HOMOGENEOUS KÄHLER AND SASAKIAN STRUCTURES RELATED TO COMPLEX HYPERBOLIC SPACES

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Abstract We study homogeneous Kähler structures on a non-compact Hermitian symmetric space and their lifts to homogeneous Sasakian structures on the total space of a principal line bundle over it, and we analyze the case of the complex hyperbolic space.

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1 Introduction

The general theory of homogeneous Kähler manifolds is well-known, as well as the relation between homogeneous symplectic and homogeneous contact manifolds (see e.g. Boothby and Wang [6], Díaz Miranda and Reventós [9], and Dorfmeister and Nakajima [10]).

As is also widely known, a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if its curvature tensor field is parallel. Ambrose and Singer [2] extended this result to obtain a characterization of homogeneous Riemannian manifolds in terms of the existence of a tensor field \( S \) of type \((1, 2)\) on the manifold, called a homogeneous Riemannian structure (see Tricerri and Vanhecke [26], where a classification of such structures is also given), satisfying certain properties (see (2.1); if \( S = 0 \) one has the symmetric case). Moreover, Sekigawa [24] obtained the corresponding result for almost Hermitian manifolds, defining homogeneous almost Hermitian structures, which were classified by Abbena and Garbiero in [1] (among them the homogeneous Kähler structures). Its odd-dimensional version, the almost contact metric case, has been also studied (see, for instance, [8, 11, 14, 19]).

In Section 2, we give the basic results about homogeneous Riemannian and homogeneous Kähler structures. In particular we consider these structures on Hermitian symmetric spaces of non-compact type. Besides the trivial homogeneous structure \( S = 0 \) associated to the description of one such space as a symmetric space, other structures can be obtained associated to other descriptions as a homogeneous space and, in particular, to its description as a solvable Lie group given by an Iwasawa decomposition (§ 2.2).
We also give a construction of homogeneous Sasakian structures on the bundle space of a principal line bundle over a Hermitian symmetric space of non-compact type, endowed with a connection 1-form that is the contact form of a Sasakian structure on the total space (Proposition 2.5).

The complex hyperbolic space $\mathbb{C}H(n) = SU(n, 1)/S(U(n) \times U(1))$ with the Bergman metric is an irreducible Hermitian symmetric space of non-compact type, and, up to homotheties, is the simply-connected complete complex space form of negative curvature. It has been characterized in [12] in terms of the existence of certain type of homogeneous Kähler structure on it, and in [7] a Lie-theoretical description of its homogeneous structure of linear type is found. In Section 3 we study the homogeneous Kähler structures on $\mathbb{C}H(n)$ from other point of view, which in particular provide an infinite number of descriptions of $\mathbb{C}H(n)$ as non-isomorphic solvable Lie groups. Moreover, we consider the principal line bundle over $\mathbb{C}H(n)$ with its Sasakian structure given in a natural way from a connection form on the bundle, and we obtain the families of homogeneous Sasakian structures on its bundle space following our previous general construction. Summarizing, we get:

(a) All the homogeneous Kähler structures on $\mathbb{C}H(n) \equiv AN$. They are given in terms of some 1-forms related by a system of differential equations on the solvable Lie group $AN$ (Theorem 3.1).

(b) The explicit description of a multi-parametric family of homogeneous Kähler structures on $\mathbb{C}H(n)$, given by using the generators of $\mathfrak{a} + \mathfrak{n}$ (Proposition 3.6), and the corresponding subgroups of the full isometry group $SU(n, 1)$ of $AN$ (Theorem 3.7).

(c) The explicit description of a one-parametric family of homogeneous Sasakian structures on the bundle space of the line bundle $\tilde{M} \to \mathbb{C}H(n)$, given in terms of the horizontal lifts of the generators of $\mathfrak{a} + \mathfrak{n}$ and the fundamental vector field $\xi$ on $\tilde{M}$ (Proposition 3.9), and their associated reductive decompositions (Propositions 3.11 and 3.12). One of them describes $\tilde{M}$ as the complete simply connected $\varphi$-symmetric Sasakian space $S\mathbb{U}(n, 1)/SU(n)$, which is also a Sasakian space form.

On the other hand, complex hyperbolic space was the first target spacetime where Nishino’s [21] alternative (i.e., neither necessarily hyper-Kähler nor quaternion-Kähler) $N = (4, 0)$ superstring theory proved to work. This model has some interesting features, among them, not to have (which is a trait common to heterotic $\sigma$-models) the incompatibility between the torsion tensor and quaternion-Kähler manifolds found by de Wit and van Nieuwenhuizen [27]. Another peculiarity is that in this case, one of the two scalars of the relevant global multiplet is promoted to coordinates on $\mathbb{C}H(n)$, while the other plays the role of a tangent vector under the holonomy group $SU(n) \times U(1)$.

2 Homogeneous Riemannian Structures

Ambrose and Singer [2] proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only there exists a tensor field $S$ of type $(1, 2)$ on $M$ such that the connection $\tilde{\nabla} = \nabla - S$ satisfies the Eqs.

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0,$$ (2.1)
where \( \nabla \) is the Levi-Civita connection of \( g \) and \( R \) its curvature tensor field, for which we adopt the conventions \( R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z, \quad R_{XYZW} = g(R_{XY}Z,W) \).

Such a tensor field \( S \) is called a homogeneous Riemannian structure ([26]). We also denote by \( S \) the associated tensor field of type \((0,3)\) on \( M \) defined by \( S_{XYZ} = g(S_XY,Z) \).

### 2.1 Homogeneous Kähler structures

An almost Hermitian manifold \((M,g,J)\) is said to be a homogeneous almost Hermitian manifold if there exists a Lie group of holomorphic isometries which acts transitively and effectively on \( M \). Sekigawa proved the following

**Theorem 2.1.** ([24]) A connected, simply connected and complete almost Hermitian manifold \((M,g,J)\) is homogeneous if and only if there is a tensor field \( S \) of type \((1,2)\) on \( M \) which satisfies Eqs. (2.1) and \( \nabla J = 0 \).

A tensor \( S \) satisfying the Eqs. (2.1) and \( \bar{\nabla} J = 0 \) is called a homogeneous almost Hermitian structure. The almost Hermitian manifold \((M,g,J)\) is Kähler if and only if \( J \) is integrable and the fundamental 2-form \( \Omega \) on \( M \), given by \( \Omega(X,Y) = g(X,JY) \), is closed, or equivalently \( \nabla J = 0 \). In this case, a homogeneous almost Hermitian structure is also called a homogeneous Kähler structure, and we have

**Proposition 2.2.** A homogeneous Riemannian structure \( S \) on a Kähler manifold \((M,g,J)\) is a homogeneous Kähler structure if and only if \( S \cdot J = 0 \), or equivalently \( S_{XYZ} = S_XJYJZ \) for all the vector fields \( X,Y,Z \) on \( M \).

**Corollary 2.3.** A connected, simply connected and complete Kähler manifold \((M,g,J)\) is a homogeneous Kähler manifold if and only if there exists a homogeneous Kähler structure on \( M \).

If \((M = G/H,g)\) is a homogeneous Riemannian manifold, where \( G \) is a connected Lie group acting transitively and effectively on \( M \) as a group of isometries and \( H \) is the isotropy group at a point \( o \in M \), then the Lie algebra \( \mathfrak{g} \) of \( G \) may be decomposed into a vector space direct sum \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \), where \( \mathfrak{h} \) is the Lie algebra of \( H \) and \( \mathfrak{m} \) is an \( \text{Ad}(H) \)-invariant subspace of \( \mathfrak{g} \). If \( G \) is connected and \( M \) is simply connected then \( H \) is connected, and the condition \( \text{Ad}(H) \mathfrak{m} \subset \mathfrak{m} \) is equivalent to \([\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m} \). The vector space \( \mathfrak{m} \) is identified with \( T_o(M) \) by the isomorphism \( X \in \mathfrak{m} \mapsto X^* \in T_o(M), \) where \( X^* \) is the Killing vector field on \( M \) generated by the one-parameter subgroup \( \{ \exp tX \} \) of \( G \) acting on \( M \). If \( X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \), we write \( X = X^* + X_m \), \((X^* \in \mathfrak{h}, X_m \in \mathfrak{m}) \). The canonical connection \( \bar{\nabla} \) of \( M = G/H \) (with regard to the reductive decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \)) is determined by

\[
(\bar{\nabla}_X Y^*)_o = [X^*, Y^*]_o = -[X, Y]_o^* = -([X, Y]_\mathfrak{m})^* \quad , \quad X, Y \in \mathfrak{m}, \tag{2.2}
\]

and \( S = \nabla - \bar{\nabla} \) satisfies the Ambrose-Singer Eqs. (2.1), and it is the homogeneous Riemannian structure associated to the reductive decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \). If \((M,g)\) is endowed with a compatible almost complex structure \( J \) invariant by \( G \) (so that \((M = G/H,g,J)\) is a homogeneous almost Hermitian manifold), restricting \( J \) to \( T_o(M) \equiv \mathfrak{m} \), we obtain a
linear endomorphism $J_o$ of $\mathfrak{m}$ such that $J_o^2 = -1$, and $J_o \text{ad}_\mathfrak{h} = \text{ad}_\mathfrak{h} J_o$. Moreover, $J$ is integrable if and only if

$$[J_o X, J_o Y]_m - [X, Y]_m - J_o[X, J_o Y]_m - J_o[J_o X, Y]_m = 0$$

for all $X, Y \in \mathfrak{m}$ (\cite{18}, Ch. 10, Prop. 6.5).

Conversely, suppose that $(M, g)$ is a connected, simply connected and complete Riemannian manifold, and let $S$ be a homogeneous Riemannian structure on $(M, g)$. We put $\mathfrak{m} = T_o(M)$, where $o \in M$. If $\tilde{R}$ is the curvature tensor of the connection $\tilde{\nabla} = \nabla - S$, the holonomy algebra $\mathfrak{h}$ of $\tilde{\nabla}$ is the Lie subalgebra of the Lie algebra of antisymmetric endomorphisms $\mathfrak{so}(\mathfrak{m})$ of $(\mathfrak{m}, g_o)$ generated by the operators $\tilde{R}_{XY}$, where $X, Y \in \mathfrak{m}$. A Lie bracket is defined (Nomizu \cite{20}) in the vector space direct sum $\tilde{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ by

$$[U, V] = UV - VU, \quad U, V \in \mathfrak{h},$$

$$[U, X] = U(X), \quad U \in \mathfrak{h}, \quad X \in \mathfrak{m},$$

$$[X, Y] = \tilde{R}_{XY} + S_x Y - S_y X, \quad X, Y \in \mathfrak{m},$$

and $\tilde{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ is the reductive decomposition corresponding to the homogeneous Riemannian structure $S$. Let $\tilde{G}$ be the connected simply connected Lie group whose Lie algebra is $\tilde{\mathfrak{g}}$ and $\tilde{H}$ the connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\mathfrak{h}$. Then $\tilde{G}$ acts transitively on $M$ as a group of isometries and $M$ is diffeomorphic to $\tilde{G}/\tilde{H}$. If $\Gamma$ is the set of the elements of $\tilde{G}$ which act trivially on $M$, then $\Gamma$ is a discrete normal subgroup of $\tilde{G}$, and the Lie group $\tilde{G} = \tilde{G}/\Gamma$ acts transitively and effectively on $M$ as a group of isometries, with isotropy group $H = \tilde{H}/\Gamma$. Then $M$ is diffeomorphic to $G/H$. Now, if $J$ is a compatible almost complex structure on $(M, g)$ and $S$ is a homogeneous almost Hermitian structure, then the holonomy algebra $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{u}(\mathfrak{m}) = \{ A \in \mathfrak{so}(\mathfrak{m}) : A \cdot J = 0 \}$ of the unitary group, and $M \approx \tilde{G}/H \approx G/H$ is a homogeneous almost Hermitian manifold.

### 2.2 Hermitian symmetric spaces of non-compact type

Suppose that $(M = G/K, g, J)$ is a connected Hermitian symmetric space of non-compact type, where $G = I_0(M)$ is the identity component of the group of (holomorphic) isometries and $K$ is a maximal compact subgroup of $G$. Then $M$ is simply connected and the Hermitian structure is Kähler. We consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of $G$, and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of $K$, $\mathfrak{a} \subset \mathfrak{p}$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ is a nilpotent subalgebra. Let $A$ and $N$ be the connected abelian and nilpotent Lie subgroups of $G$ whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The solvable Lie group $AN$ acts simply transitively on $M$, so $M$ is isometric to $AN$ equipped with the left-invariant Riemannian metric defined by the scalar product $\langle \cdot, \cdot \rangle$, induced on $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$ by a positive multiple of $B_{\mathfrak{p} \times \mathfrak{p}}$, where $B$ is the Killing form of $\mathfrak{g}$.

Now, let $\tilde{G}$ be a connected closed Lie subgroup of $G$ which acts transitively on $M$. The isotropy group of this action at $o = K \in M$ is $H = \tilde{G} \cap K$. Then $M = G/K$ has also the description $M \equiv \tilde{G}/H$, and $o \equiv H \in \tilde{G}/H$. Let $\tilde{\mathfrak{g}} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ corresponding to $M \equiv \tilde{G}/H$. 

4
We have the isomorphisms of vector spaces
\[ p \cong g / \mathfrak{k} \cong \hat{g} / \mathfrak{h} \cong m \cong T_o(M) \cong a + n, \]
with
\[ \xi: p \xrightarrow{\sim} m, \quad \mu: m \xrightarrow{\sim} T_o(M), \quad \zeta: T_o(M) \xrightarrow{\sim} a + n, \]
given by
\[ \xi^{-1}(Z) = Z_p, \quad \mu(Z) = Z_a^*, \quad \zeta^{-1}(X) = X_a^*, \quad Z \in m, \ X \in a + n. \]

For each \( X \in g \), we have \((X_p)^* = 0\) and \((\nabla(X_p)^*)_o = 0\), and since the Levi-Civita connection \( \nabla \) has no torsion, for each \( X, Y \in g \), we have
\[
(\nabla_X Y^*)_o = (\nabla_(X_p)(Y_p)^*)_o = [(X_p)^*, (Y_p)^*)_o = -[X_p, Y_p]^o. \quad (2.4)
\]

The reductive decomposition \( \hat{g} = \mathfrak{h} + m \) defines the homogeneous Riemannian structure \( S = \nabla - \tilde{\nabla} \), where \( \tilde{\nabla} \) is the canonical connection of \( M \equiv \hat{G} / H \) with respect to \( \hat{g} = \mathfrak{h} + m \), which is \( \hat{G} \)-invariant and uniquely determined by \((\nabla_X Y^*)_o = -[X, Y]^o\), for \( X, Y \in m \) (2.2). The tensor field \( S \) is also uniquely determined by its value at \( o \) because \( M \equiv \hat{G} / H \) and \( S \) is \( G \)-invariant. Since \( J \) is \( \hat{G} \)-invariant, from [18], Ch. 10, Prop. 2.7, it follows that \( \nabla J = 0 \), and by Theorem 2.1, \( S \) is a homogeneous Kähler structure.

We have
\[
(S_X Y^*)_o = (\nabla_X Y^*)_o + [X, Y]^o = \nabla_Y X^*, \quad X, Y \in m. \quad (2.5)
\]

By (2.4) and (2.5), \( S \) is given by
\[
S_{X_p} Y^*_o = [X_p, Y_p]^o, \quad X, Y \in m.
\]

Then, for each \( X, Y \in a + n \), we have
\[
S_{X_p} Y^*_o = S_{\xi(X_p)} : \xi(Y_p)^o = [[\xi(X_p)]_o, Y_p]^o.
\]

The complex structure \( J \) on \( M = G / K \) is defined by an element \( E_J \) in the center of \( \mathfrak{k} \), and it defines the complex structure \( J \in \text{End}(a + n) \) such that the following diagram is commutative, and \((a + n, (\ , \ ), J)\) becomes a Hermitian vector space isomorphic to \((T_o(M), g_o, J_o)\).

\[
\begin{array}{cccccc}
p & \xrightarrow{\xi} & m & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & a + n \\
\text{ad}_{E_J} & \downarrow & j_o & \downarrow & j_o & \downarrow & j \\
p & \xrightarrow{\xi} & m & \xrightarrow{\mu} & T_o(M) & \xrightarrow{\zeta} & a + n
\end{array}
\]

Let \( A \) and \( N \) be the connected abelian and nilpotent Lie subgroups of \( G \) whose Lie algebras are \( a \) and \( n \), respectively. The solvable Lie group \( AN \) acts simply transitively on \( M \). Then \( M \) is isometric to \( AN \) equipped with the left-invariant Riemannian metric defined by the scalar product induced on \( a + n \cong \mathfrak{g} / \mathfrak{k} \cong p \) by a positive multiple of \( B_{p \times p} \), where \( B \) is the Killing form of \( g \), so that \( AN \) equipped with the left-invariant almost complex structure defined by \( J \) is a Kähler manifold.
2.3 Homogeneous almost contact Riemannian manifolds

An almost contact structure on a \((2n + 1)\)-dimensional manifold \(M\) is a triple \((\varphi, \xi, \eta)\), where \(\varphi\) is a tensor field of type \((1,1)\), \(\xi\) a vector field (called the characteristic vector field) and \(\eta\) a differential 1-form on \(M\) such that

\[
\varphi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.
\]

Then \(\varphi \xi = 0\), \(\eta \circ \varphi = 0\), and \(\varphi\) has rank \(2n\). If \(\bar{g}\) is a Riemannian metric on \(\bar{M}\) such that \(\bar{g}(\varphi X, \varphi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y)\) for all vector fields \(X\) and \(Y\) on \(M\) then \((\varphi, \xi, \eta, \bar{g})\) is said to be an almost contact metric structure on \(\bar{M}\). In this case, \(\bar{g}(\bar{X}, \xi) = \eta(\bar{X})\). The 2-form \(\Phi\) on \(M\) defined by \(\Phi(X, Y) = \bar{g}(\bar{X}, \varphi Y)\) is called the fundamental 2-form of the almost contact metric structure \((\varphi, \xi, \eta, \bar{g})\). If \(d\eta(\bar{X}, \bar{Y}) = \bar{X}\eta(\bar{Y}) - \bar{Y}\eta(\bar{X}) - \eta([\bar{X}, \bar{Y}]) = 2\Phi(\bar{X}, \bar{Y})\), then \((\phi, \xi, \eta, \bar{g})\) is called a contact metric (or contact Riemannian) structure; in particular, \(\eta \wedge (d\eta)^n \neq 0\), that is, \(\eta\) is a contact form on \(\bar{M}\). If

\[
(D_X \varphi)Y = \bar{g}(X, Y)\xi - \eta(Y)\bar{X},
\]

where \(D\) is the Levi-Civita connection of \(\bar{g}\), then \((\varphi, \xi, \eta, \bar{g})\) is called a Sasakian structure, and the manifold \(\bar{M}\) with such a structure is a Sasakian manifold. Sasakian manifolds can also be characterized as normal contact metric manifolds and they are in some sense odd-dimensional analogues of Kähler manifolds (see Blair [3, 4]).

If \((\varphi, \xi, \eta, \bar{g})\) is an almost contact metric structure on \(\bar{M}\) and \((\bar{M} = \bar{G}/H, \bar{g})\) is a homogeneous Riemannian manifold such that \(\varphi\) is invariant under the action of the connected Lie group \(\bar{G}\) (and hence so are \(\xi\) and \(\eta\)) then \((\bar{M}, \varphi, \xi, \eta, \bar{g})\) is called a homogeneous almost contact Riemannian manifold ([8, 14, 19]). Let \(\bar{R}\) be the curvature tensor field of the Levi-Civita connection \(D\) of \(\bar{g}\). Let \(S\) be a homogeneous Riemannian structure on \(\bar{M}\), that is \(\bar{D}\bar{g} = 0, \bar{D}\bar{R} = 0\) and \(\bar{D}S = 0\), where \(\bar{D} = D - S\). If \(S\) satisfies the additional condition \(\bar{D}\varphi = 0\) (and hence \(\bar{D}\xi = 0\) and \(\bar{D}\eta = 0\)), then \(S\) is called a homogeneous almost contact metric structure on \((\bar{M}, \varphi, \xi, \eta, \bar{g})\). From the results of Kirichenko in [17] on homogeneous Riemannian spaces with invariant tensor structure, it follows

**Theorem 2.4.** A connected, simply connected and complete almost contact metric manifold \((\bar{M}, \varphi, \xi, \eta, \bar{g})\) is a homogeneous almost contact Riemannian manifold if and only if there exists a homogeneous almost contact metric structure on \(\bar{M}\).

A homogeneous almost contact metric structure on a Sasakian manifold will be also called a homogeneous Sasakian structure.

2.4 Principal 1-bundles over almost Hermitian manifolds

Let \((M, g, J)\) be an almost Hermitian manifold and let \(\bar{M}\) be the bundle space of a principal 1-bundle over \(M\). Let \(\eta\) be a connection (form) on the principal bundle \(\pi: \bar{M} \to M\), and let \(\xi\) be the fundamental vector field on \(\bar{M}\) defined by the element 1 of the Lie algebra \(\mathfrak{r}\) of the structure group of the bundle. Then \(\eta(\xi) = 1\). For each vector field \(X\) on \(M\), we denote by \(X^H\) the horizontal lift of \(X\) with respect to \(\eta\). If \(X\) is a vector field on \(M\), its vertical part is \(\eta(\bar{X})\xi\). Then, for any vector fields \(X\) and \(Y\) on \(M\), we have

\[
[X^H, Y^H] = [X, Y]^H + \eta([X^H, Y^H])\xi.
\]
Moreover, \([X^H, \xi] = 0\), because \(X^H\) is invariant under the action of the structural group. We define a tensor field \(\varphi\) of type \((1, 1)\) and a Riemannian metric \(\bar{g}\) on \(\bar{M}\) by

\[
\varphi X^H = (JX)^H, \quad \varphi \xi = 0, \quad \bar{g} = \pi^* g + \eta \otimes \eta, \tag{2.7}
\]

where \(X\) and \(Y\) are vector fields on \(M\). Clearly, \((\varphi, \xi, \eta, \bar{g})\) is an almost contact metric structure on \(\bar{M}\), and we have \(\bar{g}(X^H, Y^H) = g(X, Y) \circ \pi\), and \(\bar{g}(X^H, \xi) = 0\). Let \(\Phi\) be its 2-fundamental form. If \(\Omega\) is the fundamental 2-form of the almost Hermitian manifold \((M, g, J)\), then \(\pi^* \Omega = \Phi\).

If \(\nabla\) and \(D\) are the Levi-Civita connections of \(g\) and \(\bar{g}\), respectively, then (Ogieu [22])

\[
D_{X^H}Y^H = (\nabla_{X^H}Y^H)^H + \frac{1}{2} \eta([X^H, Y^H]) \xi = (\nabla_X Y)^H - \frac{1}{2} d\eta(X^H, Y^H) \xi,
\]

and \(D_{X^H} \xi = D_X \xi = -\varphi X^H\). Now, if \(2\Phi = d\eta\), Eq. (2.6) is satisfied as one can easily see by replacing \((\bar{X}, \bar{Y})\) by \((X^H, Y^H), (X^H, \xi),\) and \((\xi, Y^H)\), respectively. Then, if the almost contact metric structure \((\varphi, \xi, \eta, \bar{g})\) is a contact structure, it is also Sasakian.

Suppose now that the structural group of the principal 1-bundle \(\pi: \tilde{M} \to M\) is \(\mathbb{R}\) and that the base manifold is a 2n-dimensional connected Hermitian symmetric space of non-compact type \((M = G/K, g, J)\), so that \(M\) is isometric to the solvable Lie group \(AN\) as in \(\S\) 2.2. Then \(M\) is holomorphically diffeomorphic to a bounded symmetric domain, i.e., to a simply connected open subset of \(\mathbb{C}^n\) such that each point is an isolated fixed point of an involutive holomorphic diffeomorphism of itself ([15], Ch. VIII, Th. 7.1). Since \(\pi: \tilde{M} \to M\) is a principal line bundle over the paracompact manifold \(M\), then it admits a global section ([18], Ch. I, Th. 5.7), so there exists a diffeomorphism \(\tilde{M} \to M \times \mathbb{R}\), and the bundle space \(\tilde{M}\) may be identified with \(AN \times \mathbb{R}\), with \(\pi\) being the projection on \(AN\). On the other hand, since the fundamental 2-form \(\Omega\) associated to the Kähler structure \((g, J)\) is closed, \(\Omega = d\zeta\) for some real analytic 1-form \(\zeta\) on \(AN\). We consider the connection form \(\eta = 2\pi^* \zeta + dt\) on \(\tilde{M}\), where \(t\) is the coordinate of \(\mathbb{R}\). The vertical vector field \(\xi\) with \(\eta(\xi) = 1\) can be identified with \(\frac{\partial}{\partial t}\), and we consider \(\varphi\) and \(\bar{g}\) given by (2.7). Then \(2\Phi = 2\pi^* \Omega = 2\pi^* d\zeta = d\eta\), and hence \((\varphi, \xi, \eta, \bar{g})\) is a Sasakian structure on \(\tilde{M}\).

If \(\bar{S}\) is a homogeneous almost contact metric structure on \(\tilde{M}\), and \(\bar{D} = D - \bar{S}\), then \(\bar{D}\xi = 0\), and hence \(\bar{S}_{X^H} \xi = D_{X^H} \xi = -\varphi X^H\). We have

**Proposition 2.5.** Let \((M = G/K, g, J)\) be a connected Hermitian symmetric space of non-compact type. Let \(\pi: \tilde{M} \to M\) be a principal line bundle with connection form \(\eta\) such that the almost contact metric structure \((\varphi, \xi, \eta, \bar{g})\) on \(\tilde{M}\) defined by (2.7) is Sasakian.

(a) If \(S\) is a homogeneous Kähler structure on \(\tilde{M}\) then the tensor field \(\bar{S}\) on \(\tilde{M}\) defined by

\[
\bar{S}_{X^H} Y^H = (S_X Y)^H - \bar{g}(X^H, \varphi Y^H) \xi, \quad \bar{S}_{X^H} \xi = -\varphi X^H \bar{S}, \quad \bar{S} \xi = 0,
\]

for all vector fields \(X\) and \(Y\) on \(M\), is a homogeneous Sasakian structure on \(\tilde{M}\).

(b) \(\{S^t: t \in \mathbb{R}\}\) defined by

\[
S_{X^H}^t Y^H = -\bar{g}(X^H, \varphi Y^H) \xi, \quad S_{X^H}^t \xi = -\varphi X^H \bar{S}^t, \quad S^t \xi = 0,
\]

is a family of homogeneous Sasakian structures on \(\tilde{M}\).
Proof. (a) If \( \bar{D} = D - \bar{S} \), then since \( \bar{S}_{X^uY^uZ^u} = \bar{g}((S_XY)^H, Z^H) = g(S_XY, Z) \circ \pi = -g(Y, S_XZ) \circ \pi = -\bar{g}(Y^H, (S_XZ)^H) = -\bar{S}_{X^uY^uZ^u} \), and \( \bar{S}_{X^uY^u\xi} = -\bar{S}_{X^u\xi Y^u} \), the condition \( \bar{D}\bar{g} = 0 \) is satisfied. On the other hand, if \( \bar{\nabla} = \nabla - S \) we have
\[
\bar{D} X^uY^H = (\bar{\nabla} X Y)^H, \quad \bar{D} X^u\xi = \bar{D} \xi X^H = 0. \tag{2.8}
\]
We can identify \( M = G/K \) with the solvable Lie group \( AN \) in an Iwasawa decomposition \( G = KAN \) and consider the Lie algebra \( \mathfrak{a} + \mathfrak{n} \) of \( AN \). If \( U, \bar{V}, X, \bar{Y}, \bar{Z} \) are horizontal lifts of elements of \( \mathfrak{a} + \mathfrak{n} \) or some of them are the vertical vector field \( \xi \), then
\[
(\bar{D}_G R)_{\bar{X}\bar{Y}\bar{Z}\bar{V}} = -\bar{R}_{\bar{X}\bar{Y}\bar{Z}\bar{D}_G \bar{V}} + \bar{R}_{\bar{X}\bar{Y}\bar{V}\bar{D}_G \bar{Z}} - \bar{R}_{\bar{Z}\bar{V}\bar{X}\bar{D}_G \bar{Y}} + \bar{R}_{\bar{Z}\bar{V}\bar{Y}\bar{D}_G \bar{X}}, \tag{2.9}
\]
so since \( \bar{U}(\bar{R}_{\bar{X}\bar{Y}\bar{Z}\bar{V}}) = 0 \). Now, if \( X, Y, Z, V \in \mathfrak{a} + \mathfrak{n} \), then
\[
\bar{R}_{X^uY^uZ^uV^u} = (R_{XYZV} - 2g(X, JY)g(Z, JV))
+ g(X, JY)g(Y, JZ) - g(X, JZ)g(Y, JV)) \circ \pi,
\]
\[
\bar{R}_{X^uY^uZ^u\xi} = -\bar{g}([X, Y]^H, \varphi Z^H)
+ \bar{g}((\nabla_X Z)^H, \varphi Y^H) - \bar{g}((\nabla_Y Z)^H, \varphi X^H), \tag{2.10}
\]
By using (2.8) and (2.10), the conditions \( \bar{\nabla}R = 0 \) and \( \bar{\nabla}J = 0 \) for the homogeneous Kähler structure \( S \) on \( M \), and the formula \( \bar{R}_{\bar{X}\bar{Y}} \xi = \eta(\bar{X})\bar{Y} - \eta(\bar{Y})\bar{X} \) for the Sasakian manifold \( \tilde{M} \), \( \varphi, \xi, \eta, \bar{g} \) ([4], Prop. 7.3), one obtains from (2.9) that \( \bar{D}\bar{R} = 0 \). Now, \( (\bar{D}_U \bar{S})_{X^uY^u} = (\bar{g}(\bar{u}, S_XY) \circ \pi) \), \( (\bar{D}_U \bar{S})_{X^u\xi} = (\bar{g}(\bar{u}, S_X\xi) \circ \pi) \), and \( \bar{D}\bar{S} = 0 \). Moreover, \( (\bar{D}_X^u\varphi)^Y = (\bar{g}(\bar{u}, \bar{S})) \), \( (\bar{D}_X^u\varphi)\xi = 0 \), \( \bar{D}\varphi = 0 \), and \( \bar{S} \) is a homogeneous Sasakian structure on \( M \).

(b) If \( t = 1 \) the corresponding tensor \( S^1 \) coincides with \( \bar{S} \) in (a) for \( S = 0 \). For arbitrary \( t \), if \( \bar{D}^t = D - S^t \) we have \( \bar{D}^t \xi X^H = (t - 1)(JX)^H \), and we get \( \bar{D}^t \bar{g} = 0 \), \( \bar{D}^t R = 0 \), \( \bar{D}^t S^t = 0 \), \( \bar{D}^t \varphi = 0 \). \( \square \)

3 The Complex Hyperbolic Space \( \mathbb{CH}(n) \)

3.1 \( \mathbb{CH}(n) \) as a solvable Lie group

The complex hyperbolic space \( \mathbb{CH}(n) \), which may be identified with the unit ball in \( \mathbb{C}^n \) endowed with the hyperbolic metric of constant holomorphic sectional curvature \(-4\), may also be viewed as the irreducible Hermitian symmetric space of non-compact type \( SU(n, 1)/SU(n) \times U(1) \).

The Lie algebra \( \mathfrak{su}(n, 1) \) of \( SU(n, 1) \) can be described as the subalgebra of \( \mathfrak{sl}(n+1, \mathbb{C}) \) of all matrices of the form
\[
X = \begin{pmatrix} Z & P^T \\ P & ic \end{pmatrix}, \tag{3.1}
\]
where $Z \in \mathfrak{u}(n)$, $c \in \mathbb{R}$, and $P = (p_1, \ldots, p_n) \in \mathbb{C}^n$. The involution $\tau$ of $\mathfrak{su}(n, 1)$ given by $\tau(X) = -X^\tau$ defines the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{t} + \mathfrak{p}$, where

$$\mathfrak{t} = \left\{ \left( \begin{array}{cc} Z & 0 \\ 0 & ic \end{array} \right) : \text{tr} Z + ic = 0 \right\} \cong \mathfrak{g}(\mathfrak{u}(n) \oplus \mathfrak{u}(1)), \quad \mathfrak{p} = \left\{ \left( \begin{array}{cc} 0 & p^\tau \\ p & 0 \end{array} \right) \right\}.$$  

The element $A_0$ of $\mathfrak{p}$ defined by $P = (0, \ldots, 0, 1)$ generates a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{su}(n, 1)$. Let $f_0$ be the linear functional on $\mathfrak{a}$ given by $f_0(A_0) = 1$. If $n > 1$, the set of roots of $(\mathfrak{su}(n, 1), \mathfrak{a})$ is $\Sigma = \{ \pm f_0, \pm 2f_0 \}$, the set $\Pi = \{ f_0 \}$ is a system of simple roots, and the corresponding positive root system is $\Sigma^+ = \{ f_0, 2f_0 \}$. If $n = 1$, $\Sigma = \{ \pm 2f_0 \}$, and $\Pi = \Sigma^+ = \{ 2f_0 \}$.

Let $E_{ij}$ be the matrix in $\mathfrak{g}(n, \mathbb{C})$ such that the entry at the $i$-th row and the $j$-th column is 1 and the other entries are all zero. The root vector spaces are

$$\mathfrak{g}_{f_0} = \langle Z_j, Z'_j : 1 \leq j \leq n - 1 \rangle \quad (\text{if } n > 1), \quad \mathfrak{g}_{-f_0} = \langle U_j, W_j : 1 \leq j \leq n - 1 \rangle \quad (\text{if } n > 1),$$

where

$$Z_j = E_{jn} - E_{n,j+1} + E_{n,j} - E_{n+1,j}, \quad Z'_j = i(E_{jn} - E_{j,n+1} + E_{nj} + E_{n+1,j}),$$

$$W_j = E_{jn} + E_{n,j+1} + E_{nj} + E_{n+1,j}, \quad W'_j = i(E_{jn} + E_{j,n+1} + E_{nj} - E_{n+1,j}),$$

$$U = i(E_{nn} - E_{n,n+1} + E_{n+1,n} - E_{n+1,n+1}),$$

$$V = i(E_{nn} + E_{n,n+1} - E_{n+1,n} - E_{n+1,n+1}).$$

If $n > 2$, the centralizer of $\mathfrak{a}$ in $\mathfrak{t}$ is $Z_\mathfrak{t}(\mathfrak{a}) = \langle C_r, f_{jk}, H_{jk} : r, j, k = 1, \ldots, n, j < k \rangle \cong \mathfrak{u}(n-1)$, where

$$C_r = 2iE_{rr} - iE_{nn} - iE_{n+1,n+1}, \quad F_{jk} = E_{jk} - E_{kj}, \quad H_{jk} = i(E_{jk} + E_{kj})$$

and $\mathfrak{su}(n, 1) = (Z_\mathfrak{t}(\mathfrak{a}) + \mathfrak{a}) + \sum_{f \in \Sigma} \mathfrak{g}_f$ is the restricted-root space decomposition. We also have the Iwasawa decomposition $\mathfrak{su}(n, 1) = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_{f_0} + \mathfrak{g}_{-2f_0} = \langle U, Z_j, Z'_j : 1 \leq j \leq n - 1 \rangle$.

If $n = 2$, we put $C = C_{11} = \text{diag}(2i, -i, -i)$, $Z = Z_1$, $Z' = Z'_1$, and in this case $C$ generates $Z_\mathfrak{t}(\mathfrak{a})$, and $\mathfrak{a} + \mathfrak{n} = \langle A_0, U, Z, Z' \rangle$. If $n = 1$, $Z_\mathfrak{t}(\mathfrak{a}) = 0$, we have the restricted-root space decomposition $\mathfrak{su}(1, 1) = \mathfrak{a} + (\mathfrak{g}_{2f_0} + \mathfrak{g}_{-2f_0}) = \langle A_0 \rangle + \langle U, V \rangle$, and the solvable part in the Iwasawa decomposition is $\mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$.

By using the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{t} + \mathfrak{p}$, we express each element $X \in \mathfrak{su}(n, 1)$ as the sum $X = X_\mathfrak{t} + X_\mathfrak{p}$ ($X_\mathfrak{t} \in \mathfrak{t}, X_\mathfrak{p} \in \mathfrak{p}$). In particular, we have

$$U_\mathfrak{t} = i(E_{nn} - E_{n+1,n+1}), \quad U_\mathfrak{p} = i(E_{n+1,n} - E_{n,n+1}),$$

$$Z_j \mathfrak{t} = E_{jn} - E_{nj}, \quad (Z_j)_\mathfrak{p} = -(E_{n+1,j} + E_{j,n+1}),$$

$$Z'_j \mathfrak{t} = i(E_{jn} + E_{nj}), \quad (Z'_j)_\mathfrak{p} = i(E_{n+1,j} - E_{j,n+1}).$$

From the basis $\{ A_0, U, Z_j, Z'_j : 1 \leq j \leq n - 1 \}$ of $\mathfrak{a} + \mathfrak{n}$ and the generators of $Z_\mathfrak{t}(\mathfrak{a})$ above, we get the basis $\{ C_r, f_{jk}, H_{jk}, U_\mathfrak{t}, (Z_j)_\mathfrak{t}, (Z'_j)_\mathfrak{t} : r, j, k = 1, \ldots, n, j < k \}$ of $\mathfrak{t}$, and the basis $\{ A_0, U_\mathfrak{p}, (Z_j)_\mathfrak{p}, (Z'_j)_\mathfrak{p} : 1 \leq j \leq n - 1 \}$ of $\mathfrak{p}$. Notice that, if $n = 1$, $\mathfrak{t} = \langle U_\mathfrak{t} \rangle$ and $\mathfrak{p} = \langle A_0, U_\mathfrak{p} \rangle$, and if $n = 2$ we have $\mathfrak{t} = \langle C, U_\mathfrak{t}, Z_\mathfrak{t}, Z'_\mathfrak{t} \rangle$, and $\mathfrak{p} = \langle A, U_\mathfrak{p}, Z_\mathfrak{p}, Z'_\mathfrak{p} \rangle$. We
also decompose $\mathfrak{k} = \mathfrak{k}' + \mathfrak{c}$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = \langle C_r - U_t, F_{jk}, H_{jk}, (Z_r')_t, (Z_j')_t : r, j, k = 1, \ldots, n-1, j < k \rangle \cong \mathfrak{su}(n)$, and $\mathfrak{c}$ is the center of $\mathfrak{k}$, which is generated by the element $E_I = \frac{1}{2n} (C_1 + \cdots + C_n + (n+1)U_t)$ such that $\text{ad}_{E_I} : \mathfrak{p} \to \mathfrak{p}$ defines the complex structure on $\mathbb{C}H(n)$. By the isomorphisms $\mathfrak{p} \cong \mathfrak{su}(n,1)/\mathfrak{k} \cong \mathfrak{a} + \mathfrak{n}$, we obtain the complex structure $J$ acting on $\mathfrak{a} + \mathfrak{n}$ as follows.

$$J A_0 = -U, \quad JU = A_0, \quad JZ_r = Z'_r, \quad JZ'_r = -Z_r, \quad (3.2)$$

We consider the scalar product $\langle , \rangle$ on $\mathfrak{a} + \mathfrak{n}$ defined by the isomorphism $\mathfrak{a} + \mathfrak{n} \cong \mathfrak{p}$ and $\frac{1}{4(n+1)} B_{\mathfrak{p} \times \mathfrak{p}}$. Then $(\mathfrak{a} + \mathfrak{n}, \langle , \rangle, J)$ is a Hermitian vector space, and the basis $\{ A_0, U, Z_r, Z'_r : 1 \leq r \leq n-1 \}$ of $\mathfrak{a} + \mathfrak{n}$ is orthonormal. We consider the solvable factor $AN$ (with Lie algebra $\mathfrak{a} + \mathfrak{n}$) of the Iwasawa decomposition of $SU(n,1)$ with the invariant metric $g$ and almost complex structure $J$ defined by $\langle , \rangle$ and $J$, respectively.

The Lie brackets of the elements of the basis of $\mathfrak{a} + \mathfrak{n}$ are given by

$$[A_0, U] = 2U, \quad [A_0, Z_j] = Z_j, \quad [A_0, Z'_j] = Z'_j, \quad [Z_j, Z'_j] = -\delta_{jr}2U, \quad [U, Z_j] = [U, Z'_j] = [Z_j, Z'_j] = [Z'_j, Z''_j] = 0.$$ 

The Levi-Civita connection $\nabla$ is given by $2g(\nabla X,Y,Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$. So, the covariant derivatives between generators of $\mathfrak{a} + \mathfrak{n}$ are given by

$$\nabla_{A_0} A_0 = \nabla_{A_0} U = \nabla_{A_0} Z_r = \nabla_{A_0} Z'_r = 0, \quad \nabla_U U = 2U, \quad \nabla_U Z_r = Z'_r, \quad \nabla_U Z'_r = -Z_r, \quad \nabla_{Z'_j} A_0 = -Z'_j, \quad \nabla_{Z'_j} Z_r = Z''_j, \quad \nabla_{Z'_j} Z'_r = -Z'_j, \quad (3.3)$$

The components of the curvature tensor field $R$ are given by

$$R_{A_0U} A_0 = -4U, \quad R_{A_0U} U = 4A_0, \quad R_{A_0U} Z_r = 2Z'_r, \quad R_{A_0U} Z'_r = -2Z_r, \quad R_{A_0Z_j} A_0 = -Z'_j, \quad R_{A_0Z_j} Z_r = Z'_j, \quad R_{A_0Z'_j} Z_r = -\delta_{jr}A_0, \quad R_{A_0Z'_j} Z'_r = \delta_{jr}A_0, \quad R_{A_0Z'_j} Z''_j = \delta_{jr}U, \quad R_{A_0Z''_j} Z'_r = \delta_{jr}A_0, \quad R_{A_0Z''_j} Z''_j = \delta_{jr}U.$$ 

In particular we see that the invariant metric on $AN$ has constant holomorphic sectional curvature $-4$.

### 3.2 Homogeneous Kähler structures on $\mathbb{C}H(n) \equiv AN$

We will determine the homogeneous Kähler structures on $\mathbb{C}H(n) \equiv AN$ in terms of the basis of left-invariant forms $\alpha, \beta, \gamma^j, \gamma'^j, 1 \leq j \leq n-1$, dual to $A_0, U, Z_j, Z'_j$. If $S$ is
a homogeneous Riemannian structure on $A N$ and $\nabla = \nabla - S$, the condition $\nabla g = 0$ in (2.1) is equivalent to $S_{X Y Z} + S_{X Z Y} = 0$ for all $X, Y, Z \in a + n$. Moreover, $\nabla R = 0$ is equivalent to the condition

$$\left(\nabla X R\right)_{Y_1 Y_2 Y_3 Y_4} = -R_{X Y_1 Y_2 Y_3 Y_4} - R_{Y_1 Y_2 X Y_3 Y_4} - R_{Y_1 Y_2 Y_3 X Y_4} - R_{Y_1 Y_2 Y_3 Y_4} X_4,$$

for all $Y_1, Y_2, Y_3, Y_4 \in a + n$. Replacing $(Y_1, Y_2, Y_3, Y_4)$ by $(A_0, U, A_0, Z_j), (A_0, U, A_0, Z_j')$, $(A_0, U, Z_k, Z_j)$, and $(A_0, U, Z_k, Z_j')$, one gets that $S_{X U Z_j} = S_{X A_0 Z_j'}$, $S_{X U Z_j'} = -S_{X A_0 Z_j}$, $S_{X Z_k Z_j} = -S_{X Z_k Z_j'}$, and $S_{X Z_k Z_j'} = S_{X Z_k Z_j'}$, respectively. It is easy to see that the condition $\nabla R = 0$ holds if and only if the last four Eqs. are satisfied for all $X \in a + n$. These Eqs. also show (see (3.2)) that the condition $S \cdot J = 0$ of homogeneous Kähler structure (see Proposition 2.2) is fulfilled.

We put

$$\omega(X) = S_{X A_0 U}, \quad \sigma^j(X) = S_{X A_0 Z_j} = -S_{X U Z_j'}, \quad \tau^j(X) = S_{X A_0 Z_j'} = S_{X U Z_j}, \quad \theta^{kj}(X) = S_{X Z_k Z_j} = S_{X Z_k Z_j'}, \quad \psi^{kj}(X) = S_{X Z_k Z_j} = S_{X Z_k Z_j'}. \quad (3.4)$$

We have $\theta^{kj} = \theta^{kj}$ and $\psi^{kj} = -\psi^{kj}$. Now, we must determine the conditions for the 1-forms $\omega, \sigma^j, \tau^j, \theta^{kj}$ and $\psi^{kj}$ under which the condition $\nabla S = 0$ in (2.1) is satisfied. By (3.3), (3.4) and (3.5), the connection $\nabla = \nabla - S$ is given by

$$\begin{align*}
\nabla X A_0 &= -2(\beta + \omega)(X) U - \sum_j (\gamma^j + \sigma^j)(X) Z_j - \sum_k (\gamma^k + \tau^k)(X) Z_j', \\
\nabla X U &= (2\beta + \omega)(X) A_0 - \sum_j (\gamma^j + \tau^j)(X) Z_j + \sum_k (\gamma^k + \sigma^k)(X) Z_j', \\
\nabla X Z_j &= (\gamma^j + \sigma^j)(X) A_0 + (\gamma^j + \tau^j)(X) U + (\beta - \theta^j)(X) Z_j' \\
&\quad + \sum_{k \neq j} (\psi^{kj}(X) Z_k - \theta^{kj}(X) Z_k'), \\
\nabla X Z_j' &= (\gamma^j + \tau^j)(X) A_0 - (\gamma^j + \sigma^j)(X) U + (\theta^j - \beta)(X) Z_j \\
&\quad + \sum_{k \neq j} (\theta^{kj}(X) Z_k - \psi^{kj}(X) Z_k').
\end{align*}$$

Now, replacing $(V_1, V_2)$ in the Eq. $(\nabla X S)(W, V_1, V_2) = 0$ by $(A_0, U), (A_0, Z_j), (A_0, Z_j'), (Z_k, Z_j)$ and $(Z_k, Z_j')$, respectively, we obtain that the condition $\nabla S = 0$ is equivalent to the following conditions:

$$\begin{align*}
\nabla \omega &= 2 \sum_j ((\gamma^j + \sigma^j) \otimes \tau^j - (\gamma^j + \tau^j) \otimes \sigma^j), \\
\nabla \sigma^j &= -(\beta + \omega + \theta^j) \otimes \tau^j + (\gamma^j + \tau^j) \otimes (\omega + \theta^j) \\
&\quad + \sum_{k \neq j} ((\psi^{kj} \otimes \sigma^k - \theta^{kj} \otimes \tau^k + (\gamma^k + \tau^k) \otimes \theta^{kj} - (\gamma^k + \sigma^k) \otimes \psi^{kj}), \\
\nabla \tau^j &= (\beta + \omega + \theta^j) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes (\omega + \theta^j) \\
&\quad + \sum_{k \neq j} ((\theta^{kj} \otimes \sigma^k + \psi^{kj} \otimes \tau^k - (\gamma^k + \sigma^k) \otimes \theta^{kj} - (\gamma^k + \tau^k) \otimes \psi^{kj}), \\
\nabla \theta^{kj} &= (\gamma^j + \sigma^j) \otimes \sigma^k + (\gamma^k + \tau^k) \otimes \tau^j - (\gamma^j + \tau^j) \otimes \sigma^k - (\gamma^k + \tau^k) \otimes \sigma^j \\
&\quad + \sum_i \psi^{ik} \wedge \theta^{ij} + \sum_i \theta^{ik} \wedge \psi^{ij}, \\
\nabla \psi^{kj} &= (\gamma^j + \sigma^k) \otimes \sigma^j - (\gamma^j + \sigma^j) \otimes \sigma^k - (\gamma^k + \tau^k) \otimes \tau^j - (\gamma^j + \tau^j) \otimes \tau^k \\
&\quad + \sum_i \theta^{ik} \wedge \theta^{ij} - \sum_i \psi^{ik} \wedge \psi^{ij},
\end{align*}$$

where $\theta^j = \theta^{ij}$. Thus, from (3.4) and (3.5), we have
Theorem 3.1. All the homogeneous Kähler structures on $\mathbb{C}H(n) \equiv AN$ are given by

$$S = \omega \otimes (\alpha \wedge \beta) + \sum_{j=1}^{n-1} \left( \sigma^j \otimes (\alpha \wedge \gamma^j - \beta \wedge \gamma^j) + \tau^j \otimes (\alpha \wedge \gamma^j + \beta \wedge \gamma^j) + \theta^{kj} \otimes (\gamma^j \wedge \gamma^j) \right)$$

$$+ \sum_{1 \leq k < j \leq n-1} (\psi^{kj} \otimes (\gamma^k \wedge \gamma^j + \gamma^k \wedge \gamma^j) + \theta^{kj} \otimes (\gamma^k \wedge \gamma^j + \gamma^j \wedge \gamma^k)),$$

where $\omega$, $\sigma^j$, $\tau^j$, $\theta^{kj}$, and $\psi^{kj}$, $(1 \leq k, j \leq n-1)$, are 1-forms on $AN$ satisfying $\theta^{jk} = \theta^{kj}$, $\psi^{kj} = -\psi^{kj}$ and the Eqs. (3.6).

If $n = 2$, we put $\gamma = \gamma^1$, $\gamma' = \gamma'^1$, so that $\{\alpha, \beta, \gamma, \gamma'\}$ is the basis of left-invariant forms on $AN = \mathbb{C}H(2)$ dual to $\{A_0, U, Z, Z'\}$, and we have

Corollary 3.2. All the homogeneous Kähler structures on the complex hyperbolic plane $\mathbb{C}H(2) \equiv AN$ are given by

$$S = \omega \otimes (\alpha \wedge \beta) + \sigma \otimes (\alpha \wedge \gamma - \beta \wedge \gamma') + \tau \otimes (\alpha \wedge \gamma' + \beta \wedge \gamma) + \theta \otimes (\gamma \wedge \gamma'),$$

where $\omega$, $\sigma$, $\tau$ and $\theta$ are 1-forms on $AN$ satisfying

$$\tilde{\nabla} \omega = 2(\gamma + \sigma) \otimes \tau - 2(\gamma' + \tau) \otimes \sigma = \tilde{\nabla} \theta,$$

$$\tilde{\nabla} \sigma = -(\beta + \omega + \theta) \otimes \gamma + (\gamma' + \tau) \otimes (\omega + \theta),$$

$$\tilde{\nabla} \tau = (\beta + \omega + \theta) \otimes \sigma - (\gamma + \sigma) \otimes (\omega + \theta).$$

If $n = 1$, $\{\alpha, \beta\}$ is the basis of 1-invariant forms on the 2-dimensional solvable Lie group $AN = \mathbb{C}H(1)$ dual to the basis $\{A_0, U\}$ of $\mathfrak{a} + \mathfrak{n}$, and we have

Corollary 3.3. All the homogeneous Kähler structures on the complex hyperbolic line (or real hyperbolic plane) $\mathbb{C}H(1) \equiv AN$ are given by $S = \omega \otimes (\alpha \wedge \beta)$, where $\omega$ is a 1-form on $AN$ satisfying $\tilde{\nabla} \omega = 0$.

Remark 3.4. If $S = \omega \otimes (\alpha \wedge \beta)$ is a homogeneous Kähler structure on $\mathbb{C}H(1)$, and $\omega = \lambda \alpha + \mu \beta$, where $\lambda$ and $\mu$ are functions on $\mathbb{C}H(1)$, the condition $\tilde{\nabla} \omega = 0$ together with the structure Eq. $[A_0, U] = 2U$ gives $\lambda = \mu = 0$ or $\lambda^2 + \mu^2 = 4$, and we have that there are infinite homogeneous Kähler structures on $\mathbb{C}H(1)$. However (see [26], Th. 4.4), up to isomorphism, there are only two homogeneous structures on the real hyperbolic plane: one of them is $S = 0$ ($\lambda = \mu = 0$), and the other, which is given by $S_1 X = g(X, Y)\xi_0 - g(\xi_0, Y)X$, with $\xi_0 = 2A_0$ (for $X, Y \in \mathfrak{a} + \mathfrak{n} = \langle A_0, U \rangle$), corresponds to $S = \omega \otimes (\alpha \wedge \beta)$, with $\omega = -2\beta$ ($\lambda = 0$, $\mu = -2$).

Remark 3.5. For each $n > 0$, $S = 0$ is a homogeneous Kähler structure on $\mathbb{C}H(n) \equiv AN$, the corresponding canonical connection is $\tilde{\nabla} = \nabla$, its holonomy algebra is $\mathfrak{k} \cong \mathfrak{su}(n, 1)$, the associated reductive decomposition is the Cartan decomposition $\mathfrak{su}(n, 1) = \mathfrak{k} + \mathfrak{p}$, and it gives the description of $\mathbb{C}H(n)$ as symmetric space $\mathbb{C}H(n) = SU(n, 1)/SU(n) \times U(1))$. 

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Now, our purpose is to obtain nontrivial homogeneous Kähler structures on $\text{CH}(n)$, $n \geq 2$, their associated reductive decompositions, and the corresponding descriptions as homogeneous Kähler spaces.

We will seek for solutions for which $\sigma^j = -\gamma^j$, $\tau^j = -\gamma^j$. In this case, we have

$$\nabla \gamma^j = (\beta - \theta^j) \otimes \gamma^j + \sum_{k \neq j} (\psi^{k} \otimes \gamma^k - \theta^k \otimes \gamma^k),$$

$$\nabla \gamma^j = (\theta^j - \beta) \otimes \gamma^j + \sum_{k \neq j} (\theta^k \otimes \gamma^k + \psi^{k} \otimes \gamma^k).$$

(Obviously, the last summands on the right hand-side in each one of the two Eqs. above do not appear if $n = 2$.) By the second and third Eqs. in (3.6), we must have $\omega = -2\beta$, which also satisfies the first Eqs. in (3.6), because $\nabla \beta = (2\beta + \omega) \otimes \alpha - \sum_j (\gamma^j + \tau^j) \otimes \gamma^j + \sum_j (\gamma^j + \sigma^j) \otimes \gamma^j = 0$. If $n = 2$, by Corollary 3.2, we only have to determine $\theta$ such that $\nabla \theta = 0$. If we put $\theta = a \alpha + b \beta + c \gamma + c' \gamma'$, by using also the structure Eqs. of $a + n = \langle A_0, U, Z, Z' \rangle$, we obtain that $c = c' = 0$ and $a$ and $b$ are constant. For $n > 2$ we put $\theta^j = \theta^j = a_j \alpha + b_j \beta$, $\theta^k = c_k \alpha$, $\psi^k = p_k \alpha$, $(k \neq j)$, with $a_j, b_j, c_k, p_k \in \mathbb{R}$. Then, if $\sigma^j = -\gamma^j$, $\tau^j = -\gamma^j$, and $\omega = -2\beta$, Eqs. (3.6) are satisfied if and only if one has

$$p_k (b_k - b_j) = c_k (b_k - b_j) = 0.$$

Consequently, we get

**Proposition 3.6.** For $n > 2$, the space $\text{CH}(n)$ admits the multi-parametric family of homogeneous Kähler structures $S = S^{a_j, b_j, c_k, p_k}$, given in terms of the generators of $a + n$ by the following table.

<table>
<thead>
<tr>
<th>Table I</th>
<th>$A_0$</th>
<th>$U$</th>
<th>$Z_j$</th>
<th>$Z'_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{A_0}$</td>
<td>0</td>
<td>0</td>
<td>$a_j Z'<em>j + \sum</em>{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l)$</td>
<td>$-a_j Z_j + \sum_{l \neq j} (p_{jl} Z'<em>l - c</em>{jl} Z_l)$</td>
</tr>
<tr>
<td>$S_U$</td>
<td>$-2U$</td>
<td>$2A_0$</td>
<td>$b_j Z'_j$</td>
<td>$-b_j Z_j$</td>
</tr>
<tr>
<td>$S_Z_k$</td>
<td>$-Z_k$</td>
<td>$Z'_k$</td>
<td>$\delta_{kj} A_0$</td>
<td>$-\delta_{kj} U$</td>
</tr>
<tr>
<td>$S_{Z'_k}$</td>
<td>$-Z'_k$</td>
<td>$-Z_k$</td>
<td>$\delta_{kj} U$</td>
<td>$\delta_{kj} A_0$</td>
</tr>
</tbody>
</table>

The complex hyperbolic plane $\text{CH}(2)$ admits the two-parametric family of homogeneous Kähler structures $S = S^{a, b}$, given in terms of the generators of $a + n$ by the following table.

<table>
<thead>
<tr>
<th>Table II</th>
<th>$A_0$</th>
<th>$U$</th>
<th>$Z$</th>
<th>$Z'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{A_0}$</td>
<td>0</td>
<td>0</td>
<td>$aZ'$</td>
<td>$-aZ$</td>
</tr>
<tr>
<td>$S_U$</td>
<td>$-2U$</td>
<td>$2A_0$</td>
<td>$bZ'$</td>
<td>$-bZ$</td>
</tr>
<tr>
<td>$S_Z$</td>
<td>$-Z$</td>
<td>$Z'$</td>
<td>$A_0$</td>
<td>$-U$</td>
</tr>
<tr>
<td>$S_{Z'}$</td>
<td>$-Z'$</td>
<td>$-Z$</td>
<td>$U$</td>
<td>$A_0$</td>
</tr>
</tbody>
</table>

If $S = S^{a_j, b_j, c_k, p_k}$, with respect to the basis $\{ A_0, U, Z_j, Z'_j \}$ of $a + n$, the connection $\tilde{\nabla} = \nabla - S$ is given by

$$\tilde{\nabla}_{A_0} Z_j = -a_j Z'_j - \sum_{l \neq j} (p_{jl} Z_l + c_{jl} Z'_l), \quad \tilde{\nabla}_{U} Z_j = (1 - b_j) Z'_j,$$

$$\tilde{\nabla}_{A_0} Z'_j = a_j Z_j - \sum_{l \neq j} (p_{jl} Z'_l - c_{jl} Z_l), \quad \tilde{\nabla}_{U} Z'_j = (b_j - 1) Z_j,$$
with the rest vanishing. Hence, the components of the curvature tensor field are \( \bar{R}_{A_0 U} = -\bar{R}_{Z_k Z_k} = 2 \sum_j (1 - b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma^j) \), and the rest zero.

If \( b_j = 1 \) for all \( j = 1, \ldots, n-1 \), the holonomy algebra of \( \bar{\nabla} \) is trivial and the reductive decompositions associated to the homogeneous Kähler structures given in Proposition 3.6 are given by \( \bar{\mathfrak{g}}^{\lambda_j, c_{kj}, pk_j} = \{0\} + (\mathfrak{a} + \mathfrak{n}) \) with nonvanishing brackets, by (2.3), given by

\[
\begin{align*}
[A_0, U] &= 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \sum_{l \neq j} (p_{jl}Z_l + c_{jl}Z'_l), \\
[A_0, Z'_j] &= -a_j Z_j + Z'_j + \sum_{l \neq j} (p_{jl}Z'_l + c_{jl}Z_l), \quad [Z_j, Z'_j] = -2U.
\end{align*}
\]

(3.7)

On the other hand, the element \( \hat{A}_0 = \lambda_1 C_1 + \cdots + \lambda_{n-1} C_{n-1} + \sum_j (c_{kj}H_{jl} - p_{jl}F_{lj}) + A_0 \) of \( \mathfrak{su}(n, 1) \) generates a subspace \( \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} \) of \( Z_{\mathfrak{g}}(\mathfrak{a}) + \mathfrak{a} \), and the structure Eqs. of the Lie subalgebra \( \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} + \mathfrak{n} \) of \( \mathfrak{su}(n, 1) \) are

\[
\begin{align*}
[\hat{A}_0, U] &= 2U, \quad [\hat{A}_0, Z_j] = Z_j + (3\lambda_j + \sum_{l \neq j} \lambda_l) Z'_j + \sum_{l \neq j} (p_{jl}Z_l + c_{jl}Z'_l), \\
[\hat{A}_0, Z'_j] &= -(3\lambda_j + \sum_{l \neq j} \lambda_l) Z_j + Z'_j + \sum_{l \neq j} (p_{jl}Z'_l + c_{jl}Z_l), \quad [Z_j, Z'_j] = -2U.
\end{align*}
\]

(3.8)

with the rest vanishing. From (3.7) and (3.8), it follows that \( \bar{\mathfrak{g}}^{\lambda_j, c_{kj}, pk_j} \) is isomorphic to \( \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} + \mathfrak{n} \).

Now, for the structure \( S = S^{a_j, b_j, c_{kj}, pk_j} \) in Table I, suppose that \( b_j \neq 1 \) for some \( j = 1, \ldots, n-1 \). Then, \( \rho = \bar{R}_{A_0 U} = -\bar{R}_{Z_k Z_k} = 2 \sum_j (1 - b_j)(Z'_j \otimes \gamma^j - Z_j \otimes \gamma^j) \) generates the holonomy algebra \( \bar{\mathfrak{h}}^{a_j, b_j, c_{kj}, pk_j} \) of \( \bar{\nabla} = \nabla - S \), and the reductive decomposition associated to \( S \) is \( \bar{\mathfrak{g}}^{a_j, b_j, c_{kj}, pk_j} = \bar{\mathfrak{h}}^{a_j, b_j, c_{kj}, pk_j} + (\mathfrak{a} + \mathfrak{n}) \) with structure Eqs., by (2.3), given by

\[
\begin{align*}
[\rho, A_0] &= [\rho, U] = 0, \quad [\rho, Z_j] = 2(1 - b_j)Z'_j, \quad [\rho, Z'_j] = 2(b_j - 1)Z_j, \\
[A_0, U] &= \rho + 2U, \quad [A_0, Z_j] = Z_j + a_j Z'_j + \sum_{l \neq j} (p_{jl}Z_l + c_{jl}Z'_l), \\
[A_0, Z'_j] &= -a_j Z_j + Z'_j + \sum_{l \neq j} (p_{jl}Z'_l + c_{jl}Z_l), \\
\end{align*}
\]

(3.9)

If \( \mathfrak{u} \cong \mathfrak{u}(1) \) is the subspace of \( Z_{\mathfrak{k}}(\mathfrak{a}) \) generated by \( C = C_1 + \cdots + C_{n-1} \), it is easy to see that the Lie algebra \( \bar{\mathfrak{g}}^{a_j, b_j, c_{kj}, pk_j} \) is isomorphic to the Lie subalgebra \( \mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} + \mathfrak{n} = (\mathfrak{C}, A_0, U, Z_j, Z'_j) \) of \( \mathfrak{su}(n, 1) \). We deduce

**Theorem 3.7.** Let \( S = S^{a_j, b_j, c_{kj}, pk_j} \) be the homogeneous Kähler structure on \( \mathbb{CH}(n) \), \( n > 2 \), given by Table I, and let \( \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} \) be the subspace of \( Z_{\mathfrak{e}}(\mathfrak{a}) \) generated by

\[
A_0 = \sum_j \lambda_j C_j + \sum_{1 \leq j < l \leq n-1} (c_{jl}H_{jl} - p_{jl}F_{lj}) + A_0, \quad (\lambda_j = \frac{a_j - \sum_{l=1}^{l=n} a_l}{2n+2}),
\]

and \( \mathfrak{u} = (C_1 + \cdots + C_{n-1}) \). If \( b_j = 1 \) for all \( j = 1, \ldots, n-1 \), the corresponding group of isometries is the connected subgroup \( E^{\lambda_j, c_{kj}, pk_j} \) of \( SU(n, 1) \) whose Lie algebra is \( \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} + \mathfrak{n} \). If \( b_j \neq 1 \) for some \( j = 1, \ldots, n-1 \), the corresponding group of isometries is the connected subgroup \( U(1)E^{\lambda_j, c_{kj}, pk_j} \) of \( SU(n, 1) \) whose Lie algebra is \( \mathfrak{u} + \mathfrak{e}^{\lambda_j, c_{kj}, pk_j} + \mathfrak{n} \).
If $S^a,b$ is the homogeneous Kähler structure on the complex hyperbolic plane $\mathbb{CH}(2)$ given by Table II, $e^\lambda = (A_0), \lambda = \lambda C + A_0, (\lambda = a/3)$, and $u = (C)$, then the corresponding group of isometries is (i) the subgroup $E^\lambda N$ of $SU(2,1)$ generated by the Lie subalgebra $e^\lambda + n$ of $su(2,1)$, if $b = 1$; (ii) the subgroup $U(1)E^\lambda N$ of $SU(2,1)$ generated by $u + e^\lambda + n$, if $b \neq 1$.

**Remark 3.8.** Each structure $S^{a_j,b_j,c_{k_j},p_{k_j}}$, with $b_j = 1$ for all $j$, is also characterized by the fact that $\tilde{\text{Remark 3.8}}$. Each structure $S^{a_j,b_j,c_{k_j},p_{k_j}}$ is the canonical connection for the Lie group $E^{\lambda_j,c_{k_j},p_{k_j}} N$, which is the connection for which every left-invariant vector field on $E^{\lambda_j,c_{k_j},p_{k_j}} N$ is parallel. Each one of these groups acts simply transitively on $\mathbb{CH}(n)$ and it provides a description of the corresponding group of isometries is given in Theorem 3.7 defines a homogeneous Sasakian structure $(\bar{\mathbb{M}}, \bar{\mathbb{G}})$ given in Theorem 3.7 defines a homogeneous Sasakian structure $\bar{\mathbb{M}}$ on $\bar{\mathbb{M}}$ which gives a description of $\bar{\mathbb{M}}$ as either the connected subgroup $E^{\lambda_j,c_{k_j},p_{k_j}} N \times \mathbb{R}$ of $SU(n,1) \times \mathbb{R}$ (if $b_j = 1$ for all $j = 1, \ldots, n - 1$), or as the homogeneous space $(U(1)E^{\lambda_j,c_{k_j},p_{k_j}} N \times \mathbb{R})/U(1)$.

On the other hand, from (b) of Proposition 2.5, it follows

**Proposition 3.9.** The bundle space $\bar{\mathbb{M}}$ of the line bundle $\pi: \bar{\mathbb{M}} \to \mathbb{CH}(n)$ admits the family of homogeneous Sasakian structures $\{S^t : t \in \mathbb{R}\}$ given, in terms of the horizontal lifts of the generators of $\mathfrak{su}(n)$ and the fundamental vector field $\xi$, by the following table.

<table>
<thead>
<tr>
<th>Table III</th>
<th>$A^H_0$</th>
<th>$U^H$</th>
<th>$Z^H_\lambda$</th>
<th>$Z^l_H$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^a_{A^H_0}$</td>
<td>0</td>
<td>$-\xi$</td>
<td>0</td>
<td>0</td>
<td>$U^H$</td>
</tr>
<tr>
<td>$S^a_{U^H}$</td>
<td>$\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-A^H$</td>
</tr>
<tr>
<td>$S^a_{Z^l_H}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\delta_{k_j}\xi$</td>
<td>$-Z^l_H$</td>
</tr>
<tr>
<td>$S^a_{Z^H_k}$</td>
<td>0</td>
<td>0</td>
<td>$-\delta_{k_j}\xi$</td>
<td>0</td>
<td>$Z^H_k$</td>
</tr>
<tr>
<td>$S^a_{\xi}$</td>
<td>$tU^H$</td>
<td>$-tA^H$</td>
<td>$-tZ^l_H$</td>
<td>$tZ^H_k$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Remark 3.10.** For each $p \in \bar{\mathbb{M}}$, if $c_{12}(S^t)^p$ is the map from the tangent space $T_p(\bar{\mathbb{M}})$ to its dual given by $c_{12}(S^t)^p(\bar{X}) = \sum_{i=1}^{2n+1} S^t_{e_i, e_j, \bar{X}}$, then $\{e_i\}$ is an orthonormal basis
of $T_p(M)$, then $c_{12}(S^r)_{p}$ vanishes for every $t \in \mathbb{R}$. According to Tricerri-Vanhecke’s classification of homogeneous Riemannian structures in [26], each $S^r$ is of type $T_2 \oplus T_3$. Moreover, if $t = -1$, we have $S_X Y + S_Y X = 0$, then $S^{-1}$ is of type $T_3$, which means that $M$ is a naturally reductive Riemannian space. If $t = 2$, then each cyclic sum $\mathfrak{g} \mathfrak{g}^2 S \mathfrak{g} \mathfrak{g}^2$ vanishes, and hence $M$ is of type $T_2$, which may be also expressed by saying that $M$ is a cotorsionless manifold (see [13]).

We will construct the reductive decomposition $\mathfrak{g}_t = \mathfrak{h}_t + \mathfrak{m}$ associated to each homogeneous Sasakian structure $S^r$, where $\mathfrak{m} = T_o(M)$, with $o \in M$, is generated by $\tilde{A} = (A^H_0)_o$, $\tilde{U} = (U^H)_o$, $\tilde{Z}_j = (Z^H_j)_o$, $\tilde{Z}_j' = (Z_1^H)_o$, $\tilde{\xi} = \xi_o$, $(1 \leq j \leq n - 1)$, and $\mathfrak{h}_t$ is the holonomy algebra of the connection $D^t = D - S^t$. Each connection $D^t$ is given by

<table>
<thead>
<tr>
<th>Table IV</th>
<th>$A^H_0$</th>
<th>$U^H$</th>
<th>$Z^H_j$</th>
<th>$Z^H_j'$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{D}^t_{A_0^H}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{D}^t_{U^H}$</td>
<td>$-2U^H$</td>
<td>$2A^H_0$</td>
<td>$Z^H_j$</td>
<td>$-Z^H_j'$</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{D}^t_{Z^H_j}$</td>
<td>$-Z^H_j$</td>
<td>$Z^H_j'$</td>
<td>$\delta_{kj}A^H_0$</td>
<td>$-\delta_{kj}U^H$</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{D}^t_{Z^H_j'}$</td>
<td>$-Z^H_j'$</td>
<td>$-Z^H_j$</td>
<td>$\delta_{kj}U^H$</td>
<td>$\delta_{kj}A^H_0$</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{D}^t_{\xi}$</td>
<td>$(1-t)U^H$</td>
<td>$(t-1)A^H_0$</td>
<td>$(t-1)Z^H_j$</td>
<td>$(t-1)Z^H_j'$</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $\tilde{D}^t$ be the curvature of $D^t$, and let $\{ \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\gamma}^j, \tilde{\gamma}^j_k, \tilde{\gamma}^j_k' \}$ be the basis dual to the basis $\{ \tilde{A}, \tilde{U}, \tilde{Z}_j, \tilde{Z}_j', \tilde{\xi} \}$ of $\mathfrak{m}$. The holonomy algebra $\mathfrak{h}_t$ of $D^t$ is generated by the curvature operators $\rho_0, \rho_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}$ ($r, j, k = 1, \ldots, n - 1, j < k$), given by

$$
\rho_0 = \tilde{R}^t_{A_0^H} = 2(t - 3)(\tilde{\alpha} \otimes \tilde{U} - \tilde{\beta} \otimes \tilde{A}) + 2(2 - t) \sum_{j=1}^{n-1} (\tilde{\gamma}^j \otimes \tilde{Z}_j' - \tilde{\gamma}^j \otimes \tilde{Z}_j),
$$

$$
\rho_r = \tilde{R}^t_{2Z_r} = 2(2 - t)(\tilde{\alpha} \otimes \tilde{U} - \tilde{\beta} \otimes \tilde{A}) + 2(t - 3)(\tilde{\gamma}^j \otimes \tilde{Z}_j' - \tilde{\gamma}^j \otimes \tilde{Z}_j)
$$

$$
+ 2(t - 2) \sum_{j \neq r} (\tilde{\gamma}^j \otimes \tilde{Z}_j' - \tilde{\gamma}^j \otimes \tilde{Z}_j),
$$

$$
\varphi_r = \tilde{R}^t_{\tilde{A}Z_r} = -\tilde{R}^t_{\tilde{U}Z_r} = -\tilde{\alpha} \otimes \tilde{Z}_r - \tilde{\beta} \otimes \tilde{Z}_r' + \tilde{\gamma}^j \otimes \tilde{Z}_j - \tilde{\gamma}^j \otimes \tilde{Z}_j',
$$

$$
\psi_r = \tilde{R}^t_{\tilde{U}2Z_r} = -\tilde{\alpha} \otimes \tilde{Z}_r - \tilde{\beta} \otimes \tilde{Z}_r' + \tilde{\gamma}^j \otimes \tilde{U} + \tilde{\gamma}^j \otimes \tilde{A},
$$

$$
\sigma_{jk} = \tilde{R}^t_{\tilde{Z}_j2Z_k} = \tilde{R}^t_{\tilde{Z}_k2Z_j} = -\tilde{\gamma}^j \otimes \tilde{Z}_k - \tilde{\gamma}^j \otimes \tilde{Z}_k' + \tilde{\gamma}^k \otimes \tilde{Z}_j + \tilde{\gamma}^k \otimes \tilde{Z}_j',
$$

$$
\tau_{jk} = \tilde{R}^t_{\tilde{Z}_j2Z_k} = \tilde{R}^t_{\tilde{Z}_k2Z_j} = -\tilde{\gamma}^j \otimes \tilde{Z}_k + \tilde{\gamma}^j \otimes \tilde{Z}_k' + \tilde{\gamma}^k \otimes \tilde{Z}_j + \tilde{\gamma}^k \otimes \tilde{Z}_j. 
$$

(If $n = 2$, the operators $\sigma_{jk}$ and $\tau_{jk}$ do not appear, that is $\tilde{h}_t = \langle \rho_0, \rho_r, \varphi_r, \psi_r \rangle$, and if $n = 1$, $\mathfrak{h}_t$ is generated by $\rho_0 = \tilde{R}^t_{A_0} = 2(t - 3)(\tilde{\alpha} \otimes \tilde{U} - \tilde{\beta} \otimes \tilde{A})$.) The Lie structure of $\mathfrak{g}_t = \mathfrak{h}_t + \mathfrak{m}$ is defined by the Eqs. (2.3). If $t \neq (2n + 1)/n$, the subalgebra $\mathfrak{h}_t$ is isomorphic to the Lie algebra $\mathfrak{g} = \mathfrak{su}(n) + \mathfrak{u}(1)$ in $\mathfrak{g}$ in § 3.1, via the map $h: \mathfrak{h}_t \to \mathfrak{t}$ given by $h(\rho_0) = 2U_t$, $h(\rho_r) = -(C_r + U_t)$, $h(\varphi_r) = (Z_{k \ell})_r$, $h(\psi_r) = (Z'_r)_r$, $h(\sigma_{jk}) = F_{jk}$, $h(\tau_{jk}) = -H_{jk}$. If we put $\rho_0 = \frac{1}{3}(\rho_0 - 2\xi)$, $\rho_r = -\frac{1}{2} \rho_0 - \rho_r - \xi$, then $\mathfrak{su}(n, 1) = \langle \rho_0, \rho_r, \varphi_r, \psi_r, \sigma_{jk}, \tau_{jk}, A, U, Z_r, Z'_r \rangle$ is an ideal of $\tilde{h}_t$, and the map $h$ extends to a Lie algebra isomorphism $h: \mathfrak{su}(n, 1) \to \mathfrak{su}(n, 1) = \mathfrak{t} + \mathfrak{p}$, given by $h(\rho_0) = U_t$, $h(\rho_r) = C_r$, $h(\varphi_r) = (Z_r)_t$, $h(\psi_r) = (Z'_r)_t$, $h(\sigma_{jk}) = F_{jk}$, $h(\tau_{jk}) = -H_{jk}$.
\[ \hat{h}(A) = A_0, \hat{h}(U) = U_p, \hat{h}(Z) = (Z_r)_p, \hat{h}(Z') = (Z'_r)_p. \] Moreover, \( \tilde{g}_t \) is the semidirect product of \( \mathfrak{su}(n, 1) \) and the line generated by \( \xi \) under the homomorphism \( \delta_t : (\xi) \to \text{Der}(\mathfrak{su}(n, 1)) \), given by \( \delta_t(\xi)(A) = (t - 1)A, \delta_t(\xi)(U) = (1 - t)A, \delta_t(\xi)(Z) = (1 - t)Z_r, \delta_t(\xi)(Z'_r) = (1 - t)Z'_r, \) and \( \delta_t(\xi)((\rho_0, \rho_r, \varphi, \psi_r, \sigma_{jk}, \tau_{jk})) = 0. \) So, we have

**Proposition 3.11.** The reductive decomposition associated to the homogeneous Sasakian structure \( S^t, t \neq (2n + 1)/n \), on the total space of the line bundle \( M \to \mathbb{C}H(n) \) is \( \tilde{g}_t = h_t + m, \) where \( h_t \cong \mathfrak{su}(n) + \mathfrak{u}(1) \cong \mathfrak{su}(n, 1), \) and \( m = p + (\xi) = \langle A_0, U_p, (Z_r)_p, (Z'_r)_p, \xi : 1 \leq r \leq n - 1 \rangle. \) Moreover, \( \tilde{g}_t \) is the semidirect product \( \mathfrak{g}_t = \mathfrak{g}_0 \ltimes \mathfrak{su}(n, 1), \) where \( \delta_t(\xi)(A_0) = (t - 1)U_p, \delta_t(\xi)(U_p) = (1 - t)A_0, \delta_t(\xi)((Z_r)_p) = (1 - t)(Z'_r)_p, \delta_t(\xi)((Z'_r)_p) = (1 - t)(Z'_r)_p, \) and \( \delta_t(\xi)((h_t)) = 0. \)

If \( n \geq 2 \) and \( t = (2n + 1)/n, \) then it is easy to see that \( \rho_0 = \rho_1 + \cdots + \rho_{n - 1}, \) and we put \( \rho_r = \frac{1}{t} (\rho_0 + \rho_r), 1 \leq r < n - 1. \) In this case, \( \tilde{g}_{2n+1} = h_{2n+1} + m \) coincides with the reductive decomposition \( \mathfrak{su}(n, 1) = \mathfrak{t}' + \mathfrak{m}', \) where \( \mathfrak{t}' = [\mathfrak{t}, \mathfrak{t}] \cong \mathfrak{su}(n), \) and \( \mathfrak{m}' = \mathfrak{p} + (c), \) being \( c \) the center of \( \mathfrak{t}, \) which is generated by the element \( E_J \) such that \( \text{ad}_{E_J} : \mathfrak{p} \to \mathfrak{p} \) defines the complex structure of \( \mathbb{C}H(n). \) In fact, we have the isomorphism \( f : \tilde{g}_{2n+1} \to \mathfrak{su}(n, 1) \) given by \( f(\rho_r) = \frac{1}{t} (U_r - C_r), f(\varphi_r) = (Z_r)_p, f(\psi_r) = (Z'_r)_p, f(\sigma_{rk}) = F_{rk}, f(\tau_{rk}) = -H_{rk}, f(A_0) = (A_0, U_p, f(\tilde{Z}_r) = (Z'_r)_p, \) and \( f(\xi) = -\frac{n + 1}{m} E_J = -\frac{1}{2n}(C_1 + \cdots + C_{n - 1} + (n + 1)U) \), and, in particular, \( f(\tilde{h}_{2n+1}) = \mathfrak{t}' \) and \( f(m) = \mathfrak{m}'. \) If \( n = 1 \) and \( t = 3, \) then \( \rho_0 = 0. \) In this case, \( \tilde{h}_3 = 0, \mathfrak{t}' = [\mathfrak{t}, \mathfrak{t}] = 0, \) \( c = \langle E_J \rangle, E_J = \frac{1}{2} U_t, \tilde{h}_3 = \{0\} + m \) is the reductive decomposition \( \mathfrak{su}(1, 1) = \{0\} + \mathfrak{m}', \) where \( m = \langle A, U, \xi \rangle, m' = \langle A_0, U_0, U_t \rangle, \) and \( f : \tilde{h}_3 \to \mathfrak{su}(1, 1) \) such that \( f(A) = A_0, f(U) = U_0, f(\tilde{E}) = -U_t. \) Hence, we have obtained

**Proposition 3.12.** The reductive decomposition associated to the homogeneous Sasakian structure \( S^t, \) with \( t = (2n + 1)/n, \) on the total space of the line bundle \( M \to \mathbb{C}H(n) \) is \( \mathfrak{su}(n, 1) = \mathfrak{t}' + \mathfrak{m}', \) where \( \mathfrak{t}' = [\mathfrak{t}, \mathfrak{t}] \cong \mathfrak{su}(n), \) and \( \mathfrak{m}' = \mathfrak{p} + c, c = \langle E_J \rangle \) being the center of \( \mathfrak{t}. \)

**Remark 3.13.** The reductive decomposition \( \mathfrak{su}(n, 1) = \mathfrak{t}' + \mathfrak{m}' \) associated to the homogeneous Sasakian structure \( S^t, \) with \( t = (2n + 1)/n, \) provides the description of \( M \) as the homogeneous space \( \widetilde{SU}(n, 1)/K', \) where \( \widetilde{SU}(n, 1) \) is the universal covering of \( SU(n, 1), \) and \( K' \cong SU(n) \) is the connected subgroup of \( \widetilde{SU}(n, 1) \) whose Lie algebra is \( \mathfrak{t}' \cong \mathfrak{su}(n). \) (In particular, if \( n = 1, \widetilde{M} \) is the universal covering space of \( SU(2, \mathbb{R}). \)) These spaces appear in the classification by Jiménez and Kowalski [16] of complete simply connected \( \varphi \)-symmetric Sasakian manifolds, and they are also Sasakian space forms (they have constant \( \varphi \)-sectional curvature \( -7). \) Notice that for a Sasakian manifold, the condition of being a locally symmetric space is too strong, because in this case it is a space of constant curvature (Okumura [23]). For this reason, Takahashi [25] introduced \( \varphi \)-symmetric spaces in Sasakian geometry as generalizations of Sasakian space forms. They are also analogues of Hermitian symmetric spaces. A \( \varphi \)-symmetric space is a complete connected regular Sasakian manifold \( \widetilde{M} \) that fibers over a Hermitian symmetric space \( M \) so that the geodesic involutions of \( M \) lift to involutive automorphisms of the Sasakian structure on \( \widetilde{M}. \) Moreover, each complete simply connected \( \varphi \)-symmetric space is a naturally reductive homogeneous space (Blair and Vanhecke [5]).
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References


