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RESEARCH ARTICLE

Dimension of the intersection of a pair of orthogonal groups

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Let $g, h\colon V \times V \to \mathbb{C}$ be two non-degenerate symmetric bilinear forms on a finite-dimensional complex vector space V. Let G (resp. H) be the Lie group of isometries of g (resp. h). If the endomorphism $L\colon V \to V$ associated to g, h is diagonalizable, then the dimension of the intersection group $G \cap H$ is computed in terms of the dimensions of the eigenspaces of L.

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1. The group of isometries

This paper is an extended version of a preliminary statement of Theorem 2.1, which was presented [1] without proof. Here, we also include a counterexample showing that the hypothesis in the theorem cannot be improved.

Let V, W be two complex vector spaces of finite dimension and let $\mathcal{L}(V, W)$ be the space of $\mathbb{C}$-linear mappings from V into W. We write $\mathfrak{gl}(V) = \mathcal{L}(V, V)$ and we denote by $GL(V)$ the linear group of V, i.e., the group of invertible elements in $\mathfrak{gl}(V)$.

Definition 1.1 *An element $A \in GL(V)$ is said to be an* isometry *of a symmetric bilinear form $g\colon V \times V \to \mathbb{C}$ if the following equation holds:*

$$g\left(A(x), A(y)\right) = g(x, y), \quad \forall x, y \in V. \tag{1}$$

Lemma 1.2 *Let $g\colon V \times V \to \mathbb{C}$ be a symmetric bilinear form on an n-dimensional complex vector space V and let V', V'' be vector subspaces such that,*

(1) $g|_{V'}$ is non-degenerate,
(2) $g(v, v'') = 0$, $\forall v \in V, \forall v'' \in V''$,

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(3) $V = V' \oplus V''$.

Then, every isometry $A \in GL(V)$ of g can be written as

$$A = \begin{pmatrix} A' & O \\ B & C \end{pmatrix}, \quad B \in \mathcal{L}\left(V', V''\right), \; C \in \mathfrak{gl}\left(V''\right),$$

and A' is an isometry of $g|_{V'}$.

Proof We set $A = \begin{pmatrix} A' & D \\ B & C \end{pmatrix}$, with $A' \in \mathcal{L}(V', V')$, $B \in \mathcal{L}(V', V'')$, $C \in \mathfrak{gl}(V'')$, $D \in \mathcal{L}(V'', V')$. If $v_1, \ldots, v_n$ is a basis for V such that $v_1, \ldots, v_{n'}$, $n' = \dim V'$, is a basis for V' and $v_{n'+1}, \ldots, v_n$ is a basis for V'', then A is an isometry if and only if the following equations hold:

$$g\left(A(v_i), A(v_j)\right) = g(v_i, v_j), \quad i, j = 1, \ldots, n. \tag{2}$$

If $i = 1, \ldots, n'$, then $A(v_i) = A'(v_i) + B(v_i)$, with $A'(v_i) \in V'$, $B(v_i) \in V''$, and according to the item *(2)*, for $i, j = 1, \ldots, n'$ from the equation (2) we obtain

$$\begin{aligned} g\left(A(v_i), A(v_j)\right) &= g\left(A'(v_i), A'(v_j)\right) + g\left(A'(v_i), B(v_j)\right) \\ &\quad + g\left(B(v_i), A'(v_j)\right) + g\left(B(v_i), B(v_j)\right) \\ &= g\left(A'(v_i), A'(v_j)\right) \\ &= g(v_i, v_j), \end{aligned}$$

thus proving that $A' = A|_{V'}$ is an isometry for $g|_{V'}$.

Similarly, if $i = n' + 1, \ldots, n$, then $A(v_i) = D(v_i) + C(v_i)$ with $D(v_i) \in V'$, $C(v_i) \in V''$. Again from the item *(2)*, we obtain

$$0 = g(v, v_i) = g\left(A(v), A(v_i)\right) = g\left(A(v), D(v_i) + C(v_i)\right) = g\left(A(v), D(v_i)\right),$$

for every $v \in V$. As A is an isomorphism this in particular implies $g(v', D(v_i)) = 0$ $\forall v' \in V'$, and we can conclude $D = 0$ by applying the item *(1)*. ■

Consequently, the structure of the set of isometries of a degenerate symmetric bilinear form g can be recovered from the non-degenerate part of g. Because of this, below we confine ourselves to consider only non-degenerate symmetric bilinear forms. In this case, the equation (1) implies $\det A = \pm 1$, and the set of all isometries of g is a subgroup of $GL(V)$, which is denoted by G. By choosing an orthonormal basis in V, every element of G is represented by an orthogonal matrix and an isomorphism holds, $G \cong O(n, \mathbb{C})$. The Lie algebra of $O(n, \mathbb{C})$ is denoted by $\mathfrak{o}(n, \mathbb{C})$. We also remark on the fact that G is a closed subgroup in $GL(V)$ and hence, G is a Lie subgroup of the linear group of V, the Lie algebra of which is denoted by $\mathfrak{g}$.

2. The dimension of the intersection group

Theorem 2.1 *Let V be an n-dimensional complex vector space and let g, h be two non-degenerate symmetric bilinear forms on V. Let G, H be the groups of isometries of g, h, respectively and let $L \colon V \to V$ be the endomorphism associated to g, h, i.e.,*

$g(x, L(y)) = h(x,y)$, $\forall x, y \in V$. *If L is diagonalizable, then*

$$\dim(G \cap H) = \sum_{i=1}^{r} \binom{m_i}{2},$$

where m_i, $i = 1, \dots, r$, are the dimensions of the eigenspaces of L.

Proof Let α_i, $i = 1, \dots, r$, be the distinct eigenvalues of L and let $E(\alpha_i)$ be the eigenspace attached to α_i. As L is diagonalizable, we have $V = \oplus_{i=1}^{r} E(\alpha_i)$. We claim that $E(\alpha_i)$ and $E(\alpha_j)$ are orthogonal with respect to both metrics for $i \neq j$. In fact, if v_i (resp. v_j) is a non-vanishing eigenvector for α_i (resp. α_j), then taking account of the fact that L is symmetric, we obtain

$$\alpha_j g(v_i, v_j) = g(v_i, L(v_j)) = g(L(v_i), v_j) = \alpha_i g(v_i, v_j).$$

Hence, $(\alpha_i - \alpha_j)\, g(v_i, v_j) = 0$. As $\alpha_i \neq \alpha_j$, we conclude $g(v_i, v_j) = 0$. In addition, from the definition of L, we have $h(v_i, v_j) = g(v_i, L(v_j)) = \alpha_j g(v_i, v_j) = 0$. Therefore $E(\alpha_i)$ and $E(\alpha_j)$ are also h-orthogonal.

As a consequence of the g-orthogonality of the eigensubspaces we deduce that every $E(\alpha_i)$ is non-singular with respect to both bilinear forms g and h.

By choosing a g-orthonormal basis for every subspace $E(\alpha_i)$ and collecting all theses bases, we obtain a basis $(v_1, \dots, v_n)$ of eigenvectors for L which is also g-orthonormal. Hence the matrices of g and h in this basis are as follows:

$$M_g = I_n = n \times n \text{ identity matrix},$$
$$M_h = \text{diagonal}\left(\alpha_1, \overset{(m_1}{\dots}, \alpha_1, \dots, \alpha_r, \overset{(m_r}{\dots}, \alpha_r\right), \; m_1 + \ldots + m_r = n.$$

Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). The map $\exp\colon \mathfrak{g} \to G$ induces a diffeomorphism from a neighborhood of the origin in $\mathfrak{g}$ onto a neighborhood of the unit element in G ([4, Theorem 3.31]). Hence $\dim(G \cap H) = \dim(\mathfrak{g} \cap \mathfrak{h})$, and we are led to determine the Lie algebra of the intersection subgroup. Moreover, as $\mathfrak{g} = \{A \in \mathfrak{gl}(V)\colon g(x, A(y)) + g(A(x), y) = 0, \forall x, y \in V\}$, and similarly for $\mathfrak{h}$, we conclude that $\mathfrak{g} \cap \mathfrak{h}$ can be identified to the subspace of $n \times n$ skew-symmetric matrices $A = (a_{ij})$ such that,

$$A^t M_h + M_h A = 0. \tag{3}$$

We decompose A in blocks as follows:

$$A = \begin{pmatrix} A_{11} & \dots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \dots & A_{rr} \end{pmatrix},$$

each A_{ij} being a $m_i \times m_j$ matrix for $i, j = 1, \dots, r$, and the equation (3) transforms into the following system: $\alpha_i A_{ij} + \alpha_j A_{ji}^t = 0$, $i, j = 1, \dots, r$. As A is skew-symmetric, we have $A_{ij} + A_{ji}^t = 0$. Hence this system is equivalent to saying $(\alpha_i - \alpha_j) A_{ij} = 0$ for $1 \leq i < j \leq r$.

Accordingly, $A_{ij} = 0$, $i \neq j$, and the submatrices $A_{11}, \dots, A_{rr}$ are arbitrary. As $\dim \mathfrak{o}(m, \mathbb{C}) = \binom{m}{2}$, we can conclude. ■

Corollary 2.2 *Let S^2V^* be the space of symmetric bilinear forms on V and let*

$\mathcal{U} \subset S^2V^*$ be the subset of non-degenerate forms. The pairs $(g,h) \in \mathcal{U} \times \mathcal{U}$ for which the conclusion of the theorem above holds, is a dense subset in $\mathcal{U} \times \mathcal{U}$.

Proof The map $\theta\colon \mathcal{U} \times \mathcal{U} \to \mathfrak{gl}(V)$, $\theta(g,h) = L$, is analytic and the result follows taking [2, Chapter 7, Theorem 1] into account. ■

Remark 1 According to the proof of the previous theorem, the matrices of the form $\exp(\tilde{A}_{11}) \cdots \exp(\tilde{A}_{rr})$, with $A_{ii} \in \mathfrak{o}(m_i, \mathbb{C})$ for $1 \le i \le r$, and

$$\tilde{A}_{ii} = \begin{pmatrix} O_{\mu_i,\mu_i} & O_{\mu_i,m_i} & O_{\mu_i,n-\mu_{i+1}} \\ O_{m_i,\mu_i} & A_{ii} & O_{m_i,n-\mu_{i+1}} \\ O_{n-\mu_{i+1},\mu_i} & O_{n-\mu_{i+1},m_i} & O_{n-\mu_{i+1},n-\mu_{i+1}} \end{pmatrix},$$

where $\mu_i = m_1 + \ldots + m_{i-1}$, $O_{\mu\nu}$ denoting the null $\mu \times \nu$ matrix, span the intersection group $G \cap H$. Hence the problem of computing the intersection group is feasible; in fact, it reduces (up to polynomial time) to exponentiate skew-symmetric matrices of sizes $m_1, \ldots, m_r$ (see [3]).

Example 2.3 Assume $\dim V = n = 5$, and that L has two distinct eigenvalues α, β such that $\dim E(\alpha) = 2$, $\dim E(\beta) = 3$. In this case, $\mathfrak{g} \cap \mathfrak{h}$ is identified to the matrices of the form

$$A = \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix}, \qquad A_{11} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, A_{22} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

According to *Remark 1*, the intersection group is generated by $\exp \tilde{A}_{11} \exp \tilde{A}_{22}$. Exponentiating, we obtain

$$\exp \tilde{A}_{11} \exp \tilde{A}_{22} = \begin{pmatrix} \begin{pmatrix} \cos d & \sin d \\ -\sin d & \cos d \end{pmatrix} & O \\ O & |v|^{-2} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \end{pmatrix},$$

where $v = (a,b,c)$, and

$$\begin{aligned}
\lambda_{11} &= c^2 + \left(a^2 + b^2\right) \cos(|v|), \\
\lambda_{12} &= a|v| \sin(|v|) + bc \left(\cos(|v|) - 1\right), \\
\lambda_{13} &= b|v| \sin(|v|) - ac \left(\cos(|v|) - 1\right), \\
\lambda_{21} &= -a|v| \sin(|v|) - bc \left(\cos(|v|) - 1\right), \\
\lambda_{22} &= b^2 + \left(a^2 + c^2\right) \cos(|v|), \\
\lambda_{23} &= c|v| \sin(|v|) + ab \left(\cos(|v|) - 1\right), \\
\lambda_{31} &= -b|v| \sin(|v|) + ac \left(\cos(|v|) - 1\right), \\
\lambda_{32} &= -c|v| \sin(|v|) - ab \left(\cos(|v|) - 1\right), \\
\lambda_{33} &= a^2 + \left(b^2 + c^2\right) \cos(|v|).
\end{aligned}$$

3. A counterexample

The formula for the dimension of the intersection group $G \cap H$ given in the previous theorem is no longer true if the endomorphism L is not diagonalizable.

We provide a counterexample in arbitrary dimension as follows: For the metrics g, h with matrices given respectively by

$$M_g = \begin{pmatrix} \overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(k} & O \\ O & \overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(n-k} \end{pmatrix}, \quad M_h = \begin{pmatrix} \overbrace{\begin{pmatrix} 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & \dots & \alpha & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & \alpha & \dots & 0 & 0 \\ \alpha & 0 & \dots & 0 & 0 \end{pmatrix}}^{(k} & O \\ O & \overbrace{\begin{pmatrix} 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & \dots & \alpha & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & \alpha & \dots & 0 & 0 \\ \alpha & 0 & \dots & 0 & 0 \end{pmatrix}}^{(n-k} \end{pmatrix},$$

we obtain $\dim(\mathfrak{g} \cap \mathfrak{h}) = \min(k, n-k)$. In fact, assuming $k \leq n-k$, a computation shows that the $n \times n$ matrices A such that $A^t M_g + M_g A = 0$ and $A^t M_h + M_h A = 0$ are

$$A = \begin{pmatrix} O & Y \\ Z & O \end{pmatrix},$$

where

$$Y = -\overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(k} Z^t \overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(n-k}$$

and Z is the $(n-k) \times k$ matrix given by

$$Z = \sum_{h=1}^{k} \sum_{i=0}^{h-1} z_h E_{n-k-i,h-i}, \quad z_1, \dots, z_k \in \mathbb{C},$$

(E_{ij}) being the standard basis of the matrix vector space. Moreover, $\dim E(\alpha) = 2$, where α is the only eigenvalue of L. In fact,

$$M_L = M_g^{-1} M_h = \begin{pmatrix} \overbrace{\begin{pmatrix} \alpha & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & \alpha & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha \end{pmatrix}}^{(k} & O \\ O & \overbrace{\begin{pmatrix} \alpha & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & \alpha & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha \end{pmatrix}}^{(n-k} \end{pmatrix}.$$

4. Conclusions

The dimension of the intersection group of the orthogonal complex groups corresponding to two non-degenerate symmetric bilinear forms g, h is seen to depend quadratically on the dimensions of the eigenspaces of the linear transformation L associated to g, h, whenever L is semisimple. A computationally feasible procedure to obtain the intersection is provided. A counterexample in arbitrary dimension to the formula for the dimension of the intersection group in Theorem 2.1 when the nilpotent part of L does not vanish, is also included.

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