Sequencing Games without Initial Order*

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July 2004

Abstract: In this note we study uncertainty sequencing situations, *i.e.*, 1-machine sequencing situations in which no initial order is specified. We associate cooperative games with these sequencing situations, study their core, and provide links with the classic sequencing games introduced by Curiel *et al.* (1989). Moreover, we propose and characterize two simple cost allocation rules for uncertainty sequencing situations with equal processing times.

JEL classification: C71, C78.
Keywords: Sequencing, Cooperative games.

1 Introduction

In operations research, sequencing situations are characterized by a finite number of jobs lined up in front of one or more machines that have to be processed on the machines. A single decision maker wants to determine a processing order of the jobs that minimizes total costs. This single decision maker problem can be transformed into a multiple decision makers problem by taking into account agents that own (at least) one job. In such a model, a group of agents (coalition) can save costs by cooperation.

Cooperative game theory has turned out to be a useful tool for the study of cooperation in sequencing situations (*cf.* Curiel *et al.*, 2002). Curiel *et al.* (1989) started this line of research. The sequencing situations they dealt with consist of a set of agents who each have one job to be processed on a single machine. Moreover, they assumed the existence of an initial order, *i.e.*, an order that is established before the processing takes place. Next, they associated with each sequencing situation a sequencing game, a cooperative transferable utility (TU) game in which the worth of a coalition equals the maximal cost savings the coalition can obtain by reordering their positions according to admissible rearrangements. Apart from studying the properties of the games, Curiel *et al.* (1989) introduced the equal gain splitting rule, an allocation rule that assigns to each sequencing game a particular core allocation.

In many sequencing situations, however, there is no (clear) initial order because the arrival pattern can be stochastic or in batches instead of deterministic and individual. An illustrating example is the short period of time in the morning in which cars arrive at a service station for reparation. The order in which the cars are delivered does not impose any direct condition

*F. Klijn's research is supported by a Ramón y Cajal contract of the Spanish *Ministerio de Ciencia y Tecnología*. His work is also partially supported by Research Grant BEC2002-02130 from the Spanish *Ministerio de Ciencia y Tecnología* and by the Barcelona Economics Program of CREA. E. Sánchez acknowledges financial support of the Spanish *Ministerio de Ciencia y Tecnología* and FEDER through project BEC2002-04102-C02-02.

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on the work scheme for the day, nor can any customer claim a particular time slot. Another example is the arrival of travellers at a passport control of an airport. Even if passengers arrive sequentially, the rules of the airport could require to change the order based on the first-come-first-serve principle, using other criteria such as nationality, departure time, final destination, *etc.*

In this note, we study sequencing situations as in Curiel *et al.* (1989), but with the difference that now no initial order is specified. In Section 2, we recall some well-known concepts on cooperative games. In Section 3, we present uncertainty sequencing situations and associate with them two natural classes of TU games. In Section 4, we show that both games are balanced and provide relations to the classic sequencing games. Finally, in Section 5, we introduce and characterize two intuitive cost allocation rules for uncertainty sequencing situations with equal processing times.

2 Preliminaries on cooperative game theory

A cooperative TU cost game (or shortly, cost game) is a pair (N, c) where $N = \{1, ..., n\}$ is a finite set of agents and $c : 2^N \to \mathbb{R}$ is a map assigning to each coalition $S \in 2^N$, a real number $c(S)$ that represents the minimum costs that the agents of S can guarantee by themselves independently of the agents of $N \backslash S$, where $c(\emptyset) = 0$.

Let (N, c) be a cost game. The game (N, c) is concave if for all $i \in N$ and all $S \subset T \subseteq N \backslash i$, $c(T \cup i) - c(T) \leq c(S \cup i) - c(S)$.[1] The core of (N, c) consists of all vectors in $\mathbb{R}^N$ that distribute the costs $c(N)$ among the players in N in such a way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. Formally, the core is defined by $C(N, c) = \{x \in \mathbb{R}^N : x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subset N\}$.[2] Games with a non-empty core are called balanced games. Each concave game is balanced, but not every balanced game is concave.

For $S \subseteq N$, we denote by $\Pi(S)$ the set of orders of S, *i.e.*, bijective functions from S to $\{1, ..., s\}$, where $s = |S|$ is the cardinality of S. A generic order of S is denoted by $\sigma_S \in \Pi(S)$. For $i \in S$ and $\sigma_S \in \Pi(S)$, let $P(\sigma_S, i) = \{j \in S : \sigma_S(j) < \sigma_S(i)\}$ and $F(\sigma_S, i) = \{j \in S : \sigma_S(j) > \sigma_S(i)\}$ be the set of predecessors and followers of i with respect to σ_S, respectively. Finally, for $\sigma \in \Pi(N)$, $\sigma = (\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$, the reversed order σ^{-1} is given by $\sigma^{-1} = (\sigma^{-1}(n), \ldots, \sigma^{-1}(1))$.

The Shapley (1953) value of a game (N, c) is defined as the average of all marginal vectors, *i.e.*,

$$Sh(N, c) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(c),$$

where the i-th coordinate of the marginal vector $m^\sigma(c)$ is given by $m_i^\sigma(c) = c(P(\sigma, i) \cup i) - c(P(\sigma, i))$. We will denote by $ext(C(N, c))$ the set of extreme points of $C(N, c)$. If (N, c) is a concave game, then the marginal vectors $m^\sigma(c)$ are the extreme points of $C(N, c)$. Hence, for a concave game (N, c), $C(N, c) = conv\{m^\sigma(c) : \sigma \in \Pi(N)\}$ and the Shapley value is the center of gravity of the extreme points of the core taking into account multiplicities.[3]

[1] $S \subseteq N$ ($S \subset N$) denotes that S is a subset (proper subset) of N. Also, we abbreviate $\{i\}$ by i.

[2] For any $x = (x_i)_{i \in N} \in \mathbb{R}^N$ and $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$.

[3] Given a set $A \subseteq \mathbb{R}^N$, we denote by $conv(A)$ its convex hull.

3 Sequencing situations and games

A sequencing situation consists of a triple (N, p, α) and possibly some (information on the) initial order. The triple (N, p, α) describes a finite set $N = \{1, ..., n\}$ of agents, each one of them owning one job that has to be processed on a machine. With a slight abuse of notation we denote agent i's job by i. The processing times of the jobs are given by $p = (p_i)_{i \in N}$ with $p_i > 0$ for all $i \in N$. Each agent $i \in N$ has a cost function $c_i : [0, \infty) \to \mathbb{R}$ given by $c_i(t) = \alpha_i t$, $t \in [0, \infty)$, where $\alpha_i > 0$. The expression $c_i(t)$ is interpreted as the cost incurred by agent i if his job is completed at time t.

Concerning the information on the initial order, we distinguish between the following two classes of sequencing situations:

(i.s.s.) An *initial order sequencing situation* (*cf.* Curiel *et al.*, 1989) is given by a quadruple (N, p, α, σ_N) where $\sigma_N \in \Pi(N)$ is the initial order of the jobs. For $i \in N$, agent i's job is initially at position $\sigma_N(i)$.

(u.s.s.) An *uncertainty sequencing situation* is given by the triple (N, p, α).

In an initial order sequencing situation there is an initial order before the processing of the machine starts. In an uncertainty sequencing situation there is no information whatsoever on the initial order. In both frameworks, we obtain the optimal order of the jobs by ordering them in non-decreasing order of their urgency indices, defined for $i \in N$ as $u_i = \frac{\alpha_i}{p_i}$ (Smith, 1956).

A subsequent problem is how to allocate the minimal total costs among the agents. This problem can be tackled by cooperative game theory. In the classic framework of an i.s.s., a (cooperative) sequencing game is defined by assigning to each coalition the minimal costs the coalition can guarantee by reordering their positions according to admissible rearrangements.[4] Given an initial order $\sigma_N \in \Pi(N)$, we call $\sigma \in \Pi(N)$ an admissible order or rearrangement for S if the players of S do not 'jump' over players outside S. Formally, $P(\sigma, i) = P(\sigma_N, i)$ for all $i \in N \backslash S$. The set of all admissible rearrangements for a coalition S is denoted by $\Sigma_S(\sigma_N) \subseteq \Pi(N)$. For $\sigma \in \Sigma_S(\sigma_N)$, let $c(S, \sigma)$ be the aggregate costs of coalition S in the order σ, *i.e.*, $c(S, \sigma) = \sum_{i \in S} \alpha_i (p_i + \sum_{j \in P(\sigma, i)} p_j)$. Hence, formally,

- *The classic sequencing cost game* (N, c_{σ_N}) (*cf.* Curiel *et al.*, 1989):

  For an i.s.s. (N, p, α, σ_N), the cost game (N, c_{σ_N}) is defined by

$$c_{\sigma_N}(S) = \min_{\sigma \in \Sigma_S(\sigma_N)} c(S, \sigma) \quad \text{for all } S \subseteq N.$$

An order $\hat{\sigma} \in \Sigma_S(\sigma_N) \subseteq \Pi(N)$ with $c(S, \hat{\sigma}) = c_{\sigma_N}(S)$ is called an optimal order or rearrangement for coalition S. An optimal order for N is simply called an optimal order. Curiel *et al.* (1989) proved that classic sequencing games are concave, and hence have a non-empty core.

[4] In Curiel *et al.* (1989) the game is defined in terms of 'cost savings.' Note, however, that the notion of cost savings in sequencing implies the existence of an initial order. Since we want to study sequencing situations without initial order we first have to translate the classic framework of 'cost savings' into one of 'costs.'

For a sequencing situation without initial order, there are (at least) two natural ways to proceed. We illustrate this by means of an example.

Example 3.1 Let (N, p, α) be an u.s.s. given by $N = \{1, 2, 3\}$, $p = (1, 1, 1)$, and $\alpha = (7, 3, 1)$.

A first approach to construct a game follows from assuming that no coalition minds being processed first. Hence, a 'bad' case scenario would be to be processed last, *i.e.*, the coalition assumes it forms the tail of some 'artificial' initial order. For instance, for coalition $\{1, 3\}$ this means that the artificial order is $(2, 1, 3)$ or $(2, 3, 1)$. In both cases, since players 1 and 3 can cooperate we obtain a further reduction to the order $(2, 1, 3)$ with minimal costs $c_{tail}(\{1, 3\}) = 17$.

A second approach considers the worst case (pessimistic) scenario, *i.e.*, the order with high 'initial' costs and few cooperation possibilities so that final minimal costs are maximal. For instance, for coalition $\{1, 3\}$ this is the order $(3, 2, 1)$. Since in this order players 1 and 3 cannot cooperate (they cannot jump over player 2), the minimal costs are $c_{pes}(\{1, 3\}) = 22$. $\diamond$

We formalize the tail and pessimistic approach in the following definitions.

- *The tail game c_{tail} and the pessimistic game c_{pes}.* Given an u.s.s. (N, p, α), we define

$$c_{tail}(S) = \min_{\sigma_S \in \Pi(S)} [\sum_{k \in S} \alpha_k (\textstyle\sum_{l \in (N \backslash S) \cup P(\sigma_S, k) \cup k} p_l)] \quad \text{for all } S \subseteq N, \qquad \text{and}$$

$$c_{pes}(S) = \max_{\sigma_N \in \Pi(N)} [c_{\sigma_N}(S)] \quad \text{for all } S \subseteq N.$$

An order $\hat{\sigma}_S \in \Pi(S)$ with $\sum_{k \in S} \alpha_k (\sum_{l \in (N \backslash S) \cup P(\hat{\sigma}_S, k) \cup k} p_l) = c_{tail}(S)$ is called optimal for S. Note that the order of the members of $N \backslash S$ is irrelevant for the value of coalition S in c_{tail}.

Remark 3.2 Although the definition of c_{tail} is more involved than that of c_{pes}, the computation of the game c_{tail} is almost direct, while for the computation of the game c_{pes} we need to find the worst order for each coalition, which in general is a cumbersome task.

Throughout our analysis the triple (N, p, α) is fixed. The next proposition provides some first elementary relations between the games introduced above. We omit its proof.

Proposition 3.3 *(i) For all $S \subseteq N$, $c_{tail}(S) \leq c_{pes}(S)$, with equality for $S = N$.*
(ii) For all $i \in N$, $c_{tail}(i) = c_{pes}(i) = \alpha_i (\sum_{k \in N} p_k)$.

4 The core

In this section we study the core of the games introduced in the previous section. We first provide an expression for the marginal vectors of c_{tail}.

Lemma 4.1 *For $\sigma \in \Pi(N)$ and $i \in N$, $m_i^\sigma(c_{tail}) = \alpha_i \big(p_i + \sum_{l \in F(\sigma, i)} p_l\big) + \sum_{k \in P(\sigma, i): u_k > u_i} (p_k \alpha_i - p_i \alpha_k)$.*

Proof. We first prove that for all $S \subseteq N$ and all $i \in S$,

$$c_{tail}(S) - c_{tail}(S \backslash i) = \alpha_i \big(p_i + \sum_{l \in N \backslash S} p_l\big) + \sum_{k \in S: u_k > u_i} (p_k \alpha_i - p_i \alpha_k). \tag{1}$$

The lemma then follows by taking $S = P(\sigma_N, i) \cup i$.

To see (1) it is sufficient to note that there are optimal orders $\hat{\sigma}_{S\backslash i} \in \Pi(S\backslash i)$ and $\hat{\sigma}_S \in \Pi(S)$ of $S\backslash i$ and S, respectively, with

$$P(\hat{\sigma}_S, k) = \begin{cases} P(\hat{\sigma}_{S\backslash i}, k) \cup i & \text{if } u_k \leq u_i; \\ P(\hat{\sigma}_{S\backslash i}, k) & \text{if } u_k > u_i. \end{cases}$$

for all $k \in S\backslash i$. $\square$

Curiel *et al.* (1989) showed that the classic sequencing games are concave. In the following proposition we show that tail games are concave as well.

Proposition 4.2 *The game (N, c_{tail}) is concave.*

Proof. Let $S \subset T \subseteq N\backslash i$. From (1) in the proof of Lemma 4.1 it follows that for $U = S, T$,

$$c_{tail}(U) - c_{tail}(U\backslash i) = \alpha_i\big(p_i + \sum_{l \in N\backslash U} p_l\big) + \sum_{k \in U: u_k > u_i} (p_k\alpha_i - p_i\alpha_k).$$

As $\sum_{l \in N\backslash S} p_l \geq \sum_{l \in N\backslash T} p_l$ and $\{k \in T : u_k > u_i\} \supseteq \{k \in S : u_k > u_i\}$ we find $c_{tail}(S) - c_{tail}(S\backslash i) \geq c_{tail}(T) - c_{tail}(T\backslash i)$. Hence, the game (N, c_{tail}) is concave. $\square$

Now, the balancedness of both the tail as well as the pessimistic games follows readily.

Proposition 4.3 *The games (N, c_{tail}) and (N, c_{pes}) are balanced. In fact, $\emptyset \neq C(N, c_{tail}) \subseteq C(N, c_{pes})$.*

Proof. By Proposition 4.2 the game (N, c_{tail}) is concave, and hence balanced, *i.e.*, $C(N, c_{tail}) \neq \emptyset$. By Proposition 3.3 (i), $C(N, c_{tail}) \subseteq C(N, c_{pes})$, and hence the game (N, c_{pes}) is balanced. $\square$

Given that the tail game is not 'hard' to calculate, we can provide some further results for this game. We first exhibit a relation between the core of the tail game and a marginal vector of all corresponding classic sequencing games. Next, we provide a simple expression of the Shapley value of the tail game in terms of any pair of 'reversed' marginal vectors.

Proposition 4.4 *(i) $C(N, c_{tail}) = conv\{m^{\sigma^{-1}}(N, c_\sigma) : \sigma \in \Pi(N)\}$.*
(ii) For any $\sigma \in \Pi(N)$, $Sh(N, c_{tail}) = \frac{1}{2}[m^\sigma(c_{tail}) + m^{\sigma^{-1}}(c_{tail})]$.

Proof. (i) Let $x \in ext(C(N, c_{tail}))$. Since (N, c_{tail}) is a concave game, there is an order $\tau \in \Pi(N)$, $\tau = (\tau^{-1}(1), ..., \tau^{-1}(n))$, such that $x = m^\tau(c_{tail})$. We will show that $x = m^\tau(c_\sigma)$, where $\sigma = (\tau^{-1}(n), ..., \tau^{-1}(1))$. Let $i \in N$ and let $p \in \{1, \ldots, n\}$ be such that $i = \tau^{-1}(p)$. Then,

$$\begin{aligned} x_i &= m_i^\tau(c_{tail}) \\ &= c_{tail}(\{\tau^{-1}(1), ..., \tau^{-1}(p)\}) - c_{tail}(\{\tau^{-1}(1), ..., \tau^{-1}(p-1)\}) \\ &= c_\sigma(\{\tau^{-1}(1), ..., \tau^{-1}(p)\}) - c_\sigma(\{\tau^{-1}(1), ..., \tau^{-1}(p-1)\}) \\ &= m_i^\tau(c_\sigma). \end{aligned}$$

Since the game (N, c_{tail}) is concave, the result follows.

(ii) Applying Lemma 4.1 to $\sigma \in \Pi(N)$ and $i \in N$ gives

$$m_i^{\sigma}(c_{tail}) + m_i^{\sigma^{-1}}(c_{tail}) = \alpha_i\big(\sum_{l\in N} p_l + p_i\big) + \sum_{k\in N: u_k > u_i} (p_k\alpha_i - p_i\alpha_k).$$

Since the right hand side of the expression above does not depend on σ,

$$Sh(N, c_{tail}) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^{\sigma}(N, c_{tail}) = \frac{1}{2}[m^{\sigma}(c_{tail}) + m^{\sigma^{-1}}(c_{tail})].$$

□

Finally, we show how the core of the tail game and the pessimistic game are related to the core of the classic sequencing game.

Proposition 4.5 $C(N, c_{tail}) \subseteq conv\left\{ \bigcup_{\sigma\in\Pi(N)} C(N, c_{\sigma}) \right\} \subseteq C(N, c_{pes})$.

Proof. The first inclusion is a a direct consequence of Proposition 4.4 (i). We prove the second inclusion. By definition of c_{pes}, $c_{\sigma}(S) \leq c_{pes}(S)$ for all $\sigma \in \Pi(N)$ and all coalitions $S \subseteq N$ (with equality for $S = N$). Hence, $C(N, c_{\sigma}) \subseteq C(N, c_{pes})$ for all $\sigma \in \Pi(N)$. The result now follows from the convexity of $C(N, c_{pes})$. □

5 Cost allocation rules

Having described the relations between the cores of the games, we now turn to the problem of finding (intuitive) cost allocation rules for the class of uncertainty sequencing situations. Let $\mathcal{C}$ be the class of all uncertainty sequencing situations with *equal processing times*. Without loss of generality we may assume that for any problem in $\mathcal{C}$ all jobs have processing time 1. A (cost allocation) rule on $\mathcal{C}$ is a map φ that assigns to each $(N, \alpha) \in \mathcal{C}$ a vector $\varphi(N, \alpha) \in \mathbb{R}^N$. Next, we introduce two natural rules on $\mathcal{C}$. For $(N, \alpha) \in \mathcal{C}$ let $\Omega(N, \alpha)$ denote the set of optimal orders, *i.e.*, $\Omega(N, \alpha) = \{\sigma \in \Pi(N) : \alpha_{\sigma^{-1}(1)} \geq ... \geq \alpha_{\sigma^{-1}(n)}\}$.

The proportional rule PRO. For any uncertainty sequencing situation $(N, \alpha) \in \mathcal{C}$ we define

$$PRO(N, \alpha) = \left(\frac{\alpha_i}{\sum_{k\in N} \alpha_k} c(N, \hat{\sigma}) \right)_{i\in N},$$

where $\hat{\sigma} \in \Omega(N, \alpha)$.

The cost splitting rule according to optimal orders ψ. For any uncertainty sequencing situation $(N, \alpha) \in \mathcal{C}$ we define

$$\psi(N, \alpha) = \left(\frac{1}{|\Omega(N, \alpha)|} \sum_{\hat{\sigma}\in\Omega(N,\alpha)} \alpha_i \hat{\sigma}(i) \right)_{i\in N}.$$

Remark 5.1 In the special case of $\alpha_i = \delta$ for all $i \in N$ for some constant $\delta > 0$, $PRO_i(N,\alpha) = \psi_i(N,\alpha) = Sh_i(N, c_{tail}) = \frac{\delta}{2}(n+1)$ for all $i \in N$.

As the following two propositions show, both rules introduced above are 'fair' in the sense that they always yield a cost allocation that is in the core of the corresponding tail game.

Proposition 5.2 *Let (N, c_{tail}) be the tail game associated with $(N,\alpha) \in \mathcal{C}$. Then, $PRO(N,\alpha) \in C(N, c_{tail})$.*

Proof. Denote $x = PRO(N,\alpha)$. Let $S \subseteq N$, $S = \{i_1, ..., i_s\}$ with $\alpha_{i_1} \geq ... \geq \alpha_{i_s}$. Then,

$$\begin{aligned}
x(S) &= \sum_{i \in S} \frac{\alpha_i}{\sum\limits_{k \in N} \alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) \\
&= \frac{\alpha_{i_1}}{\sum\limits_{k \in N} \alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) + \\
&\quad \frac{\alpha_{i_2}}{\sum\limits_{k \in N} \alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n) + \\
&\quad \vdots \\
&\quad \frac{\alpha_{i_s}}{\sum\limits_{k \in N} \alpha_k}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n)
\end{aligned} \tag{2}$$

Let $X_S = (x_{kl})$ be the $s \times n$-matrix defined by

$$x_{kl} := \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} l\alpha_l \quad \text{for all } k = 1, ..., s \text{ and } l = 1, ..., n.$$

Note that $x(S) = \sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} x_{kl}$.

The proof proceeds now as follows. We construct an $s \times n$-matrix matrix $C_S = (c_{kl})$ such that $\sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} c_{kl} = c_{tail}(S)$, and show that $\sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} x_{kl} \leq \sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} c_{kl}$, which will complete the proof.

First we define the sets U and V of matrix coordinates as follows:

$$\begin{aligned}
U &: = \{(l,m) \in \{1, ..., s\} \times \{1, ..., n\} : m > (n-s) + l\}, \\
V &: = \{(x,y) \in \{1, ..., s\} \times \{1, ..., n\} : y \leq x + (n-s-1)\}.
\end{aligned}$$

Note that $U \cap V = \emptyset$. Let $f : U \to V$ be the map defined by $f(l,m) := (m - (n-s), l)$ for all $(l,m) \in U$. It can be easily shown that f is injective.

Then, the matrix C_S is defined as follows

$$c_{kl} := \begin{cases} \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} l\alpha_l - \frac{\alpha_{i_k}\alpha_l}{\sum\limits_{k \in N} \alpha_k}(s - k - (n-l)) & \text{if } (k,l) \in U; \\ \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} l\alpha_l + \frac{\alpha_{i_k}\alpha_l}{\sum\limits_{k \in N} \alpha_k}(k - l + (n-s)) & \text{if } (k,l) \in V; \\ \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} l\alpha_l & \text{otherwise.} \end{cases} \tag{3}$$

It can be easily shown that $c_{kl} = \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} \alpha_l (n - s + k)$ for all pairs $(k, l) \in \{1, ..., s\} \times \{1, ..., n\}$. (For $(k, l) \notin U \cup V$, take into account that $n - s + k = l$.) This implies that $\sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} c_{kl} = \sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} \frac{\alpha_{i_k}}{\sum\limits_{k \in N} \alpha_k} \alpha_l (n - s + k) = \sum\limits_{k=1}^{s} \alpha_{i_k} (n - s + k) = c_{tail}(S)$, as desired.

It remains to prove that $\sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} x_{kl} \leq \sum\limits_{k=1}^{s} \sum\limits_{l=1}^{n} c_{kl}$. Taking into account (2), (3), and the fact that f is injective, one deduces that we only need to prove that

$$\frac{\alpha_{i_k} \alpha_l}{\sum\limits_{k \in N} \alpha_k} (s - k - (n - l)) \leq \frac{\alpha_{i_{l-(n-s)}} \alpha_k}{\sum\limits_{k \in N} \alpha_k} (l - k) \quad \text{for all } (k, l) \in U.$$

This inequality follows readily since for all $(k, l) \in U$,
(i) $(s - k - (n - l)) \leq l - k$ since $s \leq n$;
(ii) $\alpha_{i_k} \leq \alpha_k$ for all $k = 1, ..., s$;
(iii) $\alpha_l \leq \alpha_{i_{l-(n-s)}}$ since $i_{l-(n-s)} \leq l$ as there are $s - (l - (n - s)) = n - l$ players in S after player $i_{l-(n-s)}$. $\square$

Proposition 5.3 *Let (N, c_{tail}) be the tail game associated with $(N, \alpha) \in \mathcal{C}$. Then, $\psi(N, \alpha) \in C(N, c_{tail})$.*

Proof. Let $\hat{\sigma} \in \Omega(N, \alpha)$. It follows from a result by Curiel *et al.* (1989) that $(\alpha_i \hat{\sigma}(i))_{i \in N} = (c_{\hat{\sigma}}(i))_{i \in N} \in C(N, c_{\hat{\sigma}})$. Moreover, it is easy to check that $C(N, c_{\hat{\sigma}}) \subseteq C(N, c_{tail})$. So, $(\alpha_i \hat{\sigma}(i))_{i \in N} \in C(N, c_{tail})$. Because the core is a convex set, we obtain that also $\psi(N, \alpha) = (\frac{1}{|\Omega(N, \alpha)|} \sum\limits_{\hat{\sigma} \in \Omega(N, \alpha)} \alpha_i \hat{\sigma}(i))_{i \in N} \in C(N, c_{tail})$. $\square$

Corollary 5.4 *Let (N, c_{pes}) be the pessimistic game associated with $(N, \alpha) \in \mathcal{C}$. Then, $PRO(N, \alpha)$, $\psi(N, \alpha) \in C(N, c_{pes})$.*

Apart from providing core allocations, the two cost allocation rules are characterized by the following properties. A rule on $\mathcal{C}$ satisfies

EFF (*efficiency*) if for all $(N, \alpha) \in \mathcal{C}$, $\sum\limits_{i \in N} \varphi_i(N, \alpha) = c(N, \hat{\sigma})$, where $\hat{\sigma} \in \Omega(N, \alpha)$.
ETE (*equal treatment of equals*) if for all $(N, \alpha) \in \mathcal{C}$ and $i, j \in N$ with $\alpha_i = \alpha_j$, $\varphi_i(N, \alpha) = \varphi_j(N, \alpha)$.
URG (*urgency*) if for all $(N, \alpha) \in \mathcal{C}$ and all $i, j \in N$, $i \neq j$ with $\alpha_i > \alpha_j$, $\varphi_i(N \backslash \{j\}, \alpha_{|N \backslash \{j\}}) = \varphi_i(N, \alpha)$.
PROP (*proportionality*) if for all $(N, \alpha) \in \mathcal{C}$ and all $i, j \in N$, $\frac{\varphi_i(N, \alpha)}{\varphi_j(N, \alpha)} = \frac{\alpha_i}{\alpha_j}$.

Proposition 5.5 *(i) PRO is the unique rule on $\mathcal{C}$ that satisfies EFF and PROP.*
(ii) ψ is the unique rule on $\mathcal{C}$ that satisfies EFF, ETE, and URG.

Proof. (i) Straightforward.
(ii) It can be easily checked that ψ satisfies EFF, ETE, and URG. We prove the uniqueness. Let φ be a rule satisfying EFF, ETE, and URG. Let $(N,\alpha) \in \mathcal{C}$. Without loss of generality, we assume that $\alpha_1 \geq \ldots \geq \alpha_n$.

Let $S_1 = \{j \in N : \alpha_j = \alpha_1\}$ and $s_1 = |S_1|$. From EFF, ETE, and URG it follows that for all $j \in S_1$,

$$\varphi_j(N,\alpha) = \varphi_j(S_1, \alpha_{|S_1}) = \frac{\alpha_1}{2}(s_1+1) = \psi_j(S_1, \alpha_{|S_1}) = \psi_j(N,\alpha).$$

If $S_1 = N$, we are done. If $S_1 \neq N$, then let $j_2 \in N$ be such that $\alpha_{j_2} = \max_{j \in N \backslash S_1} \alpha_j$. Let $S_2 = \{j \in N \backslash S_1 : \alpha_j = \alpha_{j_2}\}$ and $s_2 = |S_2|$. Then by URG, for all $j \in S_2$,

$$\varphi_j(N,\alpha) = \varphi_j(S_1 \cup S_2, \alpha_{|S_1 \cup S_2}).$$

Hence, by EFF,

$$\begin{aligned} \sum_{j \in S_2} \varphi_j(N,\alpha) &= \sum_{j \in S_2} \varphi_j(S_1 \cup S_2, \alpha_{|S_1 \cup S_2}) \\ &= \alpha_{j_2}(s_1 + \frac{s_2+1}{2})s_2 \end{aligned}$$

and by ETE, for all $j \in S_2$,

$$\varphi_j(N,\alpha) = \alpha_{j_2}(s_1 + \frac{s_2+1}{2}) = \psi_j(S_1 \cup S_2, \alpha_{|S_1 \cup S_2}) = \psi_j(N,\alpha).$$

If $S_1 \cup S_2 = N$, we are done. If $S_1 \cup S_2 \neq N$ we can repeat the same arguments for $N \backslash (S_1 \cup S_2)$ to obtain $\varphi = \psi$. □

Remark 5.6 The properties in each of the characterizations in Proposition 5.5 are logically independent. As for the first characterization, PRO satisfies EFF and ETE but not URG. (To see this, consider $(N,\alpha) \in \mathcal{C}$ with $N = \{1,2\}$ and $\alpha = (1,2)$. Then $PRO_2(N,\alpha) = \frac{8}{3} \neq 2 = PRO_2(N \backslash \{1\}, \alpha_{|\{2\}})$.) The rule ξ defined by $\xi(N,\alpha) = (\alpha_i \hat{\sigma}(i))_{i \in N}$ where $\hat{\sigma}$ is an arbitrary optimal order, satisfies EFF and URG but not ETE. The rule 2ψ satisfies ETE and URG but not EFF. As for the second characterization, note that $2PRO$ satisfies PROP but not EFF, and that ψ satisfies EFF but not PROP.

6 References

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