sandulescu, R. J. Liotta, R. Wyss, Phys. Lett. **B394** (1996) 6.

[7] V. N. Fomenko, J. Phys. A 3 (1970) 8.

[8] F. Andreozzi, A. Covello, E. E. Maqueda and R. P. J. Perazzo, Phys. Rev. **C16** (1977) 2415.

[9] Feng Pan, J. P. Draayer, W. E. Ormand, Phys. Lett. **B422** (1998) 1.

[10] A. A. Raduta and E. Moya de Guerra, submitted to Ann. Phys.

[11] P. Camiz, A. Covello and M. Jean, Nuovo Cimento **36**(1965) 663; **42** (1966) 199.

[12] A. Goswami, Nucl. Phys. **60** (1964) 228.

[13] M. Jean, Proc. Predeal Summer School, Predeal, Romania 1969.

[14] H. H. Wolter, A. Faessler and P. U. Sauer, Phys. Lett. **31B** (1970) 516.

[15] A. L. Goodman, Nucl. Phys. **A 186** (1972) 475.

[16] K. Langanke, D. J. Dean, S. E. Koonin, P.B. Radha, Nucl. Phys. **A613** (1997) 253.

[17] A. L. Goodman, Adv. Nucl. Phus. **11** (1979) 263.

[18] A. L. Goodman, Phys. Rev. C **58** (1998) R3051.

[19] Y. Takahashi, Prog. Th. Phys., **46** (1971) 1749.

[20] E. M. Muller, K. Muhlhaus, K. Neergard, U. Mosel, Nucl. Phys. **A 383** (1982) 233.

[21] M. Cheoun, A. Bobyk, A. Faessler, F. Symkovic, G. Teneva, Nucl. Phys. **A561**(1993)74.

[22] J. Engel, S. Pittel, M. Stoitsov, P. Vogel and J. Dukelsky, Phys. Rev. **55**(1997) 1781.

[23] W. Satula and R. Wyss, Phys. Lett. **B 393** (1997) 1.

[24] J. Terasaki, R. Wyss, P. H. Heenen, Phys. Lett. **B 437** (1998) 1.

[25] C. Bloch and A. Messiah, Nucl. Phys. **39** (1962) 95.

[26] A. A. Raduta, O. Haug, F. Simkovic and Amand Faessler, Nucl. Phys. A, in press.

[27] A. A. Raduta, V. Ceausescu, A. Gheorghe, M. S. Popa, Nucl. Phys. **A 427** (1984) 1.

[28] M. Sambataro and J. Suhonen, Phys. Rev. C **56** (1997) 782.

[29] F. Simkovic, A. A. Raduta, M. Veselsky and A. Faessler, Phys. Rev. C, in press.

[30] D. Cha, Phys. Rev. **C 27** (1983) 2269.

[31] A. Faessler, Prog. Part. Nucl. Phys.**21**(1988) 183.

[32] F. Villars, Nucl. Phys. **A 285** (1977) 269.

[33] E. Moya de Guerra and F. Villars, Nucl. Phys. **A 285** (1977) 297.

[34] E. Moya de Guerra, Phys. Rev. Lett. **48** (1982) 922.

[35] D. A. Kirzhnits, Field theoretical methods in many-body systems, Pergamon Press Ltd., edited by D. M. Brink, 1967, p 369.

[36] A. A. Raduta, M. C. Raduta, A. Faessler and W. A. Kaminski, Nucl. Phys. **A 634**(1998) 497.

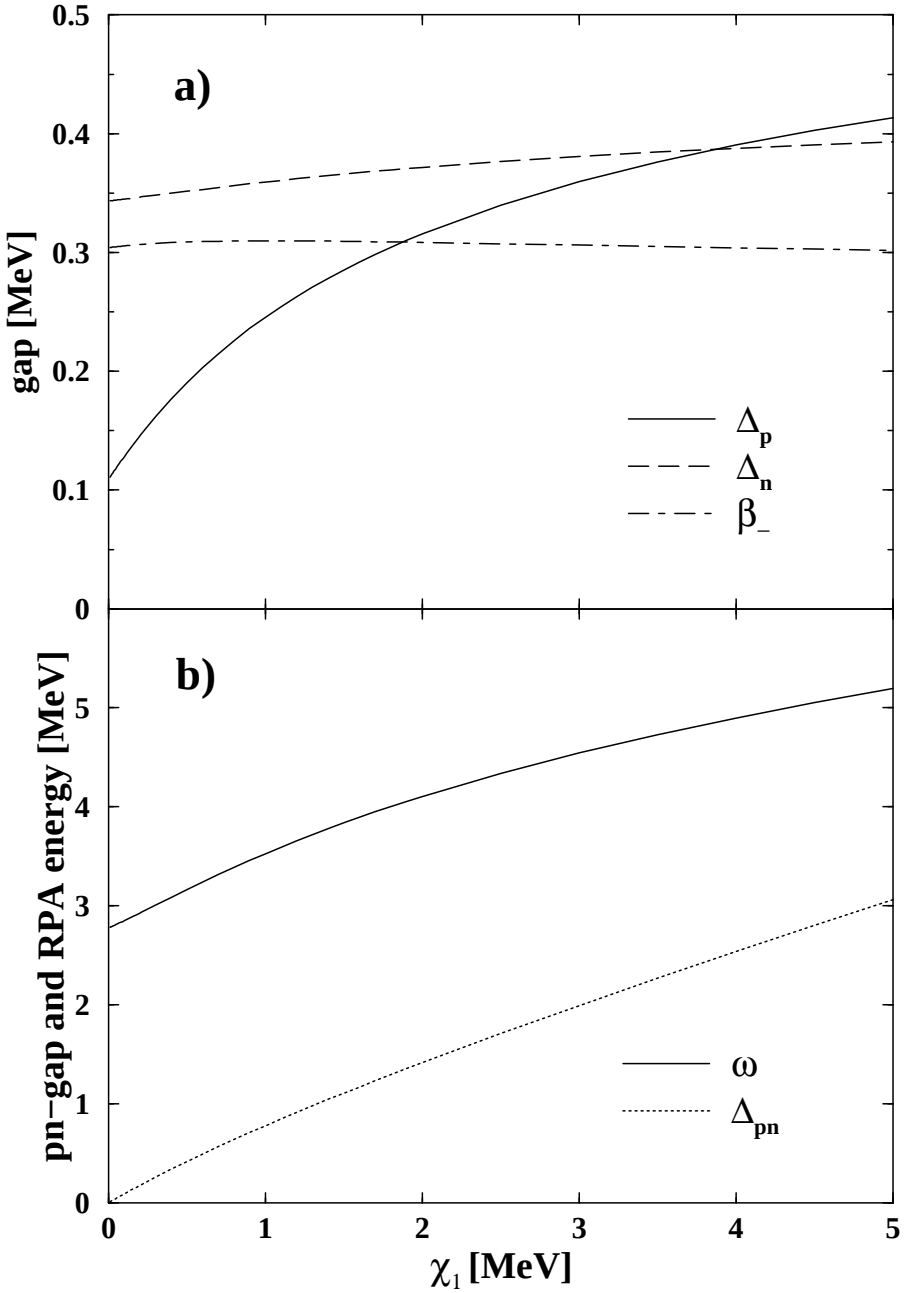
a)
b)
gap [MeV]
pn−gap and RPA energy [MeV]
χ_1 [MeV]
Δ_p
Δ_n
β_-
ω
Δ_{pn}
0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5

FIG. 1. The gap parameters $\Delta_p, \Delta_n, \beta_-$, (a), Δ_{pn} (b) obtained by solving the generalized pairing equations (2.15), (2.16) and the energy of the RPA mode (b) given by eq. (5.17) are plotted as functions of χ_1. The strengths for the two body interaction terms are $G_p = 0.25\text{MeV}, G_n = 0.12\text{MeV}, \chi = 0.20\text{MeV}$. The particle numbers are $Z = 4, N = 12.$, in $j = \frac{19}{2}$ shell.

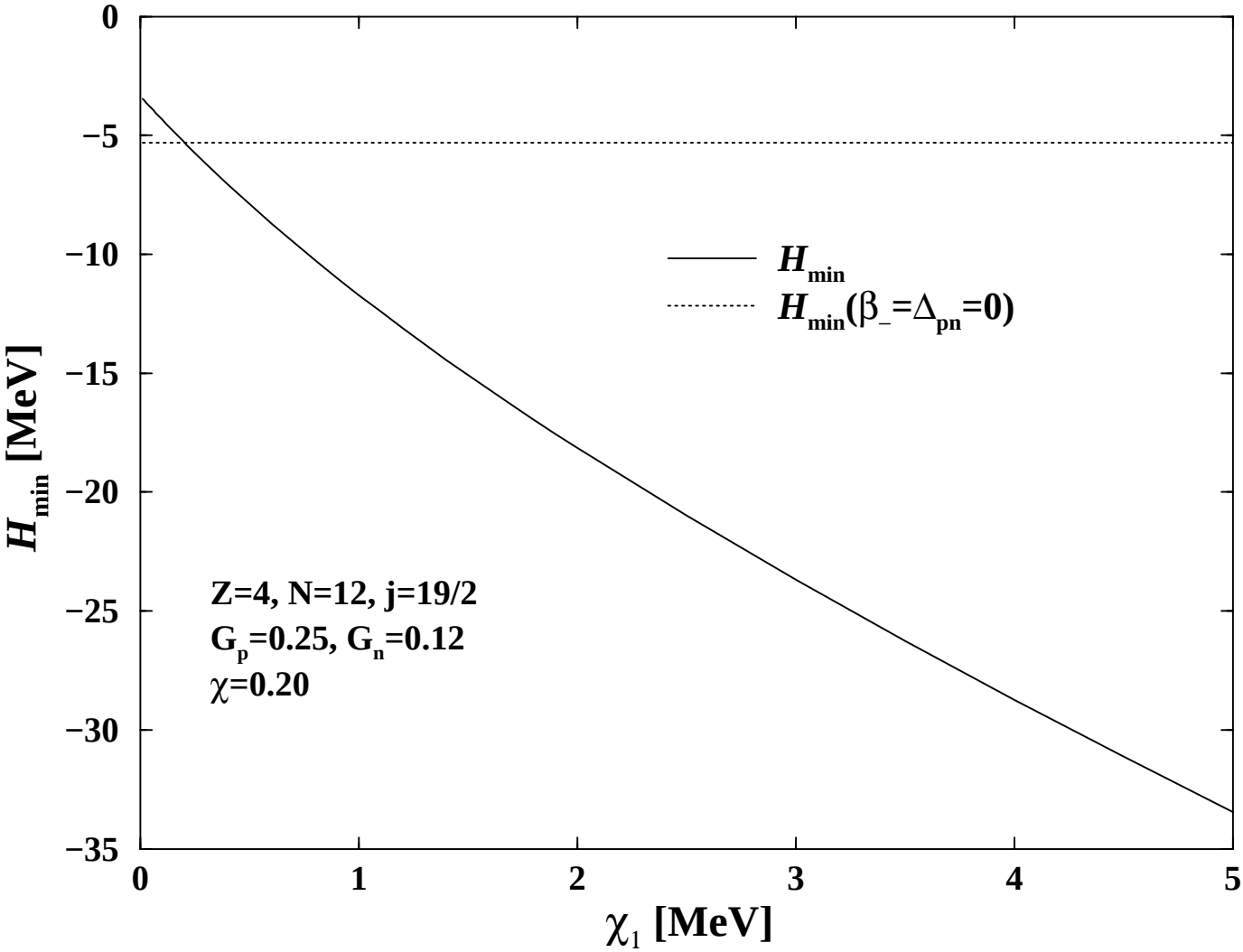

FIG. 2. The energy minima, corresponding to the solutions of the static equations, are plotted as function of χ_1. The input data are the same as in Fig. 1. Interaction strengths are given in units of MeV. The minimal classical energy corresponding to the static solution with $\Delta_{pn} = 0$ is also plotted.

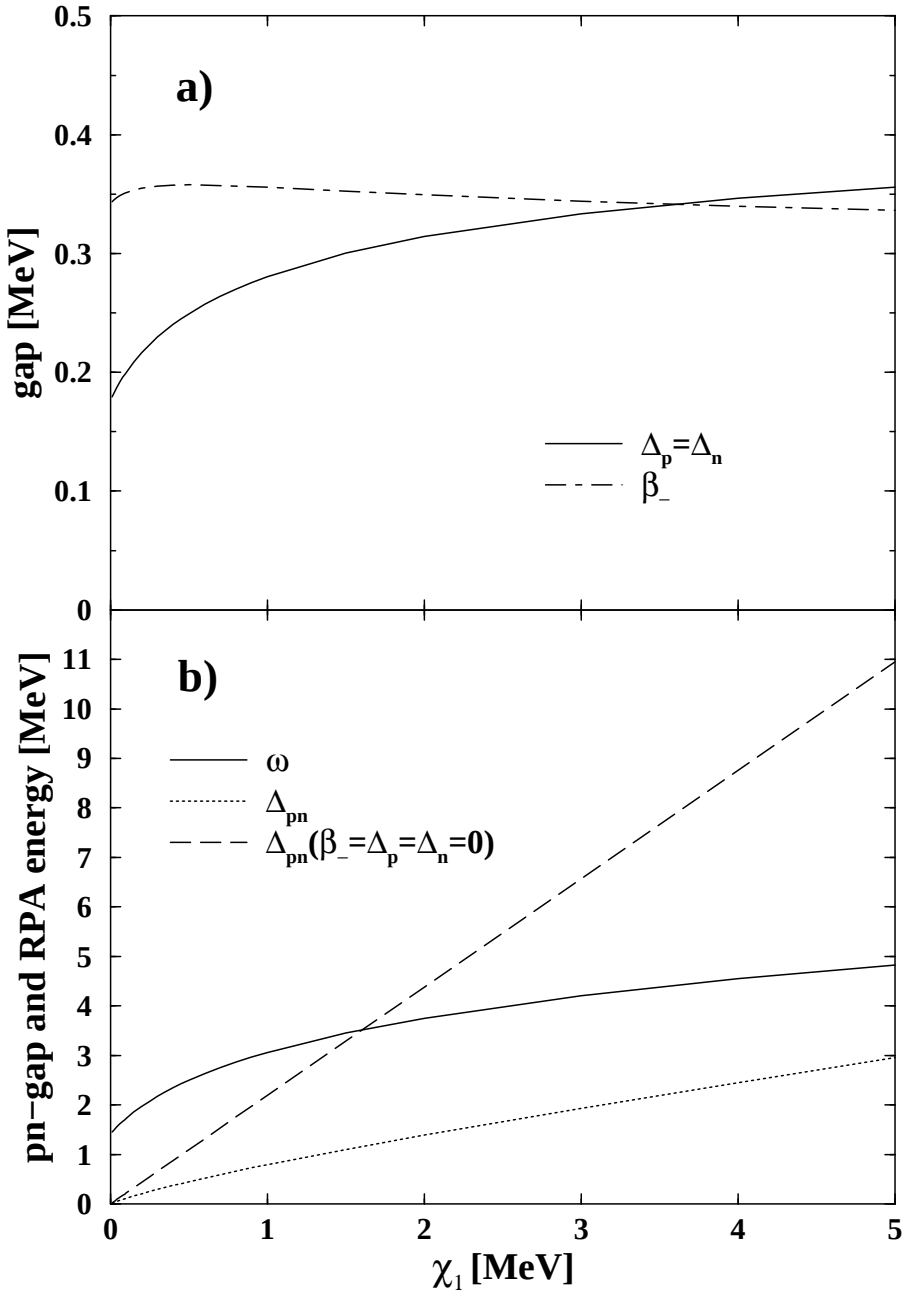

FIG. 3. The same as in Fig. 1) but for $Z = N = 8, G_p = G_n = 0.125$MeV. In this case an additional solution of the static equation appears with $\Delta_p = \Delta_n = \beta_- = 0, \Delta_{pn} \neq 0$ which is plotted by a dashed line.

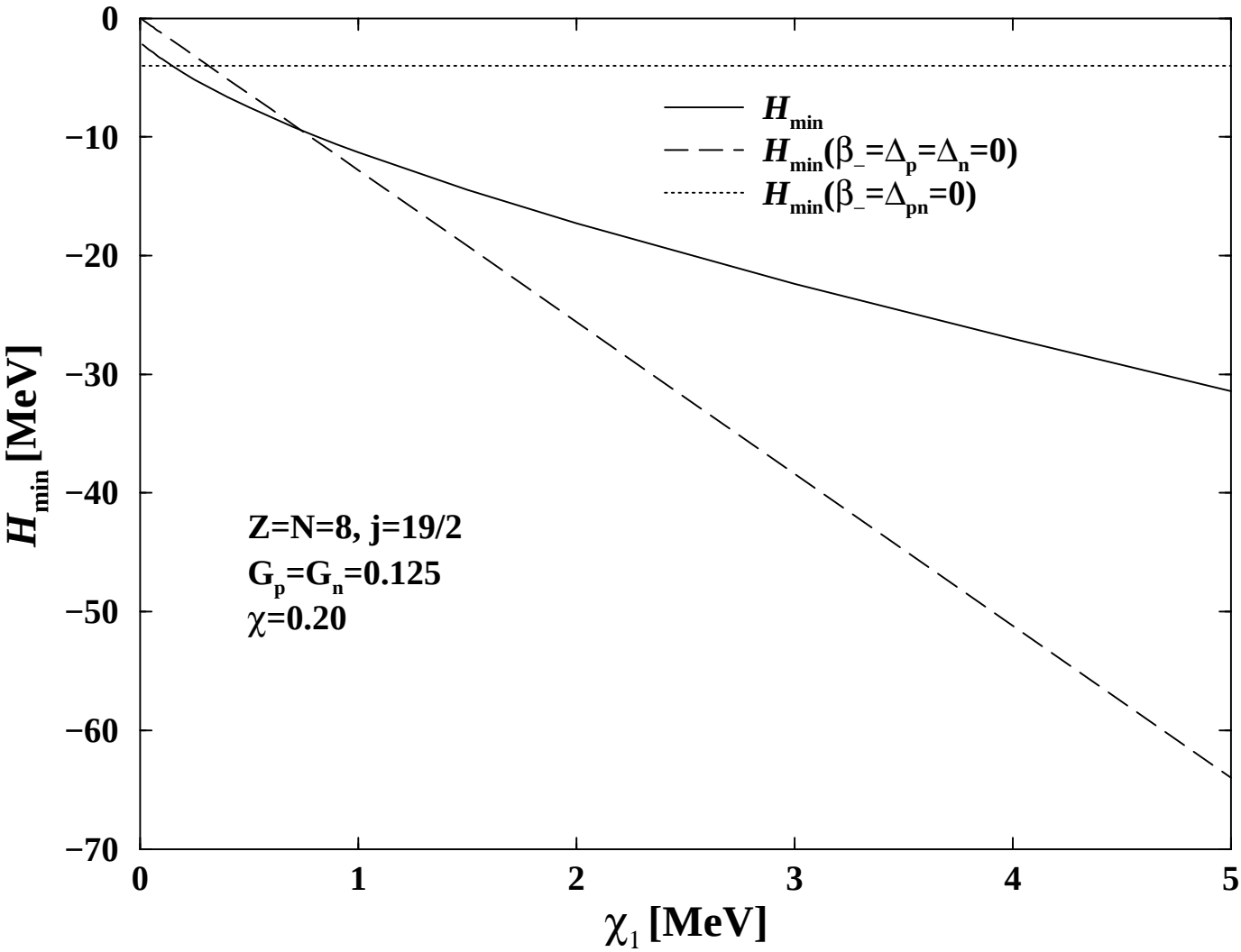

FIG. 4. The same as in Fig. 2) but for the data of Fig. 3. An additional curve corresponding to the static solutions $\Delta_p = \Delta_n = \beta_- = 0$ (dashed line) is presented.

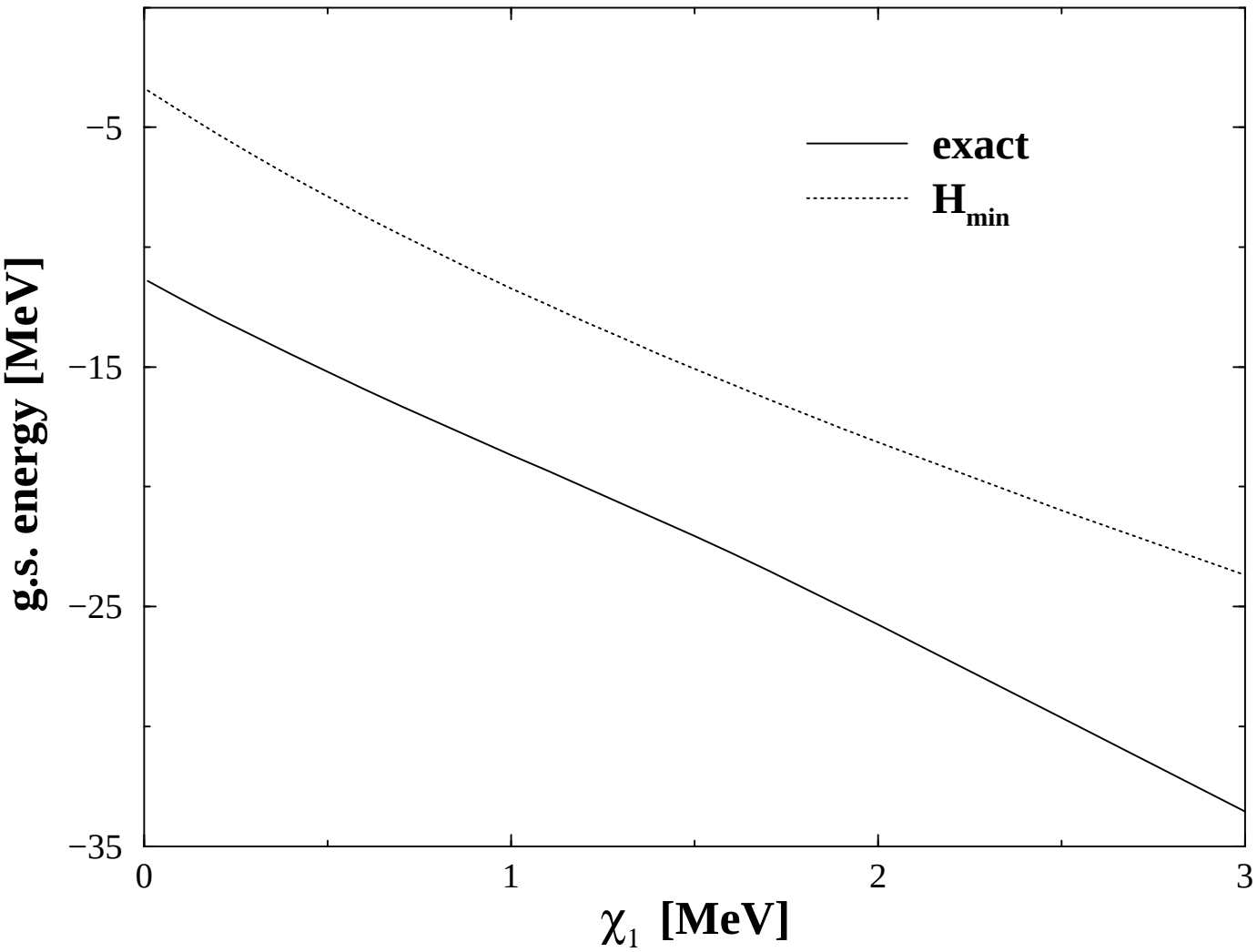

FIG. 5. Ground state energies, produced by the diagonalization procedure and by the present approach are plotted as functions of χ_1. The Fermi levels λ_p and λ_n for the exact calculation are taken equal to those given by the semi-classical treatment. The input data are the same as in Fig. 1.

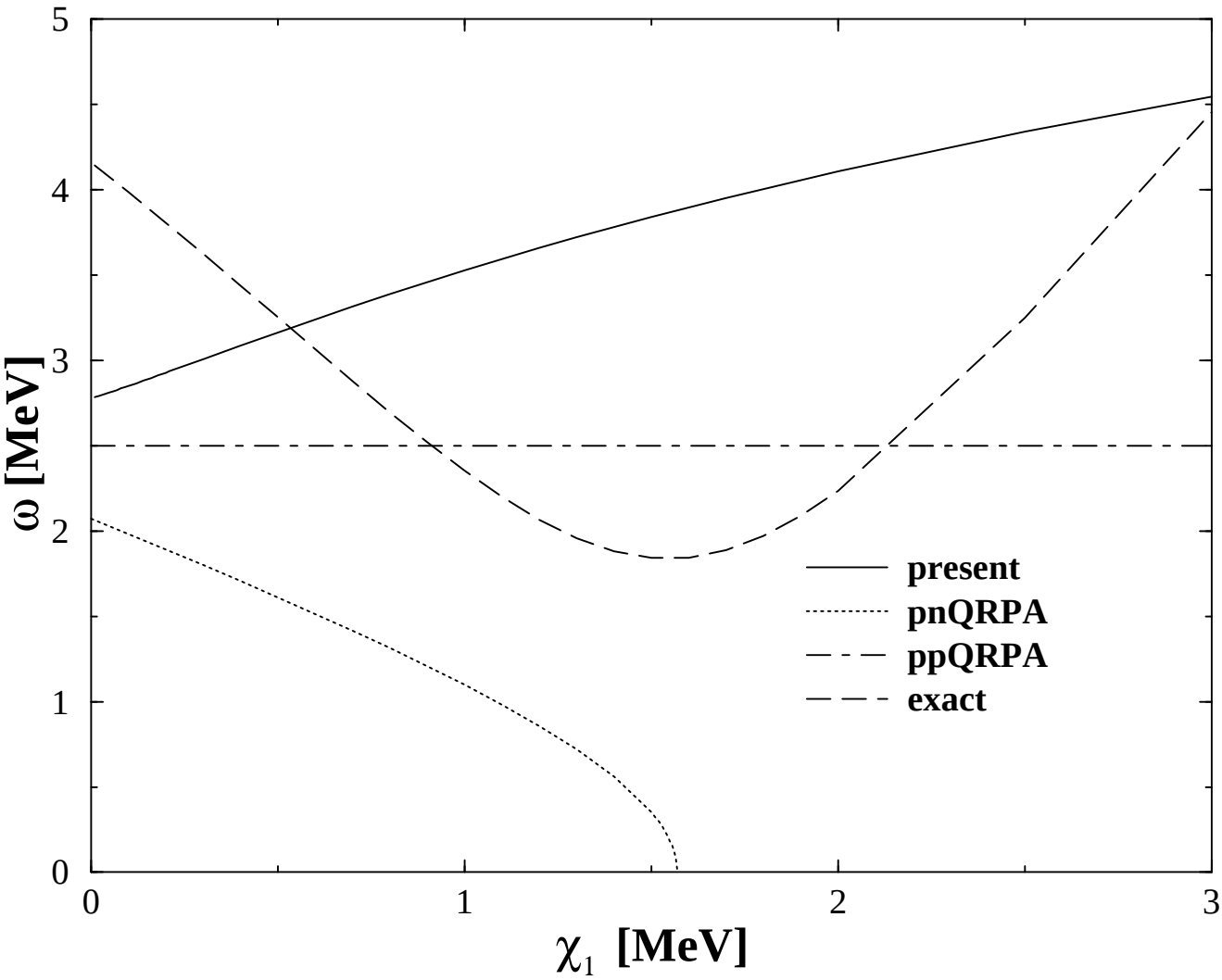

FIG. 6. The energy of the semi-classical mode is compared with the excitation energy of the first excited state obtained by diagonalization. Also shown are the energies of the standard proton-proton and proton-neutron QRPA modes, when the proton-neutron pairing is not included in the mean field($\Delta_{pn} = 0$). The input data are as shown in Fig. 1.

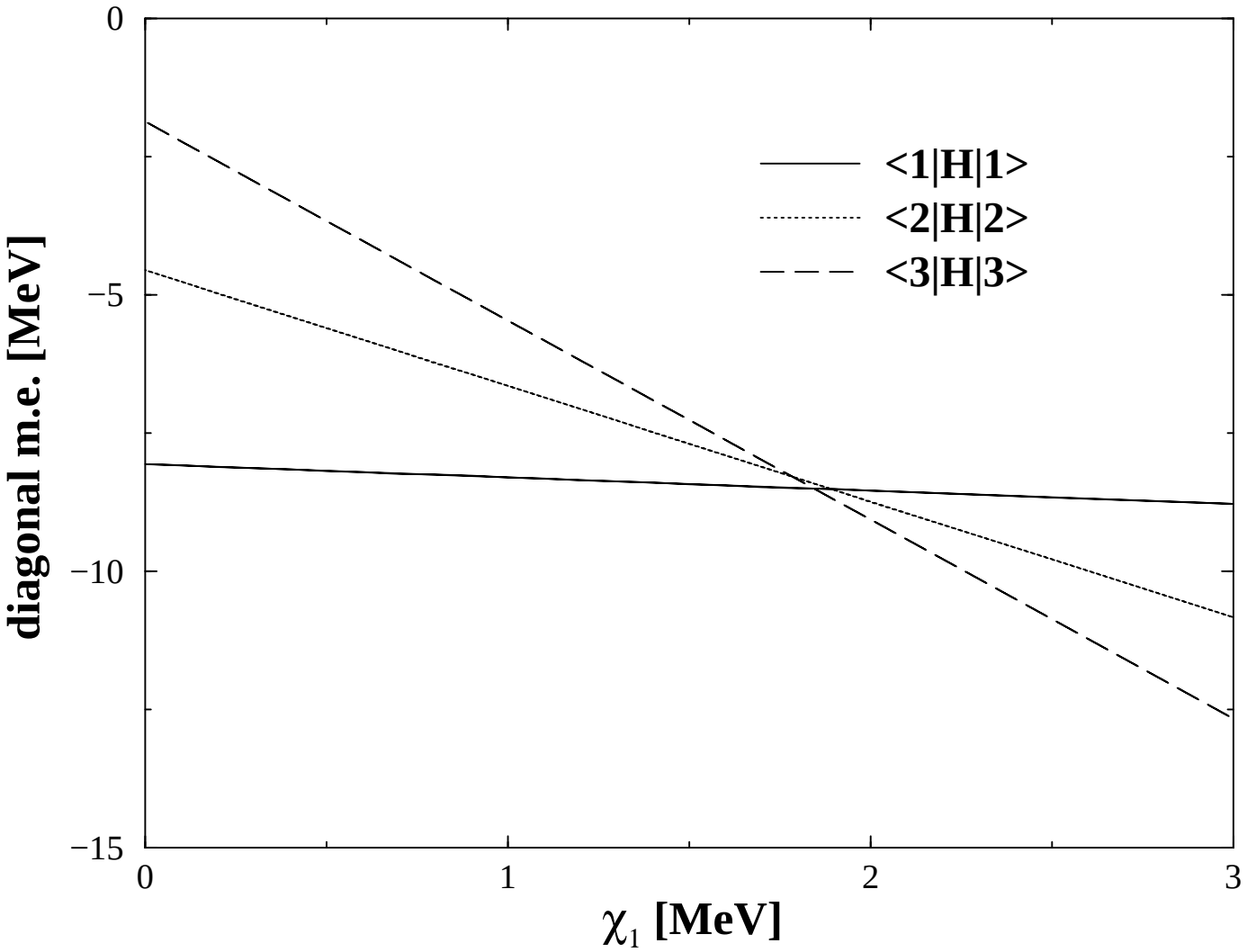

FIG. 7. The expectation values of the model Hamiltonian in the states $|k\rangle$ defined by (6.5) are plotted as function of χ_1