A note on rational L^p approximation on Jordan curves*

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Abstract

The the precise asymptotics for the error of best rational approximation of meromorphic functions in integral norm is shown to be a consequence of a result of Gonchar and Rakhmanov. This reproves and extends a recent result of Baratchart, Stahl and Yattselev.

Let T be a rectifiable Jordan curve, G and O the interior and exterior domains of T, respectively, with respect to $\overline{\mathbf{C}}$. Let $A(G)$ denote the set of functions f such that

- f vanishes at infinity and admits holomorphic and single-valued continuation from infinity to an open neighborhood of $\overline{O}$,
- f admits meromorphic, possibly multi-valued, continuation along any arc in $G \setminus E_f$ starting from T, where E_f is a finite set of points in G,
- E_f is non-empty, the meromorphic continuation of f from infinity has a branch point at each element of E_f.

Examples of such functions are algebraic functions with branch points. See the paper [1] for other examples, motivation and history.

In the recent landmark paper L. Baratchart, H. Stahl and M. Yattselev [1] have developed the theory of rational approximation of functions $f \in A(G)$ in the $L^2(s_T)$ norm on T, where s_T is the arc measure on T, and where the approximation is done from the set $\mathcal{R}_n(G)$ of rational functions p_{n-1}/q_n of degree $((n-1), n)$ which have all their poles in G. Let the error of best approximation

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in $L^p(s_T)$ be denoted by $\rho_{n,p}(f, O)$. The theory in [1] gave, besides a lot of information on the best approximants, the $p = 2$ case of the asymptotic formula

$$\lim_{n\to\infty} \rho_{n,p}^{1/2n}(f, O) = \exp\left(-\frac{1}{\mathrm{cap}(K_T, T)}\right) \tag{1}$$

(see below for the definition of the minimal condenser capacity $\mathrm{cap}(K_T, T)$). For $p = \infty$ the same formula follows from a result of A. A. Gonchar and E. A. Rakhmanov [2, Theorem 1']. As a consequence, (1) has been established for all $2 \le p \le \infty$.

In this note we derive (1) for all $1 \le p < \infty$ directly from the $p = \infty$ case proven in [2, Theorem 1'].

To have a basis of discussion, let $g_G(z, \zeta)$ denote the Green's function of G with pole at $\zeta \in G$, and if $K \subset G$ is a compact set, then consider the minimal energy

$$I_G(K) := \inf_\omega I_G(\omega) := \inf_\omega \int\int g_G(z, t) d\omega(z) d\omega(t),$$

where the infimum is taken for all unit Borel-measures on K. In the case when K is not polar (has positive logarithmic capacity) there is a unique minimizing measure $\omega_{K,T}$, called the Green equilibrium measure of K (with respect to Ω). $\mathrm{cap}(K, T) := 1/I_G(K)$ is called the condenser capacity of the condenser (K, T).

Next, we need the notion of a set of minimal condenser capacity. We say that a compact $K \subset G$ is admissible for $f \in A(G)$ if $\overline{\mathbf{C}} \setminus K$ is connected, and f has a meromorphic and single-valued extension there. The collection of all admissible sets for f is denoted by $\mathcal{K}_f(G)$. A compact $K_T \in \mathcal{K}_f(G)$ is said to be a set of minimal condenser capacity for f if

- $\mathrm{cap}(K_T, T) \le \mathrm{cap}(K, T)$ for any $K \in \mathcal{K}_f(G)$,
- $K_T \subseteq K$ for any $K \in \mathcal{K}_f(G)$ for which $\mathrm{cap}(K, T) = \mathrm{cap}(K_T, T)$.

See [1] for the existence and unicity of such a K_T. The set K_T of minimal condenser capacity is the complement of the "largest" (regarding capacity) domain containing O on which f is single-valued and meromorphic. It turns out (see [1, Theorem S]) that $K_T = E_0 \cup E_1 \cup (\cup_j \gamma_j)$, where $\cup\gamma_j$ is a finite union of open analytic arcs, $E_0 \subset E_f$, each point in E_0 is the endpoint of exactly one γ_j, while E_1 consist of those finitely many points where at least three arcs γ_j meet.

These definitions explain the notation in (1), and with these we claim

Theorem 1 (1) *holds for all* $1 \le p \le \infty$*.*

Proof. The $p = \infty$ case is covered by the Gonchar-Rakhmanov theorem from [2], so it is left to show

$$\liminf_{n\to\infty} \rho_{n,1}^{1/2n}(f, O) \ge \exp\left(-\frac{1}{\mathrm{cap}(K_T, T)}\right). \tag{2}$$

Let $G_1 \supset G_2 \supset \cdots$ be a nested sequence of Jordan domains with boundaries $T_1, T_2, \ldots$ such that $T_{j+1} \subset G_j$, each T_j lies outside $\overline{G}$, the maximal distance from a point of T_j to T is less than $1/j$ and $\mathrm{length}(T_j) \to \mathrm{length}(T)$ (say some level line of the conformal mapping of O onto the exterior of the unit disk suffices as T_j). Then there is a compact set $K \subset G$ and a j_0 such that $K_{T_j} \subset K$ for $j \geq j_0$ (see Lemma 2 below), and for $z, t \in K$ we have $g_{G_j}(z,t) \leq g_G(z,t) + \eta_j$ where $\eta_j \to 0$ (see Lemma 3 below). If $r \in \mathcal{R}_n(G)$ is any rational function from $\mathcal{R}_n(G)$ and if we apply Cauchy's formula for $(f - r_n)(z)$, $z \in T_j$, in O using integration on T, we obtain

$$\sup_{z \in T_j} |f(z) - r_n(z)| \leq \|f - r_n\|_{L^1(s_T)} \frac{1}{\mathrm{dist}(T_j, T)},$$

so

$$\liminf_{n\to\infty} \rho_{n,1}^{1/2n}(f, O) \geq \liminf_{n\to\infty} \rho_{n,\infty}^{1/2n}(f, O_j) = \exp\left(-I_{G_j}(\omega_{K_{T_j},T_j})\right),$$

where the equality follows by the aforementioned Gonchar-Rakhmanov theorem. Here for $j \geq j_0$ we have

$$I_{G_j}(\omega_{K_{T_j},T_j}) \leq I_{G_j}(\omega_{K_{T_j},T})$$

by the definition of the Green equilibrium measure $\omega_{K_{T_j},T_j}$, and clearly $g_{G_j}(z,t) \leq g_G(z,t) + \eta_j$, $t \in K$ and $K_{T_j} \subseteq K$ imply

$$I_{G_j}(\omega_{K_{T_j},T}) \leq I_G(\omega_{K_{T_j},T}) + \eta_j.$$

Finally, by the fact that K_T is the set of minimal condenser capacity for G, so it maximizes the energies $I_G(\omega_{K_S,T})$ for all $S \subset G$, it follows that

$$I_G(\omega_{K_{T_j},T}) \leq I_G(\omega_{K_T,T}).$$

Putting all these together we get

$$\liminf_{n\to\infty} \rho_{n,1}^{1/2n}(f, O) \geq \exp\left(-I_G(\omega_{K_T,T})\right) e^{-\eta_j} = \exp\left(-\frac{1}{\mathrm{cap}(K_T, T)}\right) e^{-\eta_j},$$

which proves (2) if we let $j \to \infty$.

■

The proof above used the following two quite plausible facts.

Lemma 2 *There is a compact set $K \subset G$ and a j_0 such that $K_{T_j} \subset K$ for $j \geq j_0$.*

Lemma 3 *For $z, t \in K$ we have $g_{G_j}(z,t) \leq g_G(z,t) + \eta_j$ where $\eta_j \to 0$.*

Proof of Lemma 2. Let $H_a = \{z \,|\, \Re z > a\}$, and fix a neighborhood S around T to which f has a single-valued analytic continuation.

Assume to the contrary that there is a sequence of points $P_j \in K_{T_j}$, $j = 1, 2, \ldots$, such that

$$\liminf_{j\to\infty} \mathrm{dist}(P_j, \overline{\mathbf{C}} \setminus G) = 0.$$

We may assume that here the liminf is actually a limit and $P_j \to P \in T$ (select a subsequence). Select a $\tilde{P}_j \in T_j$ with $\mathrm{dist}(P_j, \tilde{P}_j) \to 0$. Fix a $z_0 \in G$ and let φ^*, φ_j^* be the conformal maps that map the unit disk onto G, G_j such that $\varphi^*(0) = \varphi_j^*(0) = z_0$ and $(\varphi^*)'(0) > 0$, $(\varphi_j^*)'(0) > 0$. It is known (see e.g. [3, Theorem 6.12 and Exercise 6.3/4]) that $\varphi_j^* \to \varphi^*$ uniformly on the closed unit disk, therefore $(\varphi_j^*)^{-1}(P_j) \to (\varphi^*)^{-1}(P)$, $(\varphi_j^*)^{-1}(\tilde{P}_j) \to (\varphi^*)^{-1}(P)$. Combine these with some fixed mapping of the unit disk onto the right-half plane H_0 to deduce the following: if φ_j, φ are conformal maps of G_j, G onto H_0 such that $\varphi_j(z_0) = \varphi(z_0) = 1$, $\varphi_j(\tilde{P}_j) = 0$, $\varphi(P) = 0$, then $\varphi_j \to \varphi$ uniformly on compact subsets of G and $\varphi_j(P_j) \to \varphi(P) = 0$. Therefore, there is an $a > 0$ such that $\varphi_j(E_f) \subset \overline{H_a}$ for all large j and at the same time $\varphi_j(P_j) \notin \overline{H_a}$. Hence, if $B_j := \varphi_j(K_{T_j})$, then

$$B_j = \varphi_j(K_{T_j}) \not\subseteq \overline{H_a} \qquad \text{for } j \geq j_0 \tag{3}$$

with some j_0. We may also assume $a > 0$ to be so small and j_0 so large that $\varphi_j(G \setminus S) \subset H_a$ for $j \geq j_0$ (note that $\varphi(G \setminus S)$ is a compact subset of H_0). Fix a $j \geq j_0$, and with this j we get a contradiction as follows.

Consider the mapping

$$z = x + iy \to z' = \max(x, a) + iy$$

(the projection onto $\overline{H_a}$) and set $B_j' = \{z' \,\big|\, z \in B_j\}$. Then

$$g_{H_0}(z, w) = \log\left|\frac{z + \overline{w}}{z - w}\right| \leq \log\left|\frac{z' + \overline{w'}}{z' - w'}\right| = g_{H_0}(z', w') \tag{4}$$

(just note that the imaginary parts are the same, while the real parts increase resp. decrease when we go from $z + \overline{w}$ resp. $z - w$ to $z' + \overline{w'}$ resp. $z' - w'$).

We need

Lemma 4 *There is a Borel-mapping $\Phi : B_j' \to B_j$ such that $\Phi(x)' = x$ for all $x \in B_j'$. For every Borel-measure μ on B_j' this generates a Borel-measure ν on B_j via $\nu(E) = \mu(\Phi^{-1}[E])$ for all Borel-sets $E \subset B_j$ (here $\Phi^{-1}[E]$ is the complete inverse image of E) such that*

$$\int \log\left|\frac{z + \overline{w}}{z - w}\right| d\nu(z)d\nu(w) = \int \log\left|\frac{\Phi(u) + \overline{\Phi(v)}}{\Phi(u) - \Phi(v)}\right| d\mu(u)d\mu(v).$$

With this lemma at hand we continue the proof of Lemma 2. We have

$$\begin{aligned} I_{H_0}(\nu) &= \int \log\left|\frac{z+\overline{w}}{z-w}\right| d\nu(z)d\nu(w) = \int \log\left|\frac{\Phi(u)+\overline{\Phi(v)}}{\Phi(u)-\Phi(v)}\right| d\mu(u)d\mu(v) \\ &\leq \int \log\left|\frac{u+\overline{v}}{u-v}\right| d\mu(u)d\mu(v) = I_{H_0}(\mu), \end{aligned}$$

where, at the second inequality, we used (4).

Let Ω_j be the unbounded component of $\overline{\mathbf{C}}\setminus B_j'$ and $\mathrm{Pc}(B_j') : \overline{\mathbf{C}}\setminus\Omega_j$ be the so called polynomial convex hull of B_j'. Next we show that $\mathrm{Pc}(B_j')$ is an admissible set for the function $F := f(\varphi_j^{-1})$ in H_0. To see this let Γ be a polygonal curve in $\Omega_j \cap H_0$ starting and ending at the origin, i.e. Γ is a closed curve that lies in the right-half plane H_0 except for the point $0 \in \Gamma$, and Γ doe not intersect $\mathrm{Pc}(B_j')$. Let F^* be the continuation of F along (a neighborhood of) Γ as we traverse Γ once from 0 to 0. We need to show that after traversing Γ we get back to the same function element, i.e. $F^* = F$ in a neighborhood of the origin.

By assumption, F has a continuation to the strip $H_0 \setminus \overline{H_a}$ which we denote by F_0. Also, by the assumption on K_{T_j}, F has a single-valued continuation F_1 to the set $\overline{\mathbf{C}} \setminus B_j$. Note that necessarily $F_1 = F_0$ on the set $(H_0 \setminus \overline{H_a}) \setminus B_j$. We may assume that Γ does not contain a vertical segment, and for some small $\varepsilon > 0$ let $Q_1, \ldots, Q_m$ be the points of Γ (in the order of the traverse) that lie on the line $\Re z = a - \varepsilon$. Let here $\varepsilon > 0$ be so small that $\overline{H_{a-\varepsilon}} \cap \Gamma \cap B_j = \emptyset$ (there is such an $\varepsilon > 0$ since the preceding relation is true with $\varepsilon = 0$). Then the points $Q_1, \ldots, Q_m$ lie outside B_j, and let $D_k \subset H_0 \setminus \overline{H_a}$ be a small disk around Q_k not intersecting B_j. Note that, as we have just remarked, $F_1 \equiv F_0$ on all these disks. Now we can easily prove by induction that $F^* \equiv F_0 \equiv F_1$ on each D_k. Indeed, for $k = 1$ the equality $F^* \equiv F_0$ is true by the monodromy theorem in $H_0 \setminus \overline{H_a}$. Now assume that we already know the claim for D_k. The portion Γ_k of Γ in between the points Q_k and Q_{k+1} either lies in $H_{a-\varepsilon}$ or in $H_0 \setminus \overline{H_{a-\varepsilon}}$. In the former case the continuation of $F^* \equiv F_1$ along Γ_k is the same as F_1 (note that Γ_k does not intersect B_j), hence on D_{k+1} we have $F^* \equiv F_1 \equiv F_0$. On the other hand, if Γ_k lies in $H_0 \setminus \overline{H_{a-\varepsilon}}$, then the continuation $F^* \equiv F_0$ along Γ_k is the same as F_0 by the monodromy theorem in $H_0 \setminus \overline{H_a}$, hence in this case we have again $F^* \equiv F_0 \equiv F_1$ on D_{k+1}, by which the induction has been carried out. Another application of the monodromy theorem along the portion of Γ from Q_m to 0 shows that, indeed, as we get back to the origin, with F^* we arrive back to the same function element F_0 that we started with.

We have thus shown that $\mathrm{Pc}(B_j')$ is an admissible set for $f(\varphi_j^{-1})$ in H_0, hence $K_j^* := \varphi_j^{-1}(\mathrm{Pc}(B_j'))$ is an admissible set for f in G_j, and K_j^* lies in $\varphi_j^{-1}(\overline{H_a})$. If we define the measure μ on B_j' by stipulating $\mu(E) = \omega_{K_j^*,T_j}(\varphi_j^{-1}(E))$ for all Borel-sets $E \subset B_j'$, ν is the associated measure via Lemma 4, and finally ω is the measure defined by $\omega(E) = \nu(\varphi_j(E))$, then ω is supported on K_{T_j}, and has total mass 1 because $\omega_{K_j^*,T_j}$ is supported on the outer boundary of K_j^* (see

[1, Sec. 7.1.3]), and hence the interior of $\mathrm{Pc}(B_j')$ has zero μ-measure. Now we obtain from Lemma 4 and from the conformal invariance of the Green's function

$$I_{G_j}(\omega) = I_{H_0}(\nu) \le I_{H_0}(\mu) = I_{G_j}(\omega_{K_j^*,T_j}),$$

which implies

$$I_{G_j}(K_{T_j}) \le I_{G_j}(\omega) \le I_{G_j}(\omega_{K_j^*,T_j}) = I_{G_j}(K_j^*).$$

Therefore, by the extremality of K_{T_j} for G_j, we must have equality here, and then, by the definition of the set K_{T_j} of minimal condenser capacity, we must have $K_{T_j} \subseteq K_j^* \subseteq \varphi_j^{-1}(\overline{H_a})$, which contradicts (3).

This contradiction proves the claim in Lemma 3.

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Proof of Lemma 4. In this proof we use the special structure of the sets K_{T_j} described before Theorem 1.

For $z \in H_a \cap B_j' = H_a \cap B_j$ set $\Phi(z) = z$, and for $z = a+iy \in B_j' \cap \{x = a\}$ let $\Phi(z) = x(z)+iy \in B_j$ be the point in B_j with the smallest possible x-coordinate $x(z)$. In the latter case $\Phi(z) \in H_0 \setminus H_a$, and clearly $\Phi(z)' = z$ for all $z \in B_j'$, so it is left to verify that Φ is a Borel-map. To this it is sufficient to show that for a dense set of $B < C$ and for a dense set of $A \in [0, a)$ the inverse image $\Phi^{-1}[R]$ is a Borel-set, where $R = [0, A] \times [B, C]$. To get this note that if the boundary of R does not contain either endpoints of an open analytic arc $\gamma \subset B_j$which is not a vertical or horizontal segment, then $\partial R \cap \gamma$ is a finite set. Therefore, in this case $R \cap \gamma$ consists of a finite number of analytic arcs, and hence $(R \cap \gamma)'$ is the union of finitely many closed segments on ∂H_a. Since B_j is the union of finitely many points and finitely many open analytic arcs, it follows that $(R \cap B_j)'$ consists of a finite number of closed segments on ∂H_a provided ∂R does not contain any of the endpoints of these arcs. Since $\Phi^{-1}[R] = (R \cap B_j)'$, we are done.

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Proof of Lemma 3. Let $\varepsilon > 0$ and select a Jordan curve σ separating K and T so that $g_G(z, \tau) \le \varepsilon$ for all $z \in \sigma$, $\tau \in K$. (There is such a σ: if σ_1 separates T and K then $g_G(z, t) \le M$ for all $z \in \sigma_1$, $t \in K$ with some constant M. Map now the strip in between T and σ_1 into a ring $R = \{1 \le |z| \le r\}$ by a conformal map φ. Then the three-circle-theorem gives

$$g_G(z, t) \le M \frac{\log |\varphi(z)|}{\log r},$$

so

$$\sigma = \left\{ z \,\middle|\, |\varphi(z)| = \exp\left(\varepsilon \frac{\log r}{M}\right) \right\}$$

suffices for small ε.) Now $g_{G_j}(z,\tau) \searrow g_G(z,\tau)$ for all $z \in \sigma$ and $\tau \in K$, so, by Dini's theorem, this convergence is uniform in $z \in \sigma$ for all fixed $\tau \in K$, i.e. $g_{G_j}(\zeta,\tau) < 2\varepsilon$ for $j \geq j_\tau$ and all $\zeta \in \sigma$, $\tau \in K$. Then $g_{G_{j_\tau}}(z,t) < 2\varepsilon$ is true for all $z \in \sigma$ and $t \in K$ lying sufficiently close to some $\zeta \in \sigma$ and $\tau \in K$, and by compactness of σ we get $g_{G_{j_\tau}}(z,t) < 2\varepsilon$ for all $z \in \sigma$ and t lying sufficiently close to τ. Then for the same values $g_{G_j}(z,t) < 2\varepsilon$ automatically holds for $j \geq j_\tau$ because the Green's function g_{G_j} decrease. Finally, by the compactness of K there is a j_0 such that this inequality holds for all $z \in \sigma$, $t \in K$ and $j \geq j_0$.

As a consequence, $g_{G_j}(z,t) - g_G(z,t) \leq 2\varepsilon$ for $z \in \sigma$, $t \in K$ and $j \geq j_0$, and then, by the maximum theorem, this inequality persists for all $t \in K$ and z lying inside σ.

$\blacksquare$

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