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Effros, Baire, Steinhaus and non-separability

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Abstract. We give a short proof of an improved version of the Effros Open Mapping Principle via a shift-compactness theorem (also with a short proof), involving ‘sequential analysis’ rather than separability, deducing it from the Baire property in a general Baire-space setting (rather than under topological completeness). It is applicable to absolutely-analytic normed groups (which include complete metrizable topological groups), and via a Steinhaus-type Sum-set Theorem (also a consequence of the shift-compactness theorem) includes the classical Open Mapping Theorem (separable or otherwise).

Keywords: Open Mapping Theorem, absolutely analytic sets, base-σ-discrete maps, demi-open maps, Baire spaces, Baire property, group-action shift-compactness.

Classification Numbers: 26A03; 04A15; 02K20.

1 Introduction

We generalize a classic theorem of Effros [Eff] beyond its usual separable context. Viewed, despite the separability, as a group-action counterpart of the Open Mapping Theorem OMT (that a surjective continuous linear map between Fréchet spaces is open – cf. [Rud]), it has come to be called the Open Mapping Principle – see [Anc, §1]. Our ‘non-separable’ approach is motivated by a sequential property related to the Steinhaus-type Sum-set Theorem (that 0 is an interior point of $A - A$, for non-meagre $A$ with BP, the Baire property – [Pic]), because of the following argument (which goes back to Pettis [Pe]).

Consider $L : E \to F$, a linear, continuous surjection between Fréchet spaces, and $U$ a neighbourhood (nhd) of the origin. Choose $A$ an open nhd of the origin with $A - A \subseteq U$; as $L(A)$ is non-meagre (since $\{nL(A) : n \in \mathbb{N}\}$ covers $F$) and has BP (see Proposition 2 in §2.3), $L(A) - L(A)$ is a nhd of the origin by the Sum-set Theorem. But of course

$$L(U) \supseteq L(A) - L(A),$$
so \( L(U) \) is a nhd of the origin. So \( L \) is an open mapping.\(^1\)

Throughout this paper, without further comment, all spaces considered will be metrizable, but not necessarily separable. We recall the Birkhoff-Kakutani theorem (cf. [HewR, §II.8.3]), that a metrizable group \( G \) with neutral element \( e_G \) has a right-invariant metric \( d^G_R \). Passage to \( \|g\| := d^G_R(g, e_G) \) yields a (group) norm (invariant under inversion, satisfying the triangle inequality), which justifies calling these normed groups; any Fréchet space qua additive group, equipped with an F-norm ([KalPR, Ch. 1 §2]), is a natural example (cf. Auth in §2.2). Recall that a Baire space is one in which Baire’s theorem holds – see [AaL]. Below we need the following.

**Definitions 1** (cf. [Pe]). For \( G \) a metrizable group, say that \( \varphi : G \times X \to X \) is a Nikodym group action (or that it has the Nikodym property) if for every non-empty open neighbourhood \( U \) of \( e_G \) and every \( x \in X \) the set \( Ux = \varphi_x(U) := \varphi(x, U) \) contains a non-meagre Baire set. (Here Baire set, as opposed to Baire space as above, means ‘set with the Baire property’.)

2. \( A^q \) denotes the quasi-interior of \( A \) – the largest open set \( U \) with \( U \setminus A \) meagre (cf. [Ost1, §4]); other terms (‘analytic’, ‘base-\( \sigma \)-discrete’, ‘group action’) are recalled later.

Concerning when the above property holds see §2.3. Our main results are Theorems S and E below, with Corollaries in §2.3 including OMT; see below for commentary.

**Theorem S (Shift-compactness Theorem).** For \( T \) a Baire non-meagre subset of a metric space \( X \) and \( G \) a group, Baire under a right-invariant metric, and with separately continuous and transitive Nikodym action on \( X \):

for every convergent sequence \( x_n \) with limit \( x \) and any Baire non-meagre \( A \subseteq G \) with \( e_G \in A^q \) and \( A^q x \cap T^q \neq \emptyset \), there are \( \alpha \in A \) and an integer \( N \) such that \( \alpha x \in T \) and

\[
\{\alpha(x_n) : n > N\} \subseteq T.
\]

In particular, this is so if \( G \) is analytic and all point-evaluation maps \( \varphi_x \) are base-\( \sigma \)-discrete.

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\(^1\)This proof is presumably well-known – so simple and similar to that for the automatic continuity of homomorphisms – but we have no textbook reference; cf. [KalPR, Cor. 1.5].
This theorem has wide-ranging consequences, including Steinhaus’ Sumset Theorem – see the survey article [Ost4], and the recent [BinO3].

**Theorem E (Effros Theorem – Baire version).** If

(i) the normed group $G$ has separately continuous and transitive Nikodym action on $X$;
(ii) $G$ is Baire under the norm topology and $X$ is non-meagre

then for any open neighbourhood $U$ of $e_G$ and any $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of $x$, so that in particular the point-evaluation maps $g \to g(x)$ are open for each $x$. That is, the action of $G$ is micro-transitive.

In particular, this holds if $G$ is analytic and Baire, and all point-evaluation maps $\varphi_x$ are base-$\sigma$-discrete.

By Proposition B2 (§2.3) $X$, being non-meagre here, is also a Baire space.

The classical counterpart of Theorem E has $G$ a Polish group; van Mill’s version [vMil1] requires the group $G$ to be analytic (i.e. the continuous image of some Polish space, cf. [JayR], [Kec2]). The Baire version above improves the version given in [Ost3], where the group is almost complete. (The two cited sources taken together cover the literature.)

A result due to Loy [Loy] and to Hofmann-Jørgensen [HofJ, Th. 2.3.6 p. 355] asserts that a Baire, separable, analytic topological group is Polish (as a consequence of an analytic group being metrizable – for which see again [HofJ, Th. 2.3.6]), so in the analytic separable case Theorem E reduces to its classical version.

Unlike the proof of the Effros Theorem attributed to Becker in [Kec1, Th. 3.1], the one offered here does not employ the Kuratowski-Ulam Theorem (the Category version of the Fubini Theorem), a result known to fail beyond the separable context (as shown in [Pol], cf. [vMilP], but see [FreNR]).

For further commentary (connections between convexity and the Baire property, relation to van Mill’s separation property in [vMil2], certain specializations) see the extended version of this paper on arXiv.

2 Analyticity, micro-action, shift-compactness

We recall some definitions from general topology, before turning to ones that are group-related. We refer to [Eng] for general topological usage (but prefer ‘meagre’ to ‘of first category’).
2.1 Analyticity

We say that a subspace $S$ of a metric space $X$ has a *Souslin-$\mathcal{H}$ representation* if there is a determining system $\langle H(i|n) \rangle := \langle H(i|n) : i \in \mathbb{N}^n \rangle$ of sets in $\mathcal{H}$ with ([Rog], [Han2])

$$S = \bigcup_{i \in I} \bigcap_{n \in \mathbb{N}} H(i|n), \quad (I := \mathbb{N}^n, \ i|n := (i_1, ..., i_n)).$$

A topological space is an (absolutely) *analytic* space if it is embeddable as a Souslin-$\mathcal{F}$ set in its own metric completion (with $\mathcal{F}$ the closed sets); in particular, in a complete metric space $\mathcal{G}_c$-subsets (being $\mathcal{F}_c$) are analytic. For more recent generalizations see e.g. [NamP]. According to Nikodym’s theorem, if $\mathcal{H}$ above comprises Baire sets, then also $S$ is Baire (the Baire property is preserved by the Souslin operation): so analytic subspaces are Baire sets. For background – see [Kec2] Th. 21.6 (the Lusin-Sierpiński Theorem) and the closely related Cor. 29.14 (Nikodym Theorem), cf. the treatment in [Kur] Cor. 1 p. 482, or [JayR] pp. 42-43. For the extended Souslin operation of non-separable descriptive theory see also [Ost2]. This motivates our interest in analyticity as a carrier of the Baire property, especially as continuous images of separable analytic sets are separable, hence Baire.

However, the continuous image of an analytic space is not in general analytic – for an example of failure see [Han3] Ex. 3.12. But this does happen when, additionally, the continuous map is base-$\sigma$-discrete, as defined below (*Hansell’s Theorem*, [Han3] Cor. 4.2). This technical condition is the standard assumption for preservation of analyticity and holds automatically in the separable realm. Special cases include *closed surjective* maps and *open-to-analytic injective* maps (taking open sets to analytic sets). To define the key concept just mentioned, recall that for an (indexed) family $B := \{B_t : t \in T\}$:

(i) $B$ is *index-discrete* in the space $X$ (or just *discrete* when the index set $T$ is understood) if every point in $X$ has a nhd meeting the sets $B_t$ for at most one $t \in T$;

(ii) $B$ is *$\sigma$-discrete* if $B = \bigcup_n \mathcal{B}_n$, where each set $\mathcal{B}_n$ is discrete as in (i), and

(iii) $B$ is a *base for $\mathcal{A}$* if every member of $\mathcal{A}$ is the union of a subfamily of $B$.

For $\mathcal{T}$ a topology (the family of all open sets) with $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$, this reduces to $\mathcal{B}$ being simply a (topological) *base*.

**Definitions.** 1. ([Mic1], Def. 2.1) Call $f : X \to Y$ *base-$\sigma$-discrete* (or *co-$\sigma$-discrete*, [Han3, §3]) if the image under $f$ of any discrete family in $X$
2.2 Action, micro-action, shift-compactness

Recall that a normed group $G$ acts continuously on $X$ if there is a continuous mapping $'$ : $G \times X \to X$ such that $'(e, x) = x$ and $'(gh, x) = '(g, '(h, x))$ $(x \in X, g, h \in G).$ The action $'$ is separately continuous if $g : x \mapsto '(g, x)$ is continuous for each $g$, and $' : g \mapsto '(g, x)$ is continuous for each $x$; in such circumstances:

(i) the elements $g \in G$ yield autohomeomorphisms of $X$ via $g : x \mapsto g(x) := '(g, x)$ (as $g^{-1}$ is continuous), and

(ii) point-evaluation of these homeomorphisms, $' : g \mapsto '(g, x)$, is continuous. In certain situations joint continuity of action is implied by separate continuity (see [Bou] and literature cited in [Ost2]).

The action is transitive if for any $x, y$ in $X$ there is $g \in G$ such that $g(x) = y.$ For later purposes ($\S 2.3$ and 3), say that the action of $G$ on $X$ is weakly micro-transitive if for $x \in X$ and each nhd $A$ of $e_G$ the set

$$\text{cl}(Ax) = \text{cl}\{ax : a \in A\}$$

has $x$ as an interior point (in $X$). The action is micro-transitive (‘transitive in the small’ – for details see [vMil1]) if for $x \in X$ and each nhd $A$ of $e_G$ the set

$$Ax = \{ax : a \in A\}$$

is a nhd of $x.$ This (norm) property implies that $Ux$ is open for $U$ open in $G$ (i.e. that here each $' : x$ is an open mapping). We refer to $Ax$ as an $x$ orbit (the $A$-orbit of $x$). The following group action connects the Open Mapping Theorem to the present context.

**Example (Induced homomorphic action).** A surjective, continuous homomorphism $\lambda : G \to H$ between normed groups induces a transitive action of $G$ on $H$ via $\varphi^\lambda(g, h) := \lambda(g)h$ (cf. [Ost2] Th. 5.1), specializing
for $G, H$ Fréchet spaces (regarded as normed, additive groups) and $\lambda = L : G \to H$ linear (Ancel [Anc] and van Mill [vMil1]) to

$$\varphi^L(a, b) := L(a) + b.$$  

Of course for Fréchet spaces, by the Open Mapping Theorem itself, $\varphi^L$ has the Nikodym property.

**Definitions.** 1. $\text{Auth}(X)$ denotes the autohomeomorphisms of a metric space $(X, d^X)$; this is a group under composition. $\mathcal{H}(X)$ comprises those $h \in \text{Auth}(X)$ of bounded norm:

$$||h|| := \sup_{x \in X} d^X(h(x), x) < \infty.$$  

2. For a normed group $G$ acting on $X$, say that $X$ has the **crimping property** (property C for short) w.r.t. $G$ if, for each $x \in X$ and each sequence $\{x_n\} \to x$, there exists in $G$ a sequence $\{g_n\} \to e_G$ with $g_n(x) = x_n$. (This and a variant occurs in [Ban, Ch. III; Th.4]; and [ChCh]; for the term see [BinO2].)

For a subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, say that $X$ has the **crimping property** w.r.t. $\mathcal{G}$ if $X$ has the crimping property w.r.t. to the natural action $(g, x) \mapsto g(x)$ from $\mathcal{G} \times X \to X$. (This action is continuous relative to the left or right norm topology on $\mathcal{G}$ – cf. [Dug] XII.8.3, p. 271.)

3. As a matter of convenience, say that the **Effros property** (or property $E$) holds for the group $G$ acting on $X$ if the action is micro-transitive, as above.

4. For a subgroup $\mathcal{G} \subseteq \text{Auth}(X)$ say that $X$ is $\mathcal{G}$-**shift-compact** (or, shift-compact under $\mathcal{G}$) if for any convergent sequence $x_n \to x_0$, any open subset $U$ in $X$ and any Baire set $T$ co-meagre in $U$, there is $g \in \mathcal{G}$ with $g(x_n) \in T \cap U$ along a subsequence. Call the space **shift-compact** if it is $\mathcal{H}(X)$-shift-compact (cf. [MilO], [Ost5]).

In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of Prop. B2 below.

We shall prove in § 3.1 equivalence between the Effros and Crimping properties:

**Theorem EC.** The Effros property holds for a group $G$ acting on $X$ iff $X$ has the Crimping property w.r.t. $G$.

We now clarify the role of shift-compactness.

**Proposition B1.** For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if $X$ is $\mathcal{G}$-shift-compact, then $X$ is a Baire space.
Proof. We argue as in [vMil2] Prop 3.1 (1). Suppose otherwise; then
\(X\) contains a non-empty meagre open set. By Banach’s Category Theorem
(or localization principle, for which see [JayR] p. 42, or [Kel] Th. 6.35),
the union of all such sets is a largest open meagre set \(M\), and is non-empty.
Thus \(X \setminus M\) is a co-meagre Baire set. For any \(x \in M\) the constant sequence
\(x_n \equiv x\) is convergent and, since \(X \setminus M\) is co-meagre in \(X\), there is \(g \in G\) with
\(g(x) \in X \setminus M\). But, as \(g\) is a homeomorphism, \(g(M)\) is a non-empty open
meagre set, so is contained in \(M\), implying \(g(x) \in M\), a contradiction. \(\square\)

A similar argument gives the following and clarifies an assumption in
Theorem E.

Proposition B2 (cf. [vMil2]; [HofJ, Prop. 2.2.3]). If \(X\) is non-meagre
and \(G\) acts transitively on \(X\), then \(X\) is a Baire space.

Proof. As above, refer again to \(M\), the union of all meagre open sets,
which, being meagre, has non-empty complement. For \(x_0\) in this complement
and any non-empty open \(U\) pick \(u \in U\) and \(g \in G\) such that \(g(x_0) = u\). Now,
as \(g\) is continuous, \(g^{-1}(U)\) is a nhd of \(x_0\), so is non-meagre, since every nhd
of \(x_0\) is non-meagre. But \(g\) is a homeomorphism, so \(U = g(g^{-1}(U))\) is non-
meagre. So \(X\) is Baire, as every non-empty open set is non-meagre. \(\square\)

2.3 Nikodym actions

The following result generalizes one that, for separable groups \(G\), is usually a
first step in proving the weakly micro-transitive variant of the classical Effros
Theorem (cf. Ancel [Anc] Lemma 3, [Ost3] Th. 2). Indeed, one may think
of it as giving a form of ‘very weak micro-transitivity’.

Proposition 1. If \(G\) is a normed group, acting transitively on a non-
meagre space \(X\) with each point evaluation map \(\varphi_x : g \mapsto g(x)\) base-\(\sigma\)-discrete
relative – then for each non-empty open \(U\) in \(G\) and each \(x \in X\) the set \(Ux\)
is non-meagre in \(X\).

In particular, if \(G\) is analytic, then \(G\) is a Nikodym action.

Proof. We first work in the right norm topology, i.e. derived from the
assumed right-invariant metric \(d^G_R(s,t) = ||st^{-1}||\). Suppose that \(u \in U\), and
so without loss of generality assume that \(U = B_\varepsilon(u) = B_\varepsilon(e_G)u\) (open balls of
radius some \(\varepsilon > 0\)); then put \(y := ux\) and \(W = B_\varepsilon(e_G)\). Then \(Ux = Wy\). Next
work in the left norm topology, derived from \( d^G_L(s, t) = ||s^{-1}t|| = d^G_R(s^{-1}, t^{-1}) \) (for which \( W = B_e(e_G) \) is still a nhd of \( e_G \)). As each set \( hW \) for \( h \in G \) is now open (since now the left shift \( g \to hg \) is a homeomorphism), the open family \( \mathcal{W} = \{ gW : g \in G \} \) covers \( G \). As \( G \) is metrizable (and so has a \( \sigma \)-discrete base), the cover \( \mathcal{W} \) has a \( \sigma \)-discrete refinement, say \( \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \), with each \( \mathcal{V}_n \) discrete. Put \( X_n := \bigcup \{ V_y : V \in \mathcal{V}_n \} \); then \( X = \bigcup_{n \in \mathbb{N}} X_n \), as \( X = Gy \), and so \( X_n \) is non-meagre for some \( n \), for \( n = N \) say. Since \( \varphi_y \) is base-\( \sigma \)-discrete, \( \{ V_y : V \in \mathcal{V}_N \} \) has a \( \sigma \)-discrete base, say \( \mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m \), with each \( \mathcal{B}_m \) discrete. Then, as \( \mathcal{B} \) is a base for \( \{ V_y : V \in \mathcal{V}_N \} \),

\[
X_N = \bigcup_{m \in \mathbb{N}} \left( \bigcup \{ B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq V y \} \right).
\]

So for some \( m \), say for \( m = M \),

\[
\bigcup \{ B \in \mathcal{B}_m : (\exists V \in \mathcal{V}_N) B \subseteq V y \}
\]

is non-meagre. But as \( \mathcal{B}_M \) is discrete, by Banach’s Category Theorem (cf. Prop. B1), there are \( \hat{B} \in \mathcal{B}_M \) and \( \hat{V} \in \mathcal{V}_N \) with \( \hat{B} \subseteq \hat{V} y \) such that \( \hat{B} \) is non-meagre. As \( \mathcal{V} \) refines \( \mathcal{W} \), there is some \( \hat{g} \in G \) with \( \hat{V} \subseteq \hat{g}W \), so \( \hat{B} \subseteq \hat{V} y \subseteq \hat{g}W y \), and so \( \hat{g}W y \) is non-meagre. As \( \hat{g}^{-1} \) is a homeomorphism of \( X \), \( W y = U x \) is also non-meagre in \( X \).

If \( G \) is analytic, then as \( U \) is open, it is also analytic (since open sets are \( \mathcal{F}_\sigma \) and Souslin-\( \mathcal{F} \) subsets of analytic sets are analytic, cf. [JayR]), and hence so is \( \varphi_x(U) \). Indeed, since \( \varphi_x \) is continuous and base-\( \sigma \)-discrete, \( Ax \) is analytic (Hansell’s Theorem, §2.1), so Souslin-\( \mathcal{F} \), and so Baire by Nikodym’s Theorem (§2.1). □

**Definition.** (Ancel [Anc]). Call the map \( \varphi_x \) **countably-covered** if there exist self-homeomorphisms \( h^x_n \) of \( X \) for \( n \in \mathbb{N} \) such that for any open nhd \( U \) in \( G \) the sets \( \{ h^x_n(\varphi_x(U)) : n \in \mathbb{N} \} \) cover \( X \).

**Proposition 1’** (cf. Ancell [Anc]) For the action \( \varphi : G \times X \to X \) with \( X \) non-meagre, if each map \( \varphi_x \) is countably-covered and takes open sets to sets with the Baire property, then the action has the Nikodym property.

**Proof.** If \( \varphi_x \) is countably-covered, then there exist self-homeomorphisms \( h^x_n \) of \( X \) for \( n \in \mathbb{N} \) such that for any open nhd \( U \) in \( G \) the sets \( \{ h^x_n(\varphi_x(U)) : n \in \mathbb{N} \} \) cover \( X \). Then for \( X \) non-meagre, there is \( n \in \mathbb{N} \) with \( h^x_n(\varphi_x(U)) \) non-meagre, so \( U x = \varphi_x(U) \) is itself non-meagre, being a homeomorphic copy
of \( h^x_n(\varphi_x(U)) \). As \( Ux \) is assumed Baire, the action has the Nikodym property.

For \( E \) separable, an immediate consequence of \textit{continuous} maps taking open sets to analytic sets (which are Baire sets) and of Prop. 1’ is that \( \varphi^L \) is a Nikodym action.

For the general context, one needs \textit{demi-open} continuous maps, which preserve almost completeness (absolute \( G_\delta \) sets modulo meagre sets – see [Mic2] and its antecedent [Nol]), as it is not known which linear maps are base-\( \sigma \)-discrete – a delicate matter to determine, since the former include continuous linear surjections (by Lemma 1 below) and preserve almost analyticity as opposed to analyticity.

For present purposes, however, the \textit{monotonicity property} below suffices. We omit the proof of the following observation (for which see the opening step in [Rud, 2.11], or [Con, Ch. 3 §12.3], or the Appendix in the arXiv version of this paper). For the underlying translation-invariant metric of a Fréchet space denote below by \( B(a, r) \) the open \( r \)-ball with centre \( a \).

\textbf{Lemma 1.} For a continuous linear map \( L : X \rightarrow Y \) from a Fréchet space \( X \) to a normed space \( Y \), for \( s < t < r \)

\[ \text{int(cl}L(B(0,s))) \subseteq L(B(0,t)) \subseteq L(B(0,r)). \]

Hence for \( L(a, r) \) convex, either \( L(B(a, r)) \) is meagre or differs from \( \text{int}L(B(a, r)) \) by a meagre set.

\textbf{Proposition 2.} For \( L \) a continuous linear surjection from a Fréchet space \( E \) to a non-meagre normed space \( F \), the action \( \varphi^L \) has the Nikodym property.

\textbf{Proof.} As in Prop 1’ for \( L : E \rightarrow F \) a continuous linear surjection, \( \{ \varphi^L_x : x \in F \} \) are countably-covered. Indeed, fixing \( x \in F \)

\[ h^x_n(z) := n(z - x) \quad (n \in \mathbb{N} \text{ and } z \in F) \]

is on the one hand a self-homeomorphism satisfying \( h^x_n(\varphi_x(L(V))) = L(nV) \), since \( n|(L(v) + x) - x| = nL(v) = L(nv) \); on the other hand the family

\[ \{ h^x_n(L(V) + x) : n \geq 1 \} \]
covers \( F \), as \( \{ nV : n \in \mathbb{N} \} \) covers \( E \) for any open nhd of the origin in \( E \) (by the ‘absorbing’ property, cf. [Con, 4.1.13], [Rud, 1.33]). In particular, \( nL(B(0,1)) \) is non-meagre for some \( n \), and so \( L(B(0,s)) \) is non-meagre for any \( s \). By Lemma 1, \( L(B(0,t)) \) for any \( t > s \) contains the non-meagre Baire set \( \text{cl}L(B(0,s)) \). \( \Box \)

Corollary 1 below is now immediate; it is used in [Ost2, Th. 5.1] to prove the ‘Semi-Completeness Theorem’, an Ellis-type theorem [Ell, Cor. 2] (cf. [Ost6]) giving a one-sided continuity condition which implies that a right-topological group generated by a right-invariant metric is a topological group.

**Corollary 1** (cf. [Ost2, Th. 5.1], ‘Open Homomorphism Theorem’). If the continuous surjective homomorphism \( \lambda \) between normed groups \( G \) and \( H \), with \( G \) analytic and \( H \) a Baire space, is base-\( \sigma \)-discrete, then \( \lambda \) is open; in particular, for \( \lambda \) bijective, \( \lambda^{-1} \) is continuous.

**Corollary 2.** For \( L : E \to F \) a continuous surjective linear map between Fréchet spaces, the point evaluations \( \varphi^L_b \) for \( b \in F \) are open, and so \( L \) is an open mapping.

**Proof.** By surjectivity of \( L \), the action is transitive, and by Prop 2 the action \( \varphi^L \) has the Nikodym property. So by Theorem E above the point-evaluations maps \( \varphi^L_b \) are open. Hence so also is \( L \). \( \Box \)

3 Proofs

3.1 Proof that \( E \leftrightarrow C \)

In [BinO1] Th. 3.15 we showed that if the Effros property holds for the action of a group \( G \) on \( X \), then \( X \) has the crimping property w.r.t. \( G \). We recall the argument, as it is short. Suppose that \( x = \lim x_n \). For each \( n \), take \( U = B_{1/n}^G(e_G) \); then \( Ux := \{ u(x) : u \in U \} \) is an open nhd of \( x \), and so there exists \( h_{n,m} \in U \) with \( h_{n,m}(x) = x_m \) for all \( m \) large enough, say for all \( m > m(n) \). Without loss of generality we may assume that \( m(1) < m(2) < ... \). Put \( h_m := e_G \) for \( m < m(1) \), and for \( m(k) \leq m < m(k+1) \) take \( h_m := h_{k,m} \). Then \( h_m \in B_{1/k}^G(e_G) \), so \( h_m \) converges to \( e_G \) and \( h_m(e_G) = x_m \).
For the converse, suppose that the Effros property fails for $G$ acting on $X$. Then for some open nhd $U$ of $e_G$ and some $x \in X$, $Ux := \{u(x) : u \in U\}$ is not an open nhd of $x$. So for each $n$ there is a point $x_n \in B_{1/n}(x) \setminus Ux$. As $x_n$ converges to $x$ there are homeomorphisms $h_n$ converging to the identity $e_G$ with $h_n(x) = x_n$. As $U$ is an open nhd of $e_G$ and since $h_n$ converges to $e_G$, there is $N$ such that $h_n \in U$ for $n > N$. In particular, for any $n > N$, $h_n(x) = x_n \in Ux$, a contradiction.

3.2 Weak S

We view Th. S as having ‘two tasks’: to find a ‘translator of the sequence’ $\tau$, and to locate it in a given Baire non-meagre subset of the group – provided that subset satisfies a consistency condition (a necessary condition).

For clarity we break the tasks into two steps – the first delivering a weaker version of S in Proposition 3 below. The arguments are based on the following lemma. We note a corollary, observed earlier by van Mill in the case of metric topological groups ([vMil2, Prop. 3.4]), which concerns a co-meagre set, but we need its refinement to a localized version for a non-meagre set.

Separation Lemma. Let $G$ be a normed group, with separately continuous and transitive Nikodym action on a non-meagre space $X$. Then for any point $x$ and any $F$ closed nowhere dense, $W_{x,F} := \{\alpha \in G : \alpha(x) \notin F\}$ is dense open in $G$. In particular, $G$ separates points from nowhere dense closed sets.

Proof. The set $W_{x,F}$ is open, being of the form $\varphi_x^{-1}(X\setminus F)$ with $\varphi_x$ continuous (by assumption). By the Nikodym property, for $U$ any non-empty open set in $G$, the set $Ux$ is non-meagre, and so $Ux\setminus F$ is non-empty, as $F$ is meagre. But then for some $u \in U$ we have $u(x) \notin F$. □

Corollary 2. If $G$ is a normed group, Baire in the norm topology with transitive and separately continuously Nikodym action on a non-meagre space $X$ space, and $T$ is co-meagre in $X$ – then for countable $D \subseteq X$, the set $\{g : g(D) \subseteq T\}$ is a dense $G_\delta$.

In particular, this holds if $G$ is analytic and each point-evaluation map $\varphi_x : g \to g(x)$ is base-\(\sigma\)-discrete.

Proof. Without loss of generality, the co-meagre set is of the form $T = U \setminus \bigcup_{n \in \omega} F_n$ with each $F_n$ closed and nowhere dense, and $U$ open. Then, by
the Separation Lemma and as G is Baire,
\[ \{ g \in G : g(D) \subseteq T \} = \bigcap_{n \in \omega} \{ g : g(D) \cap F_n = \emptyset \} = \bigcap_{d \in D, n \in \omega} \{ g : g(d) \notin F_n \} \]
is a dense \( G_\delta \). \( \square \)

**Proposition 3.** If \( T \) is a Baire non-meagre subset of a metric space \( X \) and \( G \) a normed group, Baire in its norm topology, acting separately continuously and transitively on \( X \), with the Nikodym property – then, for every convergent sequence \( x_n \) with limit \( x_0 \) there is \( \tau \in G \) and an integer \( N \) with \( \tau x_0 \in T \) and
\[
\{ \tau(x_n) : n > N \} \subseteq T.
\]

**Proof.** Write \( T := M \cup (U \backslash \bigcup_{n \in \omega} F_n) \) with \( U \) open, \( M \) meagre and each \( F_n \) closed and nowhere dense in \( X \). Let \( u_0 \in T \cap U \). By transitivity there is \( \sigma \in G \) with \( \sigma x_0 = u_0 \). Put \( u_n := \sigma x_n \). Then \( u_n \to u_0 \). Put
\[
C := \bigcap_{m, n \in \omega} \{ \alpha \in G : \alpha(u_m) \notin F_n \},
\]
a dense \( G_\delta \) in \( G \); then, by the Separation Lemma above, as \( G \) is Baire,
\[
\{ \alpha \in G : \alpha(u_0) \in U \} \cap C
\]
is non-empty. For \( \alpha \) in this set we have \( \alpha(u_0) \in U \backslash \bigcup_{n \in \omega} F_n \). Now \( \alpha(u_n) \to \alpha(u_0) \), by continuity of \( \alpha \), and \( U \) is open. So for some \( N \) we have for \( n > N \) that \( \alpha(u_n) \in U \). Since \( \{ \alpha(u_m) : m = 1, 2, \ldots \} \subseteq X \backslash \bigcup_{n \in \omega} F_n \), we have for \( n > N \) that \( \alpha(u_n) \in U \backslash \bigcup_{n \in \omega} F_n \subseteq T \).
Finally put \( \tau := \alpha \sigma \); then \( \tau(x_0) = \alpha \sigma(x_0) \in T \) and \( \{ \tau(x_n) : n > N \} \subseteq T \). \( \square \)

### 3.3 Proof of S

We work in the right norm topology and use the notation of the preceding proof (of Proposition 3), so that \( U \) here is the quasi-interior of \( T \) and \( \sigma x_0 = u_0 \). As \( e_G \in A^q \) and \( A \) is a non-meagre Baire set, we may without loss of generality write \( A = B_{c}(e_G) \backslash \bigcup_{n} G_n \), where each \( G_n \) is closed nowhere dense with \( e_G \notin G_n \) and \( B_{c}(e_G) \) is the quasi-interior of \( A \).

As \( A^q x_0 \cap T^q \) is non-empty, there is \( \alpha_0 \in B_{c}(e_G) \) with \( \alpha_0 x_0 \in U \) (but, we want a better \( \alpha \) so that \( \alpha x_0 \in T \) and \( \alpha \in A \)). Put \( \beta_0 = \alpha_0 \sigma^{-1} \); then
\[
\beta_0 = \alpha_0 \sigma^{-1} \in B_{c}(e_G) \sigma^{-1} \cap \{ \alpha : \alpha(x_0) \in U \} \sigma^{-1} = B_{c}(e_G) \sigma^{-1} \cap \{ \beta : \beta(\sigma x_0) \in U \} = B_{c}(e_G) \sigma^{-1} \cap \{ \beta : \beta(u_0) \in U \},
\]
i.e. the open set \( \{ \beta : \beta(u_0) \in U \} \cap B_\varepsilon(e_G)\sigma^{-1} \) is non-empty. So
\[
(C \setminus \bigcup_n G_n\sigma^{-1}) \cap \{ \beta : \beta(u_0) \in U \} \cap B_\varepsilon(e_G)\sigma^{-1} \neq \emptyset,
\]
since \( G \) is a Baire space and each \( G_n\sigma^{-1} \) is closed and nowhere dense in \( G \) (as the right shift \( g \to g\sigma^{-1} \) is a homeomorphism).

So there is \( \beta \) with \( \beta(u_0) \in U \) such that \( \alpha := \beta\sigma \in B_\varepsilon(e_G) \setminus \bigcup_n G_n = A \).
That is, \( \alpha x_0 = \beta u_0 \in U \); so \( \beta(u_n) \in U \) for large \( n \), for \( n > N \) say, as \( \alpha x_0 = \lim \alpha x_n = \lim \beta\sigma x_n = \lim \beta u_n \). But \( \{ \beta(u_m) : m = 1, 2, \ldots \} \in X \setminus \bigcup_n F_n \), as \( \beta \in C \); so \( \beta(u_n) \in U \setminus \bigcup_n F_n \subseteq T \) for \( n > N \).

Finally, \( \alpha(x_0) = \beta\sigma(x_0) \in T \) and \( \{ \alpha(x_n) : n > N \} \subseteq T \). \( \square \)

### 3.4 Proof that \( S \implies E \)
Assume \( G \) acts transitively on \( X \) and that \( X \) is non-meagre. Let \( B := B_\varepsilon(e_G) \) and suppose that for some \( x \) the set \( Bx \) is not a nhd of \( x \). Then there is \( x_n \to x \) with \( x_n \notin Bx \) for each \( n \). Take \( A := B_{\varepsilon/2}(e_G) \) and note first that \( A \) is a symmetric open set \( (A^{-1} = A \) since \( ||g|| = ||g^{-1}|| \)), and secondly that by the Nikodym property \( Ax \) contains a non-meagre, Baire subset \( T \). So by Theorem S, as \( Ax \) meets \( T^\circ \), there are \( a \in A \) (which being open has the Baire property) and a co-finite \( \mathcal{M}_a \) such that \( ax_m \in Ax \) for \( m \in \mathcal{M}_a \). For any such \( m \), choose \( b_m \in A \) with \( ax_m = b_m x \). Then \( x_m = a^{-1}b_m x \in A^2 x \subseteq Bx \), a contradiction (note that \( a^{-1} \in A \), by symmetry).

As earlier, in the special case that \( G \) is (metrizable and) analytic, \( A \) is analytic, since open sets are \( \mathcal{F}_\sigma \) and Souslin-\( \mathcal{F} \) subsets of analytic sets are analytic, cf. [JayR, Th. 2.5.3], by Prop. 3 \( Ax \) is Baire non-meagre, as \( \varphi_x \) is base-\( \sigma \)-discrete.

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