Asymptotics of Christoffel functions on arcs and curves

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Abstract

For a system of smooth Jordan curves and arcs asymptotics for Christoffel functions is established. A separate new method is developed to handle the upper and lower estimates. In the course to the upper bound a theorem of Widom on the norm of Chebyshev polynomials is generalized.
1 Results

Let \( \nu \) be a finite Borel measure on the plane with compact support consisting of infinitely many points. The Christoffel functions associated with \( \nu \) are defined as

\[
\lambda_n(z, \nu) = \inf_{P_n(z)=1} \int |P_n|^2 \, d\nu,
\]

where the infimum is taken for all polynomials of degree at most \( n \) that take the value 1 at \( z \).

Christoffel functions are closely related to orthogonal polynomials (for a survey see [13] by P. Nevai and [21] by B. Simon), to statistical physics (see e.g. [15] by L. Pastur), to universality in random matrix theory (see e.g. the recent breakthrough [10] by D. Lubinsky, as well as [2],[22],[24]), to spectral theory (see e.g. [20], [21] by B. Simon and [1] by Breuer, Last and Simon) and to several other fields in mathematics. For the role and various use of Christoffel functions see [4], [6], [20], and particularly [13] by P. Nevai and [21] by B. Simon.

In this paper we consider asymptotics of Christoffel functions on smooth \((C^{1+\alpha})\) Jordan curves and arcs. Recall that a Jordan curve is the homeomorphic image of the unit circle while a Jordan arc is the homeomorphic image of \([-1,1]\). Thus, a Jordan arc has two endpoints. The asymptotics of Christoffel functions on \(C^2\)-Jordan curves was established in [25] with a systematic use of polynomial inverse images of the unit circle (lemniscates). The idea of that paper was that many things can be carried over to lemniscates from the unit circle, and a system of \(C^2\) Jordan curves can be well approximated (in a very specific sense) by lemniscates both from the inside and from the outside. This method does not work for arcs, and, in fact, except for the case when the set is a subset of the real line, no result has been known regarding Christoffel function asymptotics for arcs. In this paper we develop a method that handles both Jordan curves and arcs. We emphasize that we need a new method (actually very different ones) for both the upper and lower estimates, for previous methods do not work in either cases.

Thus, let \( \Gamma \) be the union of finitely many \( C^{1+\alpha}, \alpha > 0 \), smooth Jordan curves and arcs lying exterior to one another, and let \( s_\Gamma = s \) be the arc measure on \( \Gamma \). Let \( \Gamma_k, k = 0, 1, \ldots, k_0 \) be the disjoint components of \( \Gamma \): \( \Gamma = \bigcup_{k=0}^{k_0} \Gamma_k \). We call those \( \Gamma_k \) that are Jordan arcs the arc-components of \( \Gamma \). Since we need \( C^{1+\alpha} \) smoothness just to have higher smoothness than \( C^4 \), we may and shall always assume \( 0 < \alpha < 1 \).

Our main theorem is

**Theorem 1.1** Let \( \Gamma \) be a system of \( C^{1+\alpha} \)-smooth Jordan arcs and curves lying exterior to one another, let \( z_0 \in \Gamma \) be a point on \( \Gamma \) that is different from the endpoints of the arc components of \( \Gamma \), and assume that \( \Gamma \) is \( C^2 \)-smooth in a neighborhood of \( z_0 \). Assume that \( d\nu = w \, ds_\Gamma \) is a measure on \( \Gamma \) with density \( w \) (with respect to the arc measure \( s_\Gamma \)) which is continuous on \( \Gamma \) and positive.
\(s_\Gamma\)-almost everywhere. Then

\[
\lim_{n \to \infty} n \lambda_n(z_0, \nu) = \frac{d\nu(z_0)}{d\mu_\Gamma},
\]

(1.1)

where \(\mu_\Gamma\) denotes the equilibrium measure of \(\Gamma\), and on the right-hand side \(d\nu(z)/d\mu_\Gamma\) is the Radon-Nikodym derivative of \(\nu\) with respect to \(\mu_\Gamma\).

For the concepts from potential theory (like equilibrium measure, logarithmic capacity, Greens’ function etc.) see e.g. [5], [9], [17], or [19].

In the case that we are considering the equilibrium measure \(\mu_\Gamma\) is absolutely continuous with respect to the arc measure \(s_\Gamma\) on \(\Gamma\):

\[
d\mu_\Gamma(t) = \omega_\Gamma(t) \, ds_\Gamma(t)
\]

with \(\omega_\Gamma\) a \(C^{\alpha}\)-continuous density function (see Proposition 2.2), and with it (1.1) takes the form

\[
\lim_{n \to \infty} n \lambda_n(z_0, \nu) = \frac{w(z_0)}{\omega_\Gamma(z_0)}.
\]

(1.2)

The \(C^{1+\alpha}\)-smoothness could be replaced by piecewise \(C^{1+\alpha}\)-smoothness without cusps, in which case \(\Gamma\) could have corners, and then the result is claimed for \(z_0\) which is not an endpoint or a corner (at endpoints and at corners the order of the Christoffel function is no longer \(1/n\), see [28]).

The global positivity and continuity of \(w\) was assumed only to have an easy formulation, the proof actually gives a much more general result. To this end we recall the class \(\text{Reg}\) from [23]: a measure \(\nu\) with support \(\Gamma\) is said to be in the \(\text{Reg}\) class if

\[
\lim_{n \to \infty} \left( \sup_{P_n} \frac{\|P_n\|_{\Gamma}}{\|P_n\|_{\text{L}^2(\nu)}} \right)^{1/n} = 1,
\]

(1.3)

where the supremum is taken for all polynomials of degree at most \(n\), and where \(\|P_n\|_{\Gamma}\) stands for the supremum norm of \(P_n\) on \(\Gamma\). This is not the standard definition of the \(\text{Reg}\) class (which is in terms of the leading coefficients of orthogonal polynomials), but it is equivalent to it, see [23, Theorem 3.4.3,(v)]. See [23] for several other equivalent formulations and for general criteria implying \(\nu \in \text{Reg}\).

We only mention here that \(\nu \in \text{Reg}\) is a very weak global assumption on \(\nu\), e.g. it holds if \(d\nu(t)/ds_\Gamma(t) > 0\) \(s_\Gamma\)-almost everywhere. Therefore, Theorem 1.1 is a special case of

**Theorem 1.2** Let \(\Gamma\) be a system of \(C^{1+\alpha}\)-smooth Jordan arcs and curves lying exterior to one another, let \(z_0 \in \Gamma\) be different from the endpoints of the arc components of \(\Gamma\) and assume that \(\Gamma\) is \(C^2\)-smooth in a neighborhood of \(z_0\). Assume that \(d\nu = wd\sigma_{\Gamma} + d\nu_{\text{sing}}\) is a measure on \(\Gamma\) with density \(w\) and with singular part \(\nu_{\text{sing}}\) (with respect to the arc measure \(s_\Gamma\) which is in the \(\text{Reg}\) class. Then, if \(w\) is continuous at \(z_0\) and \(z_0\) is a Lebesgue-point for \(\nu_{\text{sing}}\), we have

\[
\lim_{n \to \infty} n \lambda_n(z_0, \nu) = \frac{w(z_0)}{\omega_\Gamma(z_0)},
\]

(1.4)

where \(\omega_\Gamma\) denotes the density of the equilibrium measure \(\mu_\Gamma\) (with respect to \(s_\Gamma\)).
The Lebesgue-point property of \( \nu_{\text{sing}} \) mentioned in the statement is
\[
\nu_{\text{sing}}(\{ \zeta \mid |\zeta - z_0| \leq \tau \}) = o(\tau) \quad \text{as } \tau \to 0.
\] (1.5)

Let us mention that some kind of global condition like \( \nu \in \text{Reg} \) is needed, e.g. if \( \nu \) vanishes on a subarc of \( \Gamma \), then (1.4) is necessarily false because then
\[
\liminf_{n \to \infty} n \lambda_n(z_0, \nu) > \frac{w(z_0)}{\omega_{\Gamma}(z_0)}.
\] (1.6)

We shall give a detailed proof for Theorem 1.1, the proof of Theorem 1.2 follows by simple changes. During the proof of the upper estimate in Theorem 1.1 we shall also verify (see Proposition 2.4)

\textbf{Theorem 1.3} Let \( \Gamma \) be a finite union of disjoint \( C^{1+\alpha} \) Jordan curves and arcs. Then there is a constant \( C \) and for every \( n = 1,2,\ldots \) there are monic polynomials \( P_n(z) = z^n + \cdots \) of degree \( n \) such that
\[
\| P_n \|_{\Gamma} \leq C \cap(\Gamma)^n,
\]
where \( \cap(\Gamma) \) denotes the logarithmic capacity of \( \Gamma \).

This should be compared to the fact (see e.g. [17, Theorem 5.5.4]) that for any \( n \) and monic polynomial \( P_n(z) = z^n + \cdots \) we have
\[
\| P_n \|_{\Gamma} \geq \cap(\Gamma)^n.
\]
Thus, the theorem says that on unions of smooth curves and arcs this theoretical lower bound can be achieved for every \( n \) disregarding a constant factor. For \( C^{2+\alpha} \) curves and arcs this follows from deep results of Widom [30]. Let us also mention that if there are at least two components, or \( \Gamma \) is a single smooth arc, then the better estimate
\[
\| P_n \|_{\Gamma} = (1 + o(1)) \cap(\Gamma)^n
\]
is impossible for all \( n \) ([27], [26]). It is a delicate problem (connected with simultaneous Diphantine approximation of the harmonic measures of the components of \( \Gamma \) ) how close one can get by the norm of monic polynomials of degree \( n \) to the theoretical lower bound \( \cap(\Gamma)^n \), see the papers [26] and [30].

First we shall deal with Theorem 1.1 in the special case when \( w \) is continuous and positive on \( \Gamma \). The general case of Theorem 1.1 will follow from this via a simple argument. The proof of the upper and lower estimates are distinctively different. The upper estimate will be obtained by a careful discretization of the equilibrium measure. That part of the proof holds at every Lebesgue-point of \( \nu \) (Lebesgue-point with respect to arc-measure) and the local \( C^2 \) property is not needed there. The lower estimate will be reduced to the case when there are no arc-components of \( \Gamma \).
2 Upper estimate for Christoffel functions

In this section we establish that

\[ \limsup_{n \to \infty} n \lambda_n(z_0, \nu) \leq \frac{d\nu(z_0)}{d\mu_\Gamma}. \]  

(2.1)

We need the concept of Lebesgue-point of a measure on \( \Gamma \). Thus, let \( \nu \) be a Borel-measure on \( \Gamma \) and \( d\nu(t) = w(t)ds(t) + d\nu_{\text{sing}} \) its decomposition into its absolutely continuous and singular parts with respect to arc measure \( s = s_\Gamma \). We say that \( z_0 \in \Gamma \), which is not an endpoint of an arc-component of \( \Gamma \), is a Lebesgue-point for \( \nu \) (with respect to arc measure) if for every \( \epsilon > 0 \) there is a \( \rho > 0 \) such that if \( 0 \leq \tau \leq \rho \) then

\[ \int_{|\zeta - z_0| \leq \tau} |w(\zeta) - w(z_0)|ds(\zeta) \leq \epsilon \tau \]  

(2.2)

and

\[ \nu_{\text{sing}}(\{\zeta \mid |\zeta - z_0| \leq \tau\}) \leq \epsilon \tau. \]  

(2.3)

Since the derivative of \( \nu_{\text{sing}} \) with respect to \( s_\Gamma \) is 0 \( s_\Gamma \)-almost everywhere (see [18, Theorem 7.13]), standard proof shows that \( s_\Gamma \)-almost every point is a Lebesgue-point for \( \nu \).

The main theorem of this part of the paper is

**Theorem 2.1** Let \( \Gamma \) be a finite union of disjoint \( C^{1+\alpha} \) Jordan curves or arcs lying exterior to one another, and \( \nu \) a Borel measure on \( \Gamma \). If \( z_0 \in \Gamma \) is not an endpoint of an arc-component of \( \Gamma \) and \( z_0 \) is a Lebesgue-point (with respect to arc measure \( s_\Gamma \) ) of \( \nu \), then

\[ \limsup_{n \to \infty} n \lambda_n(\nu, z_0) \leq \frac{d\nu(z_0)}{d\mu_\Gamma}. \]

For the proof of Theorem 2.1, let, as before, \( \Gamma_k \) be the disjoint components of \( \Gamma \) with \( \Gamma_0 \) being the one containing \( z_0 \). There is a change in the argument when \( \Gamma_0 \) is a Jordan arc as opposed to the case when it is a Jordan curve. First we consider the latter case, and return to the arc case after we have presented the proof for curves.

### 2.1 Part I: \( \Gamma_0 \) is a Jordan curve

Without loss of generality we may assume that \( z_0 = 0 \) and that the real line is the tangent line to \( \Gamma \) at 0. Then in a neighborhood of 0 the curve \( \Gamma_0 \) has a parametrization \( t + i\gamma(t) \) with \( \gamma' \in C^\alpha \) and \( \gamma(0) = 0, \gamma'(0) = 0 \); hence \( |\gamma'(t)| \leq C|t|^\alpha, |\gamma(t)| \leq C|t|^{1+\alpha} \).
Let $\theta_k = \mu_{\Gamma}(\Gamma_k)$, and for an $n$ consider the integers $n_k = [\theta_k n]$. Divide each $\Gamma_k$ into $n_k$ arcs $I_{kj}$ (for each $k$ the number of such $j$’s is $n_k$), each having equal weight $\theta_k/n_k$ with respect to $\mu_{\Gamma}$, i.e. $\mu_{\Gamma}(I_{kj}) = \theta_k/n_k$. Then

$$\left| \frac{\theta_k}{n_k} - \frac{1}{n} \right| = |\mu_{\Gamma}(I_{kj}) - 1/n| \leq C/n^2. \quad (2.4)$$

Let

$$\xi_j^k = \frac{1}{\mu_{\Gamma}(I_{kj})} \int_{I_{kj}} u \, d\mu_{\Gamma}(u) \quad (2.5)$$

be the center of mass with respect to $\mu_{\Gamma}$. Simple argument shows that on $\Gamma_0$ we can choose the $I_{0j}$’s so that the real part of one of the $\xi_0^j$’s lying close to $0$ is $0$, say $\Re \xi_0^0 = 0$. Indeed, since $\Gamma_0$ is a closed curve, the aforementioned subdivision can be started from any point on $\Gamma_0$, i.e. if $P \in \Gamma_0$ is any point then there is a unique subdivision $\sigma_P$ such that $P$ is one of the division points. Take now any subdivision $\sigma$, and in that subdivision let $0$ lie in the subarc $\hat{bc}$, with, say, $\Re b \leq 0$, $\Re c \geq 0$ (recall that at $0$ the $x$-axis is tangent to $\Gamma$), and let the two neighboring arcs of that subdivision be $\hat{ab}$ and $\hat{cd}$ with $\Re a < 0$, $\Re c > 0$. Call $a$ the left endpoint of $\hat{ab}$. Now if $P$ is moving on $\Gamma_0$ from $a$ to $c$ in a continuous manner, then the subarc $I(P)$ in $\sigma_P$ for which $P$ is its left endpoint moves from $\hat{ab}$ to $\hat{cd}$. Since the first one lies in the negative half-plane $\Re z \leq 0$, while the latter lies in the positive half-plane $\Re z \geq 0$, in the first case the center of mass lies in $\Re z < 0$, while in the second case it lies in $\Re z > 0$. Therefore, there will be a moment for which the center of mass of $I(P)$ lies on the imaginary axis, and then $\sigma_P$ is the required subdivision, and we select $I(P)$ as $I_0^0$. It then easily follows that $\xi_0^0$ lies closest to $0$ among the $\xi^j_k$’s.

Consider now the polynomial

$$R_n(z) = \prod_{j,k} (z - \xi_j^k) \quad (2.6)$$

of degree at most $n$. We claim that the polynomial

$$P_n(z) = R_n(z)/(z - \xi_0^0) \quad (2.7)$$

verifies Theorem 2.1. We prove this via a series of propositions.

In what follows $A \sim B$ means that the ratio $A/B$ is bounded away from zero and infinity.

**Proposition 2.2** $d\mu_{\Gamma}(t) = \omega_{\Gamma}(t)ds(t)$ with a positive density function $\omega_{\Gamma}$ which is $C^\alpha$-smooth away from the endpoints of the arc-components of $\Gamma$. If $E$ is an endpoint of an arc-component of $\Gamma$, then $\omega_{\Gamma}(z) \sim 1/|z - E|^{1/2}$ around $E$.

This is a standard result. When $\Gamma$ consists of one component which is a Jordan curve it immediately follows from the Kellogg-Warschawski theorem (see [16,
Figure 1: The choice of the intervals $I^k_j$ and of the points $\xi^k_j$.

Theorem 3.6). When $\Gamma$ consists of several components, we could not find it in the appropriate form in the literature, hence we will present a proof in the Appendix to this paper. Actually, the proof gives that around an endpoint $E$ of an arc-component of $\Gamma$ the function $\omega_\Gamma(t)|t - E|^{1/2}$ is a positive Lip $\alpha$ function.

Since away from endpoints of arc-components of $\Gamma$ the density $\omega_\Gamma$ is bounded away from 0 and infinity, it follows that away from the endpoints we have $s(I^k_j) \sim 1/n$, and if $a^k_j, b^k_j$ are the endpoints of the arc $I^k_j$, then in this case $|\xi^k_j - a^k_j| \sim 1/n$, $|\xi^k_j - b^k_j| \sim 1/n$ and $|\xi^k_j - \xi^k_i| \sim |j - i|/n$.

**Proposition 2.3** If $E$ is an endpoint of an arc-component of $\Gamma$, say $E \in I^k_1$ and $I^k_1, I^k_2, \ldots$ follow one another in this order on $\Gamma$, then $|\xi^k_j - E| \sim (j/n)^2$ and $s(I^k_j) \sim j/n^2$ in a neighborhood of $E$. Furthermore, if the endpoints of the arc $I^k_j$ are $a^k_j, b^k_j$ then

$$|\xi^k_j - a^k_j| \sim |\xi^k_j - b^k_j| \sim s(I^k_j) \sim j/n^2,$$

(2.8)

and

$$|\xi^k_j - \xi^k_i| \sim \frac{|j^2 - i^2|}{n^2}.$$  

(2.9)

See Figure 1.

**Proof.** Let $I^k_j$ be the arc $a^k_jb^k_j$ with $a^k_j$ lying closer to $E$. Then, by Proposition 2.2,

$$\frac{\theta^k_k}{n_k} = \int_{E\theta^k_j} \omega_\Gamma(t)ds(t) \sim \int_{E\theta^k_j} |t - E|^{-1/2}ds(t),$$

(2.10)
and since $|t - E| \sim s(\hat{E}t)$ we can continue this as
\[
\int_{\mathbb{R}^j} s(\hat{E}t)^{-1/2} ds(t) \sim s(\hat{E}b_j^k)^{1/2} \sim |E - b_j^k|^{1/2}.
\]
Therefore, $|E - b_j^k| \sim (j/n)^2$ and $s(I_j^k) \sim 1/n^2$ follow because $\theta_k/n_k \sim 1/n$.

Since $a_j^k = b_j^k - 1$, we also get for $j \geq 2$ the relation $|E - a_j^k| \sim (j/n)^2$. Therefore, for $j \geq 2$
\[
\frac{\theta_k}{n_k} = \int_{a_j^k b_j^k} \omega_t(t) ds(t) \sim \int_{a_j^k b_j^k} ((j/n)^2)^{-1/2} ds(t) \sim s(I_j^k)(n/j),
\]
which, in view again of $\theta_k/n_k \sim 1/n$, gives $s(I_j^k) \sim j/n^2$.

Since $\xi_j^k$ lies close to $I_j^k$, $|\xi_j^k - E| \sim (j/n)^2$ is immediate for $j \geq 2$. To prove it for $j = 1$ we may assume temporarily (i.e. just for the proof of this relation) that $E = 0$ and $\mathbb{R}_+$ is the half-tangent to the arc $\Gamma_k$ of $\Gamma$. Let the orthogonal projection of the arc $I_j^k$ onto the real line be $[0, d]$. Then, as we have just seen, $d \sim 1/n^2$, and $\Re \xi_j^k$ is the center of mass of a measure $\rho(t)dt$ on $[0, d]$ for which $\rho(t) \sim t^{-1/2}$. Elementary estimate shows then that $\Re \xi_j^k/d$ is bounded away from 0 and infinity (no matter how small $d$ is), which combined with $\text{diam}(I_j^k) \sim 1/n^2$ yields the desired estimate $|\xi_j^k| \sim (1/n)^2$.

The same argument verifies (2.8), while (2.9) follows from the other statements in the proposition: for example if $i < j \leq 2i, i \neq j$ then
\[
|\xi_j^k - \xi_i^k| \sim \mathcal{S}(a_j^k b_j^k) = \sum_{\tau = i}^j s(I_j^k) \sim \sum_{\tau = i}^j (\tau/n^2) \sim (j^2 - i^2)/n^2,
\]
while if $j > 2i$ then (use also the preceding relation with $j = 2i$)
\[
|\xi_j^k - \xi_i^k| \sim |E - \xi_j^k| \sim j^2/n^2 \sim (j^2 - i^2)/n^2.
\]

\[\]  

**Proposition 2.4** For the polynomials (2.6) we have
\[
\|R_n\|_1 \leq C \text{Cap}(\Gamma)^n
\]  
with some $C$ independent of $n$.

This almost proves Theorem 1.3, the only problem is that the degree of $R_n$ is $\sum_k [\theta_k n]$, which may be smaller than $n$ but at most by $k_0$. To have exact degree $n$ one should divide some of the $\Gamma_k$’s into not $[\theta_k n]$ but $[\theta_k n] + 1$ parts so as to get totally $n$ arcs, and proceed as below.
Proof. By Frostman’s theorem (see [17, Theorem 3.3.4])

$$\int \log |z - t| \, d\mu_T(t) = \log \text{cap}(\Gamma), \quad z \in \Gamma.$$  \hspace{1cm} (2.11)

Note that (with $\log^+ = \max(0, \log)$)

$$\int \log^+ |z - t| \, d\mu_T(t) \leq \log^+ \text{diam}(\Gamma),$$

hence

$$\int |\log |z - t|| \, d\mu_T(t) \leq \log^+ \text{diam}(\Gamma) - \log \text{cap}(\Gamma).$$  \hspace{1cm} (2.12)

Now we write in view of (2.11)

$$n \log \text{cap}(\Gamma) = \sum_{j,k} \left( n - \frac{1}{\mu_T(I^l_j)} \right) \int_{I^l_j} \log |z - t| \, d\mu_T(t)$$

$$+ \sum_{j,k} \frac{1}{\mu_T(I^l_j)} \int_{I^l_j} \log |z - t| \, d\mu_T(t) = \Sigma_1 + \Sigma_2.$$  \hspace{1cm} (2.13)

Here, by (2.4) and (2.12),

$$|\Sigma_1| \leq \sum_{j,k} O(1) \left| \int_{I^l_j} \log |z - t| \, d\mu_T(t) \right| = O(1).$$  \hspace{1cm} (2.14)

Therefore, to prove the claim we have to show that on $\Gamma$

$$\log |R_n(z)| - \Sigma_2 = \sum_{j,k} \frac{1}{\mu_T(I^l_j)} \int_{I^l_j} \log \left| \frac{z - \xi_j^k}{z - t} \right| \omega_T(t) \, ds(t) \leq C.$$  \hspace{1cm} (2.15)

The proof uses the idea of [19, Theorem VI.4.2]. It is more involved around endpoints of arc-components of $\Gamma$, so we give it only there. Thus, let $z$ lie in an arc $I^l_{j_0}$ that lies around an endpoint $E$ of an arc-component $\Gamma_l$ of $\Gamma$, on which, say, the arcs $I^l_j$ are following each other in the order $I^l_1, \ldots, I^l_{j_0}, \ldots$ with $I^l_{j_0}$ containing $E$. $z$ and $(j_0, l)$ will always have this meaning below. We consider the sum

$$\sum_{(j,k) \neq (j_0,l)} \frac{1}{\mu_T(I^l_j)} \int_{I^l_j} \log \left| \frac{z - \xi_j^k}{z - t} \right| \omega_T(t) \, ds(t) =: \sum_{(j,k) \neq (j_0,l)} L_{j,k}(z),$$  \hspace{1cm} (2.16)

and prove that it is uniformly bounded (both from below and above). Note that this sum differs from the one on the right of (2.15) in one term (the term with
Figure 2: The position of $z, a, b$

integral over $I_{j_0}^l$ is missing), and we shall actually show that not just the sum, but also the sum consisting of the absolute values $|L_{j,k}|$ is uniformly bounded, i.e.

$$\sum_{(j,k) \neq (j_0,l)} |L_{j,k}(z)| = O(1). \quad (2.17)$$

First we verify that the individual terms $L_{j,k}(z)$ in (2.16) are uniformly bounded. This is clear for $k \neq l$ (i.e. when $I_k^l$ is on a different component of $\Gamma$ than $z$) or for $k = l$ but $j \neq j_0 \pm 1$ (the $j = j_0$ term is not in the sum), for then in the integrand

$$|z - \xi_{j,k}^l| \sim \text{dist}\{I_{j_0}^l, I_k^l\} \sim |z - t| \quad \text{for all } t \in I_k^l.$$

So let $j = j_0 \pm 1$, say $j = j_0 + 1$. Then we know from Proposition 2.3 that $|z - \xi_{j_0+1}^l| \sim s(I_{j_0+1}^l) \sim j_0/n^2$, and from Propositions 2.2 and 2.3 that $\omega_\Gamma(t) \leq Cn/j_0$ on $I_{j_0+1}^l$. Let $I_{j_0+1}^l$ be the arc $\hat{ab}$, see Figure 2. Clearly

$$L_{j_0+1,l}(z) = \frac{1}{\mu_\Gamma(I_{j_0+1}^l)} \int_{I_{j_0+1}^l} \log \left| \frac{z - \xi_{j_0+1}^l}{z - t} \right| \omega_\Gamma(t) ds(t)$$

$$\leq Cn \frac{n}{j_0} \int_{I_{j_0+1}^l} \left( \log |z - \xi_{j_0+1}^l| + \log \frac{1}{|a - t|} \right) ds(t). \quad (2.18)$$

Here

$$\int_{I_{j_0+1}^l} \log \frac{1}{|a - t|} ds(t) \leq \int_{I_{j_0+1}^l} \log \frac{C_0}{s(at)} ds(t) = s(I_{j_0+1}^l) (\log C_0 + 1 - \log s(I_{j_0+1}^l)).$$

Therefore, the integral on the right of (2.18) equals

$$s(I_{j_0+1}^l) \log \frac{|z - \xi_{j_0+1}^l|}{s(I_{j_0+1}^l)} + O(s(I_{j_0+1}^l)) \leq Cs(I_{j_0+1}^l) \leq C \frac{j_0}{n^2}.$$
If we substitute this into (2.18) then we obtain the boundedness of $L_{j_0+1,l}(z)$ from above. Its boundedness from below is clear since for $z \in I_{j_0}^1$, $t \in I_{j_0+1}^1$ we have

$$\left| \frac{z - \xi_{j_0+1}}{z - t} \right| \geq c > 0$$  \hspace{1cm} (2.19)

by (2.8).

The case $j = j_0 - 1$ is similar provided $j_0 - 1 > 1$, but for $j_0 - 1 = 1$, we must proceed somewhat differently, for then $\omega_T(t) \leq Cn/j_0$ is no longer true on $I_1^1$. In this case (i.e. when $I_{j_0-1}^1 = I_1^1$) we have $\mu(I_1^1) \sim 1/n \sim s(I_1^1)^{1/2}$, $|z - t| \sim s(\hat{z}t)$, so

$$L_{1,l} \leq \frac{C}{s(ab)^{1/\alpha}} \int_{ab} \log \frac{Cs(ab)}{s(zt)} \frac{s(at)^{-1/2}}{ds(t)},$$

and the right-hand side will be shown to be bounded from above in the proof of (2.24)–(2.25) (the boundedness from below of $L_{1,l}$ follows again from (2.19)).

These prove the uniform boundedness of the individual terms $L_{j,k}$, $(j,k) \neq (j_0,l)$.

It follows from Proposition 2.3 that there is an $M$ such that if either $k \neq l$ or $k = l$ but $|j - j_0| \geq M$ then for $z \in I_{j_0}^1$ and $t \in I_{j_0}^k$ we have

$$\left| \frac{\xi_j - t}{z - \xi_j} \right| \leq \frac{1}{2}$$

(a closer look at the proof of Propositions 2.2 and 2.3 reveals that $M = 4$ suffices for large $n$, but we do not need the best value of $M$).

Thus, in this case for the integrands in $L_{j,k}(z)$ we get (use that log $|1-u| = \Re \log(1-u)$ with any local branch of the log)

$$\log \left| \frac{z - \xi_j^k}{z - t} \right| = -\log \left| 1 - \frac{\xi_j^k - t}{z - \xi_j^k} \right| = \Re \frac{\xi_j^k - t}{z - \xi_j^k} + O \left( \left| \frac{\xi_j^k - t}{z - \xi_j^k} \right|^2 \right).$$

Therefore, for such $j$ and $k$ we have

$$|L_{j,k}(z)| = \frac{1}{\mu_T(I_j^k)} \int_{I_j^k} O \left( \left| \frac{\xi_j^k - t}{z - \xi_j^k} \right|^2 \right) d\mu_T(t) = O \left( \frac{s(I_j^k)^2}{|\xi_j^k - \xi_{j_0}^k|^2} \right), \hspace{1cm} (2.20)$$

because the integral

$$\int_{I_j^k} \Re \frac{\xi_j^k - t}{z - \xi_j^k} d\mu_T(t) = \Re \int_{I_j^k} \frac{\xi_j^k - t}{z - \xi_j^k} d\mu_T(t)$$
vanishes by the choice of $\xi^k_j$.

The expression on the right of (2.20) is bounded by a constant times $s(I^k_j)^2$ when $k \neq l$ or $k = l$ but $I^k_j$ is far from $E$ (say farther than a fixed constant $\delta > 0$), and for $k = l$ and $I^k_j$ close to $E$ (say for $|\xi^k_j - E| \leq \delta$) it is at most (see Proposition 2.3) a constant times

$$\frac{s(I^k_j)^2}{|j/n|^2 - (j_0/n)^2|^2} \sim \frac{|j/n|^2 - (j_0/n)^2|^2}{|j^2 - j_0^2|^2}.$$  

All in all, if we take into account the uniform boundedness of the terms $L_{j,k}$ we obtain that the sum in (2.17) is at most

$$\sum_{|j-j_0| \leq M, j \neq j_0} |L_{j,l}| + \sum_{|j-j_0| > M} |L_{j,l}| + \sum_{j,k, k \neq l} |L_{j,k}|$$

$$\leq (2M)C + C \sum_{|j-j_0| > M} \frac{j^2}{|j^2 - j_0^2|^2} + C \sum_{j,k} s(I^k_j)^2 \leq C.$$  

To complete the proof of the proposition we have to show that the additional term

$$\frac{1}{\mu_{\Gamma}(I^j_{j_0})} \int_{I^j_{j_0}} \log \left|\frac{z - \xi^j_{j_0}}{z - t}\right| \omega_{\Gamma}(t) ds(t) \quad (2.21)$$

in (2.15) is also bounded from above (from below we cannot claim boundedness for $z$ can be very close to $\xi^j_{j_0}$). As before, we get from Proposition 2.3 that for $j_0 > 1$ this term is at most

$$Cn \int_{I^j_{j_0}} \left(\log \frac{Cs(I^j_{j_0})}{s(zt)}\right) \left(\frac{j_0^2}{n^2}\right)^{-1/2} ds(t),$$

which, with $I^j_{j_0} := \widehat{ab}$, equals

$$C \int_{j_0} \left(\log \frac{s(\widehat{ab})}{s(zt)}\right) \left(\frac{j_0^2}{n^2}\right)^{-1/2} ds(t).$$

Now we use for $0 \leq x \leq y \leq 1$ the inequality

$$-\frac{2}{e} (x + y) \leq x \log x + y \log y - (x + y) \log(x + y) \leq 0,$$  

(2.23)

which is immediate from the concavity of $\log$ and from the fact that on the interval $(0, 1)$ the minimum of $t \log t$ is $-1/e$. Apply (2.23) with $s(\widehat{z}b), s(\widehat{a}z)$ in place of $x, y$ (in which case $x + y = s(\widehat{ab})$) to continue (2.22) as

$$\leq C \int_{j_0} \left(\log \frac{s(\widehat{ab})}{s(\widehat{a}z)}\right) \left(\frac{j_0^2}{n^2}\right)^{-1/2} ds(t) \leq C.$$  

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where, in the last step we used that by Proposition 2.3, \( s(\hat{ab}) = s(I_{j_0}^l) \sim j_0/n^2 \). This gives the required estimate for (2.21) when \( j_0 > 1 \).

When \( j_0 = 1 \) then \( E \) is an endpoint of the arc \( I_{j_0}^l \), e.g. \( E = a \). In that case \( \omega_I \) is not bounded on \( I_{j_0}^l \), so we have to proceed differently than before. Similarly as above, now we have with \( s(\hat{ab}) = s(I_{j_0}^l) \sim 1/n^2, \mu_I(\hat{I}_1^l) \sim 1/n \sim s(\hat{ab})^{1/2} \), the bound

\[
C \frac{s(ab)^{1/2}}{s(ab)} \frac{Cs(ab)}{s(zt)} s(zt)^{-1/2} ds(t) =: I \quad (2.24)
\]

for the expression in (2.21). Recall that \( z \) lies on the arc \( \hat{ab} = \hat{E}b \), and let \( w \) be the midpoint on the arc \( \hat{E}z \) in the sense that \( s(\hat{E}w) = s(\hat{E}z) \), see Figure 3. Now we split the integral in (2.24) over \( \hat{ab} \) into three parts: the integrals over \( \hat{zb}, \hat{wz} \) and \( \hat{Ew} \). For the first we have (use that the antiderivative of \( t^{-1/2} \log t \) is \( 2t^{1/2} \log t - 4t^{1/2} \))

\[
\int_{\hat{z}b} \log \frac{Cs(ab)}{s(zt)} s(zt)^{-1/2} ds(t) \leq \int_{\hat{z}b} \log \frac{Cs(ab)}{s(zt)} s(zt)^{-1/2} ds(t)
\]

\[
= 2 \log(Cs(ab))s(zt)^{1/2} - 2s(zt)^{1/2} \log s(zt) + 4s(zt)^{1/2} \leq Cs(ab)^{1/2}
\]

because, for any \( C_0 > e^2 \) (by the monotonicity of \( x^{1/2} \log 1/x \) on \( (0, e^{-2}) \)),

\[
2s(zt)^{1/2} \log C_0 s(ab) / s(zt) \leq 2s(ab)^{1/2} \log C_0 s(ab) / s(ab) = 2s(ab)^{1/2} \log C_0.
\]

The integral over \( \hat{w}z \) can be similarly handled. Finally, for the integral over \( \hat{Ew} \) we have the bound

\[
\int_{\hat{E}w} \log \frac{Cs(ab)}{s(Ew)} s(Ew)^{-1/2} ds(t) \leq \log \frac{Cs(ab)}{s(Ew)} 2s(Ew)^{1/2} \leq \log \frac{Cs(ab)}{s(ab)} 2s(ab)^{1/2}
\]

\[
= 2(\log C)s(ab)^{1/2}.
\]
Substituting all these into (2.24) we get

\[ I \leq C, \tag{2.25} \]

and with this the upper boundedness of (2.21) for \( j_0 = 1 \), as well.

\[ \square \]

**Proposition 2.5** For the polynomials from (2.7) we have

\[ |P_n(z)| \sim n \cap(\Gamma)^n \tag{2.26} \]

uniformly in \( n \) and \( z \in I_0^0 \), in particular

\[ |P_n(0)| \sim n \cap(\Gamma)^n. \tag{2.27} \]

For \( z \in \Gamma \setminus I_0^0 \)

\[ |P_n(z)| \leq C \cap(\Gamma)^n \frac{1}{|z|} \tag{2.28} \]

with some \( C \) independent of \( n \).

**Proof.** Let \( z \in I_0^0 \), i.e. with the notations of the preceding proof we have \( l = i_0 = 0 \). By the proof of Proposition 2.4 (see in particular (2.11)–(2.14)) and (2.17)) we have uniformly in \( n \) and \( z \in I_0^0 \)

\[ \log |P_n(z)| - n \log \cap(\Gamma) + \frac{1}{\mu(\Gamma(I_0^0))} \int_{I_0^0} \log |z - t|\omega(\Gamma(t))ds(t) = O(1). \tag{2.29} \]

Now use that \(|z - t| = (1 + o(1))s(\hat{z}t) \) for \( z - t \sim 0 \) to get with \( I_0^0 =: \hat{a}b \) for the last term in (2.29)

\[ \frac{1}{\mu(\Gamma(I_0^0))} \int_{I_0^0} \log |z - t|\omega(\Gamma(t))ds(t) = \frac{1}{\mu(\Gamma(I_0^0))} \int_{I_0^0} \log \left(1 + o(1)\right)s(\hat{z}t)\omega(\Gamma(t))ds(t) \]

\[ = o(1) + \frac{1}{\mu(\Gamma(I_0^0))} \int_{I_0^0} \log s(\hat{z}t)\omega(\Gamma(t))ds(t). \]

Here we need that for \( t \in I_0^0 \) Proposition 2.2 yields

\[ |\omega(\Gamma(t) - \omega(\Gamma(0)) \leq C|t|^\alpha \leq Cn^{-\alpha} \]

to continue the preceding estimates as

\[ = o(1) + \frac{\omega(\Gamma(0))(1 + O(n^{-\alpha}))}{\mu(\Gamma(I_0^0))} \int_{I_0^0} \log s(\hat{z}t)ds(t) \]
\[ \frac{\omega_\Gamma(0)}{\mu_\Gamma(I_0^0)}(s(\hat{a}z) \log s(\hat{a}z) + s(\hat{b}) \log s(\hat{b}) - s(\hat{ab})) = o(1) + (1 + O(n^{-\alpha}))(s(\hat{a}z) \log s(\hat{a}z) + s(\hat{b}) \log s(\hat{b}) - s(\hat{ab})) \]

where, in the last step we used again (2.23) with \(s(\hat{a}z)\) and \(s(\hat{b})\) playing the role of \(x, y\) (note that then \(x + y = s(\hat{ab})\)).

Since \(\omega_\Gamma(0) = 1 + O(n^{-\alpha})\) and \(\log s(\hat{ab}) = O(1) - \log n\) are also true (the latter one follows from the first one in view of \(\mu_\Gamma(I_0^0) = (1 + o(1))/n\)), finally we can conclude

\[ \frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log |z - t| \omega_\Gamma(t) ds(t) = O(1) - \log n. \]

This and (2.29) prove (2.26).

The claim (2.28) follows immediately from Proposition 2.4, for \(|z - \xi_0^0| \sim |z|\) when \(z \in \Gamma \setminus I_0^0\).

Label the points \(\xi_0^0\) around \(0\) in such a way that, as their real part increases, they follow each other in the order

\[ \cdots < \Re \xi_{-2} < \Re \xi_{-1} < 0 = \Re \xi_0^0 < \Re \xi_1^0 < \Re \xi_2^0 < \cdots. \]

We may also assume that this labeling is such that the range of \(j\) includes all integers in \([-\tau n, \tau n]\) for some \(\tau > 0\).

**Proposition 2.6** For all \(j\) we have

\[ \left| \frac{\xi_0^0 - j}{n \omega_\Gamma(0)} \right| \leq C \left( \frac{|j|}{n} \right)^{1+\alpha}. \]  

**Proof.** Enough to prove this for \(|\xi_0^0| \leq \delta\) with some small \(\delta > 0\) (otherwise the discussion below gives \(|j| \geq \epsilon \delta n\) and then the statement is obvious).

Recall that the real line is the tangent line to \(\Gamma\) at \(0\) and in a neighborhood of \(0\) the curve \(\Gamma\) has a parametrization \(t + i\gamma(t)\) with \(\gamma' \in C^\alpha\) and \(\gamma(0) = 0, \gamma'(0) = 0, |\gamma'(t)| \leq C|t|^\alpha, |\gamma(t)| \leq C|t|^{1+\alpha}\).
Let $u = t + i\gamma(t) \in \Gamma$. Then
\[ ds(u) = \sqrt{1 + (\gamma'(t))^2}dt = dt + O(|u|^{2\alpha})dt. \tag{2.31} \]

Let $a, b$ be the endpoints of $I_0^0$. We know that $|a|, |b| \sim 1/n$ (this is immediate from the facts that $\Re a_0 = 0$, the equilibrium density $\omega_{\Gamma}$ is continuous and positive at 0, and $ds(u) \sim dt$ by (2.31)). We can write
\[
0 = \Re a_0 = \frac{n_0}{\theta_0} \int_{\Re a}^{\Re b} t \omega_{\Gamma}(0) dt + O(n^{-1-\alpha})
\]
and hence
\[
\Re a_0 = \frac{n_0}{\theta_0} \frac{1}{2} ((\Re b)^2 - (\Re a)^2) + O(n^{-1-\alpha}),
\]
from which it follows that (note $n \sim n_0$, $\Re b - \Re a \sim 1/n$)
\[
|\Re b + \Re a| = O(n^{-1-\alpha}).
\]

Now let $t_0 = \Re(a + b)/2 + i\gamma(\Re(a + b)/2)$. (2.31) implies
\[
s(t_0b) = \Re b - \Re t_0 + O(n^{-1-\alpha}); \quad s(a t_0) = \Re t_0 - \Re a + O(n^{-1-\alpha})
\]
and hence
\[
s(t_0b) - s(a t_0) = O(n^{-1-\alpha}).
\]

Therefore, if $\xi_j^0$ is the midpoint of the arc $I_j^0$ with respect to arc length, then $|t_0 - \xi_0| \leq C n^{-1-\alpha}$.

Since $|t_0 - \xi_0| \leq |t_0| + |\xi_0| \leq C n^{-1-\alpha}$ is also true, finally we obtain $|\xi_j^0 - \xi_0^j| \leq C n^{-1-\alpha}$. Note that by the definition of $\xi_j^0$ and the $C^\alpha$-smoothness of $\omega_{\Gamma}$ we also have
\[
\mu_{\Gamma}(\widehat{a \xi_0}) = \frac{1}{2} \mu_{\Gamma}(\widehat{ab}) + O(n^{-\alpha})
\]
\[
\mu_{\Gamma}(\widehat{\xi_0 b}) = \frac{1}{2} \mu_{\Gamma}(\widehat{ab}) + O(n^{-\alpha})
\]

Since $\xi_j^0, \xi_0^j$ are geometric quantities defined in terms of $\omega_{\Gamma}$ and $s_{\Gamma}$, the same argument can be given for all $j$ and we obtain
\[
|\xi_j^0 - \xi_0^j| \leq C n^{-1-\alpha}, \quad |\xi_j| \leq \delta, \tag{2.32}
\]
and (with \(a_j, b_j\) being the endpoints of \(I_j^0\))

\[
\mu_{\Gamma}(a_j, \xi_j^0) = \frac{1}{2} \mu_{\Gamma}(a_j, b_j) + O(n^{-\alpha})
\]

\[
\mu_{\Gamma}(\xi_j^0, b_j) = \frac{1}{2} \mu_{\Gamma}(a_j, b_j) + O(n^{-\alpha}).
\]

These latter imply for \(j \neq 0\), say for \(j > 0\),

\[
\frac{j}{n_0} = \mu \left( \bigcup_{j=0}^{j-1} I_j^0 \right) = \mu_{\Gamma}(\xi_0, \xi_j^0) + O(n^{-1-\alpha})
\]

\[
= \int_{\xi_0}^{\xi_j^0} \omega_{\Gamma}(0) ds(u) + O(\|\xi_j^0\|^{1+\alpha}) + O(n^{-1-\alpha})
\]

\[
= \int_{\xi_0}^{\Re \xi_j^0} \omega_{\Gamma}(0) dt + O(\|\xi_j^0\|^{1+\alpha}) = (\Re \xi_j^0 - \Re \xi_0) \omega_{\Gamma}(0) + O(\|\xi_j^0\|^{1+\alpha})
\]

\[
= (\xi_j^0 - \xi_0) \omega_{\Gamma}(0) + O(\|\xi_j^0\|^{1+\alpha}),
\]

which, in view of \(\|\xi_0\| \leq C n^{-1-\alpha}\), \(\|\xi_j^0\| \leq C j/n\) and (2.4) implies

\[
\left| \frac{j}{n} - \frac{\Re \xi_j^0}{\omega_{\Gamma}(0)} \right| \leq C \left( \frac{|j|}{n} \right)^{1+\alpha}.
\]

The argument for negative \(j\) is just the same. Finally, this inequality combined with (2.32) gives (2.30).

Fix a large integer number \(M\) and a small \(\rho > 0\), so small that even \(\rho^\alpha M\) is small. Let

\[
Q_n(z) = \prod_{M^\alpha < |j| \leq n \rho} (z - \xi_j^0).
\]  

(2.33)

**Proposition 2.7.** For \(z \in \Gamma\), \(|z| \leq M/n\) we have

\[
\left\| \frac{Q_n(z)}{Q_n(0)} \right\| - 1 \leq \left( C M \rho^\alpha + \frac{1}{M} \right)
\]

(2.34)

with a \(C\) that depends only on \(\Gamma\).

**Proof.**

\[
\log \left| \frac{Q_n(z)}{Q_n(0)} \right| = \sum_{M^\alpha < |j| \leq n \rho} \log \left| 1 - \frac{z}{\xi_j^0} \right|.
\]
and on applying (2.30) this can be written as

\[
\sum_{M^3 < j \leq \rho n} \log \left| \frac{z}{j/n\omega_T(0) + O((j/n)^{1+\alpha})} \right| \left( 1 - \frac{z}{j/n\omega_T(0) + O((j/n)^{1+\alpha})} \right) = \sum_{M^3 < j \leq \rho n} \log \left| 1 + \frac{O(|j/n|^{1+\alpha}) + O(|z|^2)}{(j/n)^2} \right| \log \left| 1 - \frac{z}{j/n\omega_T(0) + O((j/n)^{1+\alpha})} \right| \left( 1 - \frac{z}{j/n\omega_T(0) + O((j/n)^{1+\alpha})} \right)
\]

\[
= \sum_{M^3 < j \leq \rho n} O \left( |j/n|^{\alpha-1} + \frac{|z|^2}{|j/n|^2} \right) = O \left( (M/n)n^{1-\alpha}(\rho n)^{\alpha} + \frac{(M/n)^2}{M^3/n^2} \right),
\]

from which the claim follows.

\[\Box\]

The key statement in the proof of Theorem 2.1 is

**Proposition 2.8** Let \( \Gamma_\delta \) be the part of \( \Gamma \) that lies of distance \( \geq \delta \) from the origin. Then

\[
\lim_{\delta \to 0} \Re \int_{\Gamma_\delta} \frac{d\mu_\Gamma(u)}{u} = 0. \tag{2.35}
\]

The statement is that the real part of the principal value integral

\[
\text{PV} \int_{\Gamma} \frac{d\mu_\Gamma(u)}{u} \tag{2.36}
\]

is zero at 0. This is due to the fact that the tangent line to \( \Gamma \) at 0 is horizontal.

**Proof.** Let \( \Gamma_\delta \) be the complementary arc, i.e. the set of points on \( \Gamma \) which are closer to 0 than \( \delta \). With some local branch of \( \log \) we have to show that

\[
\lim_{\delta \to 0} \Re \int_{\Gamma_\delta} (\log(z - u))' \big|_{z = 0} d\mu_\Gamma(u) = 0.
\]

Here, with \( z = x + i\gamma(x) \in \Gamma, u = t + i\gamma(t) \in \Gamma \) (with some global \( t + i\gamma(t) \) parametrization of \( \Gamma \) that extends the local parametrization \( t + i\gamma(t) \) around the origin, see the discussion before (2.4))

\[
\Re \int_{\Gamma_\delta} (\log(z - u))' \big|_{z = 0} d\mu_\Gamma(u) =
\]

\[
= \lim_{x \to 0} \frac{1}{x + i\gamma(x)} \int_{t + i\gamma(t) \in \Gamma_\delta} \log \left| \frac{(x + i\gamma(x)) - (t + i\gamma(t))}{t + i\gamma(t)} \right| d\mu_\Gamma(t + i\gamma(t)).
\]

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Since the whole integral
\[ \int_{\Gamma} \log |z - u| d\mu_{\Gamma}(u) \] (2.37)
is constant on \( \Gamma \) (see (2.11)), what we need to show is that the previous expression with \( \Gamma_\delta \) replaced by \( \Gamma_{\delta_1} \) tends to 0 as \( \delta \to 0 \) (in this case the existence of the limit/derivative follows from what we have just done and from the constancy of (2.37)).

Since \( x/(x + i\gamma(x)) \to 1 \) as \( x \to 0 \), we need to show that
\[ \frac{1}{x} \int_{t + i\gamma(t) \in \Gamma_{\delta}} \log \frac{|(x + iy(x)) - (t + iy(t))|}{|t + iy(t)|} d\mu_{\Gamma}(t + iy(t)) =: \frac{1}{x} I \] (2.38)
is as small in absolute value as we wish for small \( |x| \) and small, but fixed \( \delta > 0 \).

Without loss of generality assume \( x > 0 \). Let the endpoints of \( \Gamma_{\delta} \) be \( -\delta_1 + i\gamma(-\delta_1) \) and \( \delta_2 + i\gamma(\delta_2) \), \( \delta_1, \delta_2 > 0 \). Then \( \delta_1^2 + \gamma(\delta_1)^2 = \delta_2^2 \), and hence, in view of \( \gamma(\delta_j) = O(\delta_1^{1+\alpha}) \), we have
\[ \delta_j = \delta + O(\delta_1^{1+\alpha}), \quad j = 1, 2. \] (2.39)

With some large \( N \)
\[ I = \int_{-\delta_1}^{\delta_2} \log \frac{|(x + iy(x)) - (t + iy(t))|}{|t + iy(t)|} \omega_{\Gamma}(t + iy(t)) \sqrt{1 + (\gamma'(t))^2} dt \]
\[ = \int_{-\delta_1}^{-N_x} + \int_{-N_x}^{N_x} + \int_{N_x}^{\delta_2} = I_1 + I_2 + I_3. \]

First we deal with \( I_2 \). It is the sum of
\[ I_{21} = \int_{-N_x}^{N_x} \log \frac{|(x + iy(x)) - (t + iy(t))|}{|x - t|} \omega_{\Gamma}(t + iy(t)) \sqrt{1 + (\gamma'(t))^2} dt, \]
\[ I_{22} = -\int_{-N_x}^{N_x} \log \frac{t + iy(t)}{|t|} \omega_{\Gamma}(t + iy(t)) \sqrt{1 + (\gamma'(t))^2} dt \]
and
\[ I_{23} = \int_{-N_x}^{N_x} \log \frac{|x - t|}{|t|} \omega_{\Gamma}(t + iy(t)) \sqrt{1 + (\gamma'(t))^2} dt. \]

In \( I_{21} \) the log term is \( \log(1 + O((N_x)^\alpha)) = O((N_x)^\alpha)) \) because, with some \( \zeta \in [-N_x, N_x], \)
\[ |\gamma(x) - \gamma(t)| = |x - t||\gamma'(\zeta)| \leq |x - t|C(N_x)^\alpha, \]

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so $I_{21} = O(N^{1+\alpha}x^{1+\alpha}) = o(x)$ as $x \to 0$. Similarly, $I_{22} = o(x)$. Finally,

$$I_{23} = \int_{-N x}^{N x} \log \frac{|x - t|}{|t|} \left( \omega_T(t + i\gamma(t))\sqrt{1 + (\gamma'(t))^2} - \omega_T(0) \right) dt + \omega_T(0) \int_{-N}^{N} x \frac{dt}{t} = I_{231} + I_{232}.$$  

The factor after the log term in $I_{231}$ is in absolute value $\leq C|t|\alpha \leq C(Nx)^\alpha$ and

$$\int_{-N x}^{N x} \left| \log \frac{|x - t|}{|t|} \right| dt = x \int_{-N}^{N} \left| \log \frac{1 - t}{|t|} \right| dt \leq Cx \log N,$$

hence $I_{231} = O(N^\alpha(\log N)x^{1+\alpha}) = o(x)$. For $I_{232}$, as simple calculation shows, we can write

$$\frac{|I_{232}|}{\omega_T(0)} = \int_{-N x}^{N x} \log \frac{x + t}{t} dt \leq \int_{(N - 1)x}^{N x} \frac{x}{t} dt \leq \frac{x}{N - 1}.$$  

So $|I_2| \leq \omega_T(0)x/(N - 1) + o(x)$.

For $I_1 + I_3$ we set $J = [-\delta_1, -Nx] \cup [Nx, \delta_2]$ and note that the log term in the integrals in $I_1$ and $I_3$ is

$$\Re \log \left( 1 - \frac{x + i\gamma(x)}{t + i\gamma(t)} \right) = -\Re \frac{x + i\gamma(x)}{t + i\gamma(t)} + O \left( \left( \frac{x}{t} \right)^2 \right)$$

$$= -\frac{xt + \gamma(x)\gamma(t)}{t^2 + (\gamma(t))^2} + O \left( \left( \frac{x}{t} \right)^2 \right)$$

$$= -\frac{x}{t} + xO \left( \frac{(\gamma(t))^2}{t^3} \right) + O \left( \frac{(\gamma(x)\gamma(t))}{t^2} \right) + O \left( \left( \frac{x}{t} \right)^2 \right)$$

Here on the right $\gamma(t)^2/t^3$ and $\gamma(t)/t^2$ are integrable, so the contribution to the integral over $\Gamma_\delta$ of the corresponding terms is $x_0(1)$ and $\gamma(x)\gamma_0(1) = x_0(1)$, respectively, where $o(1)$ means a quantity tending to 0 as $\delta \to 0$. The contribution of the term $O(x^2/t^2)$ is

$$\int_J O \left( \left( \frac{x}{t} \right)^2 \right) dt = O \left( \frac{x^2}{Nt} \right) = O \left( \frac{x}{N} \right).$$

Finally, the integral over $J$ of the term $-x/t$ is equal to

$$-\int_J \left( \omega_T(t + i\gamma(t))\sqrt{1 + (\gamma'(t))^2} - \omega_T(0) \right) dt + \omega_T(0) \int_{-N}^{N} \frac{x}{t} dt = I_4 + I_5.$$
In $I_4$ we have

$$\omega_T(t + i\gamma(t))\sqrt{1 + (\gamma'(t))^2} - \omega_T(0) \leq C|t|^\alpha,$$

so exactly as before $I_4 = xo_\delta(1)$. Finally,

$$|I_5| = \omega_T(0)x\int_{\min(\delta_1, \delta_2)}^{\max(\delta_1, \delta_2)} \frac{1}{t}dt \leq Cx \frac{\delta^{1+2\alpha}}{\delta} = xo_\delta(1)$$

where we used (2.39). All in all, we have

$$|I| \leq xo_\delta(1) + o(x) + O(x/N)$$

which shows that the term in (2.38) is as small as we wish if we select $N$ large and then $\delta > 0$ small (and also $x$ sufficiently small after these selections).

\[\square\]

**Proposition 2.9** Let

$$S_n(z) = \prod_{|\xi^k_j| \geq \delta} (z - \xi^k_j).$$

Then, for fixed $M$ and $z \in \Gamma$, $|z| \leq M/n$, we have $|S_n(z)/S_n(0)| = 1 + o_\delta(1)$ uniformly in $n$.

**Proof.** As always, we set $z = x + i\gamma(x)$.

$$\frac{1}{n} \log \left| \frac{S_n(z)}{S_n(0)} \right| = \sum_{|\xi^k_j| \geq \delta} \frac{1}{n} \log \left| 1 - \frac{z}{\xi^k_j} \right|$$

is easily seen to converge to

$$\int_{\Gamma_\delta} \log \left| 1 - \frac{z}{u} \right| d\mu_{\Gamma}(u)$$

(recall, that $\Gamma_\delta$ is the part of $\Gamma$ that lies of distance $\geq \delta$ from the origin). Indeed, the same sum on the right of (2.41) with $1/n$ replaced by $\mu_{\Gamma}(I^k_j)$ and $\xi^k_j$ replaced by $\xi^k_j$ (that was the midpoint of $I^k_j$ with respect to arc length) is essentially a Riemannian sum for the integral (2.42), and the sums with $\xi^k_j$, $1/n$ and with $\xi^k_j$, $\mu_{\Gamma}(I^k_j)$ are very close because of (2.4) and (2.32) and its analogue for other intervals. The integral in (2.42) is

$$\int_{\Gamma_\delta} \left( \Re \left( -\frac{z}{u} \right) + O \left( \left( \frac{|z|}{u} \right)^2 \right) \right) d\mu_{\Gamma}(u),$$

and here

$$\int_{\Gamma_\delta} O \left( \left( \frac{|z|}{u} \right)^2 \right) d\mu_{\Gamma}(u) = O \left( \frac{|z|^2}{\delta} \right)$$

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Here the second term is $O(\gamma(x)/\delta) = o(|x|) = o(|z|)$ as $z \to 0$, and for the first term Proposition 2.8 gives that it is $x a\delta(1)$.

Therefore,

$$\log \left| \frac{S_n(z)}{S_n(0)} \right| = n(1 + o(1)) \left[ |z| a\delta(1) + O(|z|^2/\delta) + o(|z|) \right] \leq 2Ma\delta(1)$$

for large $n$ and $|z| \leq M/n, z \in \Gamma$. This proves the claim.

In the polynomial $Q_n$ in (2.33) we put all factors $(z - \xi^0_j)$ with $M^3 < |j| \leq \rho n$, while $S_n$ in (2.40) contained the factors $(z - \xi^k_j)$ with $|\xi^k_j| \geq \delta$. For sufficiently small $\delta$ these latter include all $\xi^k_j$ with $k > 0$ (i.e. which are created for the components $\Gamma_k, k > 0$). Furthermore, for small $\delta > 0$ if, with a sufficiently large fixed $L$ we select $\rho = \omega_T(0)\delta - L\delta^{1+\alpha}$, then (2.30) shows that $Q_n$ and $S_n$ have no common factors. On the other hand, if we selected $\rho = \omega_T(0)\delta + L\delta^{1+\alpha}$, then (2.30) shows that all the factors $(z - \xi^k_j)$ except for $(z - \xi^0_j)$ with $|j| \leq M^3$ appear either in $Q_n$ or in $S_n$. We make the former selection, i.e. we set $\rho = \omega_T(0)\delta - L\delta^{1+\alpha}$, and let $\tilde{S}_n$ be the product of all factors $(z - \xi^0_j)$ for which $|j| > \rho n$ but $|\xi^0_j| < \delta$ (these are the ones with $|j| > M^3$ that appear neither in $Q_n$ nor in $S_n$). According to what we have just said, their number is at most $4L\delta^{1+\alpha}n$.

Proposition 2.10 We have $|\tilde{S}_n(z)/\tilde{S}_n(0)| = 1 + o(1)$ uniformly in $n$ and $|z| \leq M/n, z \in \Gamma$.

Proof. Let $H$ be the set of $j$'s for which $|j| > \rho n$ but $|\xi^0_j| < \delta$. Note that all such $\xi^0_j$'s satisfy $|\xi^0_j| \geq \delta/2$ (see (2.30) and the definition of $\rho$). Now

$$\log \left| \frac{\tilde{S}_n(z)}{\tilde{S}_n(0)} \right| = \sum_{j \in H} \log \left| 1 - \frac{z}{\xi^0_j} \right| = \sum_{j \in H} O \left( \frac{|z|}{|\xi^0_j|} \right) = O \left( \frac{M 4L \delta^{1+\alpha} n}{\delta} \right),$$

from which the claim follows.

After these preparations we turn to the proof of Theorem 2.1.

From the definition of our polynomials it follows that

$$P_n(z) = Q_n(z)S_n(z)\tilde{S}_n(z) \prod_{-M^3 \leq j \leq M^3, j \neq 0} (z - \xi^0_j) =: Q_n(z)S_n(z)\tilde{S}_n(z)V_n(z),$$

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and Propositions 2.7, 2.9 and 2.10 show that here the first three factors change little (i.e. \((1+o(1))\)) as \(z\) varies on the arc of \(\Gamma\) with \(|z| \leq M/n\). The idea of the proof is to compare the remaining factor \(V_n(z)\) to something the behavior of which we already know. This something is the unit circle and the polynomial \(1 + z + \cdots + z^{m-1} = (z^m - 1)/(z - 1)\), but with \(m = [2\pi\omega_T(0)n]\) (sic!). Apply the transformation \(T(z) = -i(z - 1)\) to the unit circle under which \(1\) gets into the point 0 and the real line becomes the tangent line to the transformed circle at 0. Let \(\Gamma^*: \{z^*\mid |z^* - i| = 1\}\) denote this rotated/translated circle, and let \(\xi_j^* = -i(e^{i2\pi/m} - 1), j = -[m/2], \ldots, [(m + 1)/2], j \neq 0\) be the images under \(T\) of the \(m\)-th roots of unity different from 1. Their enumeration is such that

\[
\left| \xi_j^* - \frac{2\pi j}{m} \right| \leq C \left( \frac{|j|}{m} \right)^2
\]  

(2.43)

for all \(j\).

To a \(z \in \Gamma, \ |z| \leq \delta\) we associate the point \(z^* \in \Gamma^*\) via \(\Re z = \Re z^*\) (and of course of the two possibilities for \(z^*\) we take the one lying closer to the real line \(\mathbf{R}\)). If \(\nu\) is a measure on \(\Gamma\), then we define a measure \(\nu^*\) on \(\Gamma^*\) in a neighborhood of the origin by stipulating \(d\nu^*(z^*) = d\nu(z)\); in other words, \(\nu^*\) is the pull-back of the measure \(\nu\) under the mapping \(z^* \to z\). Away from the origin let \(\nu^*\) be the arc measure on \(\Gamma^*\). Assume that 0 is a Lebesgue-point for \(\nu\) (with respect to \(s_T\)). Then 0 is also a Lebesgue-point for \(\nu^*\) (with respect to \(s_{T^*}\)), see (2.31). Let \(d\nu = wds_T + d\nu_{\text{sing}}\) be the decomposition of \(\nu\) into its absolutely continuous and singular part with respect to \(s_T\), and let \(d\nu^* = w^*ds_{T^*} + d\nu_{\text{sing}}^*\) be the similar decomposition of \(\nu^*\). Then, using the just mentioned Lebesgue-point property, we obtain \(w(0) = w^*(0)\).

Let \(P_m^*(z^*) = \prod_j (z^* - \xi_j^*)\) be the transform of the polynomial

\[
1 + z + \cdots + z^{m-1} = (z^m - 1)/(z - 1)
\]

under the transformation \(T\). The expression

\[
\frac{1}{m} |1 + z + \cdots + z^{m-1}|^2
\]

is the \(m\)-th Fejér kernel on the unit circle, and it is known (see [11, Lemma 2] and make the transformation \(T^*\)) that

\[
\int_{\Gamma^*} \left| \frac{P_m^*}{P_m^*(0)} \right|^2 d\nu^* \leq (1 + o(1)) \frac{2\pi w^*(0)}{m}.
\]  

(2.44)

Now the idea of the proof is that for \(|z| \leq M/n\) the ratio \(|P_m^*(z^*)/P_m^*(0)|\) looks just like \(|P_n(z)/P_n(0)|\). To show that we write

\[
P_m^*(z^*) = U_m^*(z^*) \prod_{M^2 \leq j \leq M^n, j \neq 0} (z^* - \xi_j^*) = U_m^*(z^*)V_m^*(z^*).
\]

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Note that here \( U^*_m(z^*) \) corresponds to the factor \( Q_n(z)S_n(z)\tilde{S}_n(z) \) in \( P_n \). For that factor we proved in Propositions 2.7, 2.9 and 2.10 that

\[
\left| \frac{Q_n(z)S_n(z)\tilde{S}_n(z)}{Q_n(0)S_n(0)\tilde{S}_n(0)} \right| = 1 + o(1) \tag{2.45}
\]

as \( n \to \infty \) uniformly in \( z \in \Gamma, |z| \leq M/n \). Since the \( \xi_j^* \) have the same property as the \( \xi_j^0 \) had, notably (2.30), the same proof (or direct verification) shows that

\[
\left| \frac{U^*_m(z^*)}{U^*_m(0)} \right| = 1 + o(1) \tag{2.46}
\]

as \( m \to \infty \) uniformly in \( |z| \leq M^*/m \) for any fixed \( M^* \). The choice \( m = [2\pi\omega_T(0)n] \) and formulae (2.30) and (2.43) show that for \( j \in [-M^3, M^3], j \neq 0 \) we have

\[
|\xi_j^0 - \xi_j^*| \leq C \left( \frac{M^3}{n} \right)^{1+\alpha}. \tag{2.47}
\]

Also, for \( z \in \Gamma, |z| \leq M/n \) we have \( |z - z^*| \leq C(M/n)^{1+\alpha} \). These imply that if \( J_n \) is the set of those \( \Re z \in \Gamma \) for which \( |z| \leq M/n \) but \( \text{dist}(\Re z, j/n\omega_T(0)) \geq n^{-1-\alpha/2} \) for all \( j = -M^3, \ldots, M^3, j \neq 0 \), then

\[
\left| \frac{V_n(z)}{V^*_m(z^*)} \right| = \prod_{-M^3 \leq j \leq M^3, j \neq 0} \left| 1 + \frac{z - z^*}{z^* - \xi_j^*} \right| \leq \left( 1 + O \left( \frac{(M^3)^{1+\alpha}}{n^{\alpha/2}} \right) \right)^{2M^3} = 1 + o(1), \quad z \in J_n
\]

as \( n \to \infty \). This implies

\[
\left| \frac{V_n(z)}{V_n(0)} \right| = (1 + o(1)) \left| \frac{V^*_m(z^*)}{V^*_m(0)} \right|
\]

for all such \( z \), and hence (see (2.45) and (2.46))

\[
\left| \frac{P_n(z)}{P_n(0)} \right| = (1 + o(1)) \left| \frac{P^*_m(z^*)}{P^*_m(0)} \right|.
\]

Therefore, the integral of \( |P_n(z)/P_n(0)|^2 \) against \( \nu \) over \( J_n \) is

\[
\int_{J_n \cap \Gamma} \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq (1 + o(1)) \int_{J_n \cap \Gamma^*} \left| \frac{P^*_m(z^*)}{P^*_m(0)} \right|^2 d\nu^*(z^*) \tag{2.48}
\]

\[
\leq (1 + o(1)) \frac{2\pi w^*(0)}{m} = (1 + o(1)) \frac{w(0)}{n\omega_T(0)},
\]

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where, in the second inequality, we used (2.44).

For the integrals over the sets
\[ \left\{ z \mid \Re z - \frac{j}{n\omega^T(0)} \leq n^{-1-\alpha/2} \right\} \]
with \( j = -M^3 \cdots, M^3 \), \( j \neq 0 \) we get from the Lebesgue-point property at 0 and from the fact that \( |P_n(z)/P_n(0)| \) is uniformly bounded (see Proposition 2.5)
\[
\int_{|R_2 - j/n\omega^T(0)| \leq n^{-1-\alpha/2}} \left| \frac{P_n(z)}{P_n(0)} \right|^2 w(z) ds_T(z)
\]
\[
\leq C \int_{|R_2 - j/n\omega^T(0)| \leq n^{-1-\alpha/2}} |w(z) - w(0)| ds_T(z)
\]
\[
+ C \int_{|R_2 - j/n\omega^T(0)| \leq n^{-1-\alpha/2}} |w(0)| ds_T(z)
\]
\[
= o(|j|/n) + O(n^{-1-\alpha/2}) = o(1/n).
\]
There are \( \leq CM \) such sets intersecting \( \{|z| \leq M/n\} \), so their contribution to the whole integral of \( |P_n/P_n(0)|^2 \) against \( w ds_T \) over \( \Gamma \cap \{|z| \leq M/n\} \) is \( o(1/n) \).

The same can be done for the singular part, and with this and (2.48) we have verified
\[
\int_{|z| \leq M/n} \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq (1 + o(1)) \frac{w(0)}{n\omega^T(0)}. \tag{2.49}
\]
As for the integral over \( |z| > M/n \), we use that there \( |P_n(z)/P_n(0)|^2 \leq C/n^2|z|^2 \) (see Proposition 2.5), as well as the fact that by the Lebesgue-point property
\[
\int_{2^{k-1}M/n \leq |z| \leq 2^k M/n} d\nu(z) \leq C \frac{2^k M}{n^2}.
\]
Therefore, we can write
\[
\int_{|z| \geq M/n} \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq \sum_{k \geq 1} \int_{2^{k-1}M/n \leq |z| \leq 2^k M/n} \frac{C}{n^2|z|^2} d\nu(z)
\]
\[
\leq \sum_{k \geq 1} \frac{C}{n^2(2^{k-1}M/n)^2} \frac{2^k M}{n} \leq \frac{C}{Mn}.
\]
This, together with (2.49), gives
\[
\limsup_{n \to \infty} \frac{1}{n^2} \int \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq \frac{w(0)}{\omega^T(0)} + \frac{C}{M},
\]
24
and since here $M$ is arbitrary, finally we obtain
\[
\limsup_{n \to \infty} n \int \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq \frac{w(0)}{\omega_T(0)},
\]
as was to be proved.

\[\square\]

### 2.2 Part II: $\Gamma_0$ is a Jordan arc

Let again $\theta_k = \mu_T(\Gamma_k)$, and consider the integers $n_k = [\theta_k n]$, and divide again each $\Gamma_k$, $k > 0$, into $n_j$ arcs $I_j^k$ each having equal weight $\theta_k/n_k$ with respect to $\mu_T$, i.e. $\mu_T(I_j^k) = \theta_k/n_k$. If we do the same division on $\Gamma_0$, then, unfortunately, we cannot guarantee any more that we can achieve that one of the $\xi_l^0$'s has zero real part, i.e. in this case we cannot guarantee (with the previous notations) $\Re \xi_l^0 = 0$, which was crucial in the proof in Part I. So, when $\Gamma_0$ is a Jordan arc, we make the division of $\Gamma_0$ in such a way that this property hold: let $I_0^0$ be the unique arc (at least for large $n$ it is unique) with the property that $0 \in I_0^0$, $\mu_T(I_0^0) = \theta_0/n_0$, and if $\xi_l^0$ is the center of mass of $\mu_T$ on $I_0^0$, then we have $\Re \xi_l^0 = 0$. The unicity follows, since for $a \in \Gamma_0$, $Ra < 0$ lying sufficiently close to 0 there is a unique $b \in \Gamma_0$, $\Re b > 0$ such that the arc $ab$ has $\mu_T$-mass equal to $\theta_0/n_0$, and, by the $C^{1+\alpha}$-smoothness of $\Gamma_0$, the real part of the center of mass of the arc $ab$ is strictly increasing as $Ra$ does so. Now to the “left” resp. to the “right” of $I_0^0$ (in the direction of the two endpoints of $\Gamma_0$) consider the arcs $I_0^{0-1}, I_0^{0-2}, \ldots$ resp. $I_1^0, I_2^0, \ldots$ that continuously fill $\Gamma_0$ and have the property $\mu_T(I_j^0) = \theta_0/n_0$. We can select (including $I_0^0$) at least $n_0 + 1$ such arcs (we get stuck in the selection only when the remaining part around one of the endpoints of $\Gamma_0$ has $\mu_T$-mass smaller than $\theta_0/n_0$), however, it may happen that with this selection around the endpoints of $\Gamma_0$ there still remain two “little” arcs, say $I_{-l_0}^0$ and $I_l^0$ with $0 < \mu_T(I_{-l_0}) < \theta_0/n_0$ and $0 < \mu_T(I_l^0) < \theta_0/n_0$. We include these two small arcs also into our subdivision of $\Gamma_0$, so in this case we divide $\Gamma_0$ into $n_0 + 1$ arcs $I_j^0$, $j = -l_0, \ldots, l_1$.

Then $|\mu_T(I_j^0) - 1/n| \leq C/n^2$ except for $k = 0$ and $j = -l_0$ or $j = l_1$, in which case $\mu_T(I_j^0)$ can be very small compared to $1/n$. Let $\xi_l^k$ be the center of mass from (2.5) of the arc $I_j^k$ with respect to $\mu_T$, and consider the polynomial
\[
R_n(z) = \prod_{j,k} (z - \xi_l^k)
\]
(2.50)
of degree at most $n + 1$ (note that now the degree is not necessarily at most $n$ since from $\Gamma_0$ we may get $n_0 + 1$ zeros namely the $\xi_l^0$'s). Since $\Re \xi_0^0 = 0$, it is still true that $\xi_l^0$ lies closest to 0 among the $\xi_l^k$'s. We claim that the polynomial
\[
P_n(z) = R_n(z)/(z - \xi_l^0)
\]
(2.51)
verifies Theorem 2.1.

Most of the proof remains the same, except for the proof of Proposition 2.4, which needs modifications when \(z\) belongs to the intervals \(I_{-l_0}, I_{-l_0+1}, I_{l_1-1}, I_{l_1}\) – these require some substantial modifications because \(I_{-l_0}\) or \(I_{l_1}\) can be very short. Thus, let again \(z \in I_{l_0}\) and we need to prove that

\[ |P_n(z)| \leq C \text{cap}(\Gamma)^n. \]

First of all now (2.13) contains the terms

\[ \left( n - \frac{1}{\mu_T(I^0_j)} \right) \int_{I^0_j} \log |z - t|d\mu_T(t) \]  

with \(j = -l_0, l_1\), and for these terms the coefficient

\[ n - \frac{1}{\mu_T(I^0_j)} \]

is not bounded due to the fact that \(\mu_T(I^0_j)\) can be very small. However, this coefficient is bounded from above, and for \(\text{dist}(z, I^0_j) \leq 1/2\) the integrand in (2.52) is negative, hence in this case

\[ \left( n - \frac{1}{\mu_T(I^0_j)} \right) \int_{I^0_j} \log |z - t|d\mu_T(t) \geq -C \int_{I^0_j} \log |z - t|d\mu_T(t) \geq -C, \]

while for \(\text{dist}(z, I^0_j) \geq 1/2\) we just have the bound

\[ \left| \left( n - \frac{1}{\mu_T(I^0_j)} \right) \int_{I^0_j} \log |z - t|d\mu_T(t) \right| \leq C \left| n - \frac{1}{\mu_T(I^0_j)} \right| \int_{I^0_j} d\mu_T(t) \leq C \]

since \(\mu_T(I^0_j) \leq 2/n\). Therefore, \(\Sigma_1\) in (2.13) is bounded from below: \(\Sigma_1 \geq -C\), and then (see (2.11))

\[ \log |R_n(z)| - n \log \text{cap}(\Gamma) = \log |R_n(z)| - \Sigma_1 - \Sigma_2 \leq C + \log |R_n(z)| - \Sigma_2, \]

so it is sufficient to prove again (2.15).

The boundedness of the individual terms \(L_{i,k}\) follows as before except for \(L_{-l_0,0}, L_{-l_0+1,0}\) when \(z \in I_{-l_0}^0 \cup I_{-l_0+1}^0\) or for \(L_{l_1,0}, L_{l_1+1,0}\) when \(z \in I_{l_1-1}^0 \cup I_{l_1}^0\), in which cases it may not be true. But, as we shall show below, we can still claim the boundedness of these terms from above. We shall show this for \(L_{-l_0,0}, L_{-l_0+1,0}\) when \(z \in I_{-l_0}^0 \cup I_{-l_0+1}^0\), the other case is similar.

For simpler notation let \(J_1 = I_{-l_0}^0, \zeta_1 = \xi_{-l_0}^0\) and \(J_2 := I_{-l_0+1}^0, \zeta_2 = \xi_{-l_0+1}^0\), hence \(E\) is an endpoint of the “short” arc \(J_1\), and \(J_2\) is the neighboring arc in the subdivision. The arc \(ab\) plays different roles in different parts of the proof below;
we shall always indicate its meaning. Let first \( z \in J_1, \widehat{ab} := J_1 \cup J_2 \). Then (since \( a = E \) is an endpoint of the arc \( \Gamma_0 \) around which we have \( \omega_\Gamma(t) \sim |t - a|^{-1/2} \) by Proposition 2.2)

\[
L_{-l_0 + 1,0} = \frac{1}{\mu_\Gamma(J_2)} \int_{J_2} \log \left| \frac{z - \zeta_1}{z - t} \right| \omega_\Gamma(t) ds(t)
\]

\[
\leq \frac{C}{s(ab)^{1/2}} \int_{ab} \log \frac{Cs(ab)}{s(zt)} s(\widehat{at})^{-1/2} ds(t)
\]

\[
L_{-l_0,0} \leq \frac{C}{s(ab)^{1/2}} \int_{ab} \log \frac{Cs(J_1)}{s(zt)} s(\widehat{at})^{-1/2} ds(t),
\]

and the expression on the right was treated in (2.24)–(2.25), so the same argument gives \( L_{-l_0 + 1,0} \leq C \). Next consider with \( \widehat{ab} = J_1 \) (\( z \) still being on \( J_1 \))

\[
L_{-l_0,0} \leq \frac{C}{\mu_\Gamma(J_1)} \int_{J_1} \log \frac{Cs(J_1)}{s(zt)} s(\widehat{at})^{-1/2} ds(t)
\]

which is again what we handled in (2.24)–(2.25) (that argument works for short arcs like \( \widehat{ab} = J_1 \), as well) and we can conclude \( L_{-l_0,0} \leq C \).

When \( z \in J_2 \), the reasoning is similar for \( L_{-l_0 + 1,0} \). Finally, let \( z \in J_2 \) and consider \( L_{-l_0,0} \). Let \( W \) be the point on the arc \( J_2 := AB \), for which \( s(\overline{AW}) = s(J_1) = s(\overline{EA}) \) (there is such a \( W \) since \( s(J_1) \leq s(J_2) = \theta_0/n_0 \), see Figure 4. If \( z \in J_2 \) but \( z \notin \overline{AW} \) then in

\[
L_{-l_0,0} = \frac{1}{\mu_\Gamma(J_1)} \int_{J_1} \log \left| \frac{z - \zeta_1}{z - t} \right| \omega_\Gamma(t) ds(t)
\]
we have
\[ \frac{1}{3} \leq \left| \frac{z - \zeta_1}{z - t} \right| \leq 3, \]
so in this case \( L_{-t_0,0} \leq C \) is obvious. However, if \( z \in \tilde{AW} \) then with \( \tilde{ab} = \tilde{EW} \) we get
\[ L_{-t_0,0} \leq \frac{C}{s(ab)^{1/2}} \int_{\tilde{ab}} \log \frac{Cs(\tilde{ab})}{s(\tilde{zt})} s(\tilde{at})^{-1/2} ds(t), \]
and \( L_{-t_0,0} \leq C \) follows again from the bound (2.25) for (2.24).

Once we have established the upper boundedness of the individual terms \( L_{j,k} \) in (2.15), the rest of the argument in Proposition 2.4 remains the same.

Note also that there is no problem whatsoever with the lower and upper boundedness of the sum in (2.15)–(2.17) when we are not close to the endpoints of arc-components of \( \Gamma \), and certainly this is the case for \( z \in I_0^0 \). Hence, the proof of Proposition 2.5 is unchanged, and then so is the rest of the proof of Theorem 2.1.

\[ \square \]

3 The lower estimate for the Christoffel functions in Theorem 1.1 for positive weights

The following theorem together with Theorem 2.1 completes the proof of Theorem 1.1 in the case when \( w \) is strictly positive on \( \Gamma \).

**Theorem 3.1** Let \( \Gamma \) be a system of \( C^{1+\alpha} \)-smooth Jordan arcs and curves lying exterior to one another, \( z_0 \in \Gamma \) not an endpoint of an arc-component of \( \Gamma \) and assume that \( \Gamma \) is \( C^2 \)-smooth in a neighborhood of \( z_0 \). Assume that \( d\nu = wd\mu_{\Gamma} \) is a measure on \( \Gamma \) with continuous and positive density \( w \). Then
\[ \liminf_{n \to \infty} n\lambda_n(z_0, \nu) \geq \frac{d\nu(z_0)}{d\mu_{\Gamma}}. \] (3.1)

Indeed, the definition of the Christoffel functions shows that \( \nu_1 \geq \nu_2 \) implies \( \lambda_n(z, \nu_1) \geq \lambda_n(z, \nu_2) \), so if the \( w \) in Theorem 1.1 is strictly positive, then we can just drop the singular part \( \nu_{\text{sing}} \) from \( \nu = \nu_\nu + \nu_{\text{sing}} \) and apply (3.1) to the absolutely continuous part \( d\nu_{\nu}(t) = w(t)ds(t) \) to conclude from (3.1)
\[ \liminf_{n \to \infty} n\lambda_n(z_0, \nu) \geq \liminf_{n \to \infty} n\lambda_n(z_0, \nu) \geq \frac{d\nu_{\nu}(z_0)}{d\mu_{\Gamma}} = \frac{d\nu(z_0)}{d\mu_{\Gamma}}. \]

In the proof of Theorem 3.1 let \( \Omega \) be the unbounded component of \( \mathbb{C} \setminus \Gamma \), and we denote by \( g_{\mathbb{C}\setminus \Gamma} \) the Green’s function of \( \Omega \) with respect to the pole at infinity (see e.g. [17, Sec. 4.4]).
Proof of Theorem 3.1. Without loss of generality assume \( z_0 = 0 \). Assume to the contrary that there are infinitely many \( n \) and for each \( n \) a polynomial \( Q_n \) of degree at most \( n \) such that

\[
|Q_n(0)|^2 < (1 - \beta) \frac{d\nu(0)}{d\mu^*}.
\]

with some \( \beta > 0 \). Our aim will be to show that this implies the following: there exists another system \( \Gamma^* \) of Jordan curves (no arcs!) such that \( \Gamma \subseteq \Gamma^* \), in a neighborhood \( \Delta_0 \) of 0 we have \( \Gamma \cap \Delta_0 = \Gamma^* \cap \Delta_0 \), and there is a measure \( \nu^* \) on \( \Gamma^* \) with positive and continuous density which coincides with \( \nu \) on \( \Gamma \) for which,

\[
\liminf_{n \to \infty} n \lambda_n(0, \nu^*) < d\nu^*(0).
\]

Since this contradicts [25, Theorem 1.1], it follows that (3.2) cannot be true, i.e.

\[
(3.1)
\]

holds.

Let \( \Gamma_0, \ldots, \Gamma_{k_0} \) be the connected components of \( \Gamma \), \( \Gamma_0 \) being the one that contains 0. First we deal with the case when \( \Gamma_0 \) is a Jordan arc—after that we shall indicate what changes are necessary when \( \Gamma_0 \) is a Jordan curve. Let \( n_{\pm} \) be the two normals to \( \Gamma_0 \) at 0, and let \( A_{\pm} = \partial g_{C \setminus \Gamma}(0)/\partial n_{\pm} \) be the corresponding normal derivatives of the Green’s function of \( C \setminus \Gamma \) with pole at infinity. Assume, for example, that \( A_+ \geq A_- \). Note that necessarily \( A_- > 0 \). In fact, there is a small closed disk \( D \) containing 0 that lies on the side of \( \Gamma \) (i.e. lies outside except for the point 0) which is determined by the direction of the normal \( n_- \).

For simplicity assume that \( D \) is the disk \( \{z \mid \|z - 1\| = 1\} \). Then \( g_{C \setminus \Gamma}(z + 1) \) is harmonic in the unit disk and continuous on its boundary (this follows from the \( C^2 \)-property of \( \Gamma_0 \), hence from Poisson’s formula we easily get

\[
g_{C \setminus \Gamma}(z + 1) \geq (1 - |z|)g_{C \setminus \Gamma}(1), \quad |z| < 1.
\]

Therefore, with \( z = -1 + tn_- \), for small \( t > 0 \) we have \( g_{C \setminus \Gamma}(tn_-) \geq tg_{C \setminus \Gamma}(1) \), from which \( A_- \geq g_{C \setminus \Gamma}(1) \) follows.

Let \( \epsilon > 0 \) be an arbitrarily small number. For each \( \Gamma_j \) that is a Jordan arc (i.e. NOT a Jordan curve), connect the two endpoints of \( \Gamma_j \) by another Jordan arc \( \Gamma_j' \) that lies close to \( \Gamma_j \) so that we obtain a system \( \Gamma' \) of \( k_0 + 1 \) Jordan curves with boundary \( (\cup_j \Gamma_j) \cup (\cup_j \Gamma_j') \). Assume also that \( \Gamma_0' \) is selected so that \( n_+ \) is the outer normal to \( \Gamma_0' \) at 0. This can be done in such a way that

\[
\frac{\partial g_{C \setminus \Gamma'}(0)}{\partial n_+} > \frac{1}{1 + \epsilon} \frac{\partial g_{C \setminus \Gamma}(0)}{\partial n_+}.
\]

(note that since the unbounded component of \( C \setminus \Gamma' \) is part of the unbounded component of \( C \setminus \Gamma \), we necessarily have \( \partial g_{C \setminus \Gamma'}(0)/\partial n_+ \leq \partial g_{C \setminus \Gamma}(0)/\partial n_+ \)). Indeed, to see (3.4) if \( \Gamma' \) is sufficiently close to \( \Gamma \), we can apply [12, Lemma 7.1] since, as \( \Gamma' \) tends to \( \Gamma \), we have \( g_{C \setminus \Gamma'}(z) \to g_{C \setminus \Gamma}(z) \) locally uniformly on compact subsets of the unbounded component \( \Omega \) of \( C \setminus \Gamma \).
Figure 5: Illustration of the case of a single Jordan arc

Select a small disk $\Delta_0$ about 0 for which $\Gamma' \cap \Delta_0 = \Gamma \cap \Delta_0$. By [12, Theorem 1.2] we can choose a lemniscate $\sigma = \{z \mid |T_N(z)| = 1\}$ (with some polynomial $T_N$ of degree equal to some integer $N$) such that $\Gamma'$ lies in the interior of $\sigma$ (i.e. in the union of the bounded components of $\mathbb{C} \setminus \sigma$) except for the point 0, where $\sigma$ and $\Gamma'$ touch each other, and

$$\frac{\partial g_{C\setminus \sigma}(0)}{\partial n_+} > \frac{1}{1 + \varepsilon} \frac{\partial g_{C \setminus \Gamma}(0)}{\partial n_+}. \quad (3.5)$$

By [12, Theorem 1.2] this $\sigma$ can be chosen so that it has precisely $k_0 + 1$ components each containing one-one component of $\Gamma'$, and if $\tau'_0$ denotes the signed curvature of $\Gamma'$ at 0 seen from the outside, then in a neighborhood of 0 the signed curvature $\tau_0$ of $\sigma$ is smaller than $\tau'_0$. Since the Green's function $g_{C\setminus \sigma}(z)$ is just $(\log |T_N(z)|)/N$, simple computation shows (see formula [25, (2.2)]) that

$$\frac{\partial g_{C\setminus \sigma}(0)}{\partial n_+} = \frac{|T_N'(0)|}{N}. \quad (3.6)$$

Let, for a small $a$ to be determined later, $\sigma_a$ be the lemniscate $\sigma_a := \{z \mid |T_N(z)| = e^{-a}\}$. If $\Delta \subset \Delta_0$ is a fixed small neighborhood of 0, then for sufficiently small $a$ this $\sigma_a$ contains $\Gamma \setminus \Delta$ in its interior (i.e. in the interior of its components), while in $\Delta$ the two curves $\Gamma_0$ and $\sigma_a$ intersect in two points $U, V$; see Figure 5. In fact, this is due to the fact that for small $\Delta$ and $a$ the maximal signed curvature of $\sigma_a$ in $\Delta$, which is close to $\tau_0$, is smaller than the minimal curvature of $\Gamma'$ in $\Delta$, which is close to $\tau'_0 > \tau_0$ (and recall also that $\Gamma$ and $\Gamma'$ coincide in $\Delta_0$). Now the points $U$ and $V$ are connected by the arc $\overline{UV}$ on $\Gamma_0$ (which is the same as on $\Gamma$) and also by the arc $\overline{UV}_{\sigma_a}$ on $\sigma_a$ (there are
Figure 6: The case of one Jordan curve and the formation of $\Gamma$

actually two such arcs on $\sigma_a$, we take the one lying in $\Delta$). For each $\Gamma_j$ which is a Jordan arc connect the two endpoints of $\Gamma_j$ by a new $C^2$ Jordan arc $\Gamma_j^*$ going inside $\Gamma'$ so that on $\Gamma_j^*$ we have

$$g_{C \setminus \Gamma}(z) \leq a^2, \quad z \in \Gamma_j^*.$$  \hspace{1cm} (3.7)

In addition, $\Gamma_0^*$ can be selected so that in $\Delta$ it intersects $\sigma_a$ in two points $U^*, V^*$; see Figure 6. Then $U^*V^*_{\sigma_a}$ is a subarc of $\sigma_a$. Let now $\Gamma^*$ be the union of $\Gamma$, of the $\Gamma_j^*$’s with $j > 0$, of $\Gamma_0 \setminus U^*V^*_{\Gamma_0}$ and of $U^*V^*_{\sigma_a}$; see Figure 6. This $\Gamma^*$ is the union of $k_0 + 1$ Jordan curves, and it is contained in $\sigma_a$ and its interior except for the arc $U^*V^*_{\Gamma_0}$. Furthermore, $\Gamma^*$ lies within $\sigma$ and contains $\Gamma$, so

$$\frac{\partial g_{C \setminus \Gamma}(0)}{\partial n_+} \leq \frac{\partial g_{C \setminus \Gamma^*}(0)}{\partial n_+} \leq \frac{\partial g_{C \setminus \Gamma^*}(0)}{\partial n_+}.$$  \hspace{1cm} (3.8)

Clearly, for any $m = 1, 2, \ldots$

$$|T_N(z)|^m \leq \begin{cases} e^{-am} & z \in \Gamma^* \setminus U^*V^*_{\Gamma_0} \\ 1 & z \in U^*V^*_{\Gamma_0}. \end{cases} \hspace{1cm} (3.9)$$

For the $Q_n$ from (3.2) we get

$$\int_\Gamma |Q_n|^2 ds \leq C_0/n$$
with some $C_0$ (recall that $w$ is continuous and positive), and hence the inequality from Lemma 3.2 below gives that for its supremum norm $\|Q_n\|_{\Gamma}$ we have

$$\|Q_n\|_{\Gamma} \leq C_1 n^{1/2}$$  \hspace{1cm} (3.10)

with some $C_1$ independent of $n$. Therefore, by the Bernstein-Walsh inequality [29, p. 77] the estimate

$$|Q_n(z)| \leq C_1 n^{1/2} e^{\alpha g_{C_1}(z)}$$  \hspace{1cm} (3.11)

follows everywhere on the complex plane. In particular, in view of (3.7)

$$|Q_n(z)| \leq C_1 n^{1/2} e^{\alpha a_2}, \quad z \in \Gamma^* \setminus U^* V^* \sigma_a$$  \hspace{1cm} (3.12)

(note that the part $U^* V^* \sigma_a$ of $\Gamma^*$ may lie outside $\Gamma_0 \cup \Gamma_0^*$, so there (3.7) is not applicable).

We shall also need to estimate $g_{C_1, \Gamma}$ on $U^* V^* \sigma_a$ to get a bound for the polynomials $Q_n$ there ((3.12) is not applicable there). We shall actually do the estimate on $\hat{U} \hat{V} \sigma_a$, which contains $\hat{U} \hat{V}^* \sigma_a$. The lens-shaped region enclosed by $\hat{U} \hat{V} \Gamma_0 \cup \hat{U} \hat{V} \sigma_a$ is contained in a neighborhood $\Delta_a$ of 0 where this $\Delta_a$ shrinks to 0 as $a \to 0$ (here $a$ is not the radius of $\Delta_a$, just signals that $\Delta_a$ depends on $a$). For small $a$ we have uniformly in $z \in \Delta_a \cap \Gamma_0$

$$\frac{\partial g_{C_1, \Gamma}(z)}{\partial n_z} \leq (1 + \varepsilon) A_-, \quad z \in \hat{U} \hat{V} \sigma_a$$

which easily implies (note also that $g_{C_1, \Gamma}$ is a $C^{1+\alpha}$ smooth function—see the reasoning in the Appendix below) that for small $a$

$$g_{C_1, \Gamma}(z) \leq (1 + \varepsilon)^2 b A_-, \quad z \in \hat{U} \hat{V} \sigma_a$$  \hspace{1cm} (3.13)

where $b$ is the largest distance from a point $z \in \hat{U} \hat{V} \sigma_a$ to $\Gamma_0$. This $b$ is at most as large as the largest distance $b'$ from a point $z \in \hat{U} \hat{V} \sigma_a$ to $\sigma$. Next, we estimate this $b'$. Since for small $a$

$$|T_N'(t) - T_N'(0)| = O(|t|) \leq \varepsilon |T_N'(0)|$$

in $\Delta_a$, it follows that

$$b \leq b' \leq \frac{1 - \varepsilon}{1 - \varepsilon} \frac{1 - e^{-a}}{|T_N'(0)|} \leq (1 + \varepsilon)^2 \frac{a}{|T_N'(0)|}$$  \hspace{1cm} (3.14)

Indeed, for a $z \in \hat{U} \hat{V} \sigma_a$ let $Z$ be the closest point on $\sigma$ such that (modulo $2\pi$) $\arg(T_N(z)) = \arg(T_N(Z))$. Then

$$1 - e^{-a} = |T_N(Z) - T_N(z)| = \left| \int_z^Z T_N'(t) dt \right|$$
\[ \geq \left| \int_z^Z T_N'(0) dt \right| - \left| \int_z^Z |T_N'(t) - T_N'(0)| dt \right| \]

\[ \geq (1 - \varepsilon) |T_N'(0)||z - Z|, \]

from which we get for small \( a \) and appropriate \( z \in \widehat{UV}_{\sigma_a} \)

\[ b' \leq |z - Z| \leq \frac{1}{1 - \varepsilon} \frac{1 - e^{-a}}{|T_N'(0)|}. \]

In view of (3.13) and (3.14) we have on \( \widehat{UV}_{\sigma_a} \) the estimate

\[ gC \gamma(z) \leq (1 + \varepsilon)^4 aA_- |T_N'(0)|, \]

and hence, by (3.11),

\[ |Q_n(z)| \leq C_1 n^{1/2} \exp \left( n (1 + \varepsilon)^4 aA_- / |T_N'(0)| \right) , \quad z \in \widehat{UV}_{\sigma_a} . \numberthis \tag{3.15} \]

Now consider with \( m = \left[ (1 + \varepsilon)^{7} A_- n / N A_+ \right] \)

the polynomial

\[ P_{n+mN}(z) = Q_n(z) T_N(z)^m \numberthis \tag{3.17} \]

on \( \Gamma^* \), and let the measure \( \nu^* \) be equal to \( \nu \) on \( \Gamma \) and equal to the arc measure \( s_{\Gamma} \) on \( \Gamma^* \setminus \Gamma \). The density \( w^* \) of \( \nu^* \) with respect to \( \sigma_{\Gamma} \) may not be continuous at the endpoints of those components of \( \Gamma \) that are Jordan arcs, but this will not bother us below (alternatively, one could easily choose a continuous \( w^* \)). For this polynomial we have on \( \Gamma^* \setminus \left( \widehat{UV}_{\Gamma_0} \cup \widehat{V^*}_{\sigma_a} \right) \) (see (3.9) and (3.12))

\[ |P_{n+mN}(z)| \leq C_1 n^{1/2} e^{na^2 - ma}, \numberthis \tag{3.18} \]

on \( \widehat{UV}_{\Gamma_0} \) the bound

\[ |P_{n+mN}(z)| \leq |Q_n(z)| \numberthis \tag{3.19} \]

and on \( \widehat{V^*}_{\sigma_a} \) the estimate

\[ |P_{n+mN}(z)| \leq C_1 n^{1/2} \exp \left( n (1 + \varepsilon)^4 aA_- / |T_N'(0)| - ma \right) \numberthis \tag{3.20} \]

(see (3.15) and (3.9)). Here, by the choice of \( m \) in (3.16) and by (3.5) and (3.6) the quantity in the exponent is at most

\[ n \left( \frac{(1 + \varepsilon)^5 aA_-}{A_+ N} - \frac{(1 + \varepsilon)^6 aA_-}{NA_+} \right) = -\varepsilon n \left( \frac{(1 + \varepsilon)^5 aA_-}{NA_+} \right) . \]
Fix $a$ so small that we have $a^2 - aA_- / NA_+ < 0$. Then the estimates (3.18)–(3.20) yield
\[ \lambda_{n+mN}(0, \nu^*) \leq \int |P_{n+mN}|^2 w^* ds_{\Gamma^*} \leq \int |Q_n|^2 w ds_{\Gamma} + O(n^{-2}). \]
Hence, by (3.2), if $\omega_{\Gamma} := d\mu_{\Gamma} / ds_{\Gamma}$ is the density of the equilibrium measure $\mu_{\Gamma}$ of $\Gamma$ with respect to arc measure, then for infinitely many $n$
\[ (n + mN)\lambda_{n+mN}(0, \nu^*) \leq \frac{n + mN}{n} (1 - \beta) \frac{w(0)}{\omega_{\Gamma}(0)} + o(1). \] (3.21)
It is well known (see e.g. (5.3) below) that
\[ \omega_{\Gamma}(0) = \frac{1}{2\pi} \left( \frac{\partial g_C_{\Gamma}}{\partial \mathbf{n}_+} + \frac{\partial g_C_{\Gamma}}{\partial \mathbf{n}_-} \right) = \frac{1}{2\pi} (A_+ + A_-) \] (3.22)
and
\[ \omega_{\Gamma^*}(0) = \frac{1}{2\pi} \frac{\partial g_C_{\Gamma^*}(0)}{\partial \mathbf{n}_+} \leq \frac{1}{2\pi} \frac{\partial g_C_{\Gamma}(0)}{\partial \mathbf{n}_+} = \frac{1}{2\pi} A_+, \] (3.23)
so
\[ \frac{n + mN}{n} (1 - \beta) \frac{w(0)}{\omega_{\Gamma}(0)} \leq \left( 1 + (1 + \varepsilon) \frac{A_-}{A_+} \right) (1 - \beta) \frac{w(0)}{\omega_{\Gamma^*}(0)} \frac{A_+}{A_+ + A_-} + A_+ \]
\[ \leq \left( 1 - \frac{\beta}{2} \right) \frac{w(0)}{\omega_{\Gamma^*}(0)} \]
if $\varepsilon$ is sufficiently small. Therefore, (3.21) implies
\[ \liminf_{n \to \infty} (n + mN)\lambda_{n+mN}(0, \nu^*) \leq \left( 1 - \frac{\beta}{2} \right) \frac{w(0)}{\omega_{\Gamma^*}(0)}, \]
which is impossible, since, according to [25, Theorem 1.1] (applicable to the family $\Gamma^*$ of finitely many Jordan curves and to the measure $\nu^*$ on it)
\[ \lim_{n \to \infty} (n + mN)\lambda_{n+mN}(0, \nu^*) = \frac{w(0)}{\omega_{\Gamma^*}(0)}, \]
This contradiction emerged since we assumed (3.2), and so (3.1) has been proven.

Next, consider the proof of Theorem 3.1 in the case when $\Gamma_0$ is a Jordan curve. In that case $A_- = 0$. We construct $\Gamma^*$ as before, and select again a lemniscate $\sigma = \{ z \mid \left| T_N(z) \right| = 1 \}$ that contains $\Gamma'$ in its interior except for the point 0 where it touches $\Gamma'$, and for which (3.5) is true. Now construct $\Gamma^*$ inside
as before satisfying (3.7), and let \( \nu^* \) agree with \( \nu \) on \( \Gamma \) and with \( s_{\Gamma^*} \) on \( \Gamma^* \setminus \Gamma \). There is an \( a > 0 \) such that \( |T_N(z)| \leq e^{-a} \) for \( z \in \Gamma^* \setminus \Gamma \) (recall that \( \sigma \) contains \( \Gamma' \) in its interior except for the point 0, and now there is no \( \Gamma^*_0 \) because the \( \Gamma^*_j \)'s were constructed only for those \( j \) for which \( \Gamma^*_j \) is a Jordan arc).

With \( m = \lfloor n\beta/N \rfloor \) (recall that \( \beta \) is from 3.2) consider the polynomial \( P_{n+mN} \) from (3.17). For it we have on \( \Gamma \) the inequality \( |P_{n+mN}(z)| \leq |Q_n(z)| \), while on \( \Gamma^* \setminus \Gamma \) we have

\[
|P_{n+mN}(z)| \leq C_1 n^{1/2} e^{na^2 - ma},
\]

which implies for small \( a \) just as before

\[
\lambda_{n+mN}(0, \nu^*) \leq \int |P_{n+mN}|^2 w^* ds^* \leq \int |Q_n|^2 w ds + O(n^{-2}).
\]

Hence

\[
\liminf_{n \to \infty} (n + mN) \lambda_{n+mN}(0, \nu^*) \leq (1 + \beta)(1 - \beta) \frac{w(0)}{\omega_\Gamma(0)} \leq (1 - \beta^2) \frac{w^*(0)}{\omega_{\Gamma^*}(0)},
\]

since \( w^*(0) = w(0) \), and for the density \( \omega_\Gamma = d\mu_\Gamma/ds_\Gamma \) we have (see (3.22) and (3.23))

\[
\omega_\Gamma(0) = \frac{1}{2\pi} \frac{\partial g_{\Gamma \setminus \Gamma_0}(0)}{\partial n_+} \geq \frac{1}{2\pi} \frac{\partial g_{\Gamma \setminus \Gamma_0}(0)}{\partial n_+} = \omega_{\Gamma^*}(0).
\]

This again contradicts [25, Theorem 1.1], and the proof is complete.

The proof above used the following lemma.

**Lemma 3.2** With the assumptions of Theorem 3.1, there is a constant \( C \) such that if \( Q_n \) is a polynomial of degree at most \( n \) then

\[
\|Q_n\|_\Gamma \leq C_n \|Q_n\|_{L^2(\sigma_\Gamma)},
\]

(3.24)

where, on the left-hand side, the norm is the supremum norm on \( \Gamma \).

**Proof.** Let \( M \) be the maximum of \( |Q_n(z)| \) on \( \Gamma \). By [25, Corollary 7.2] (applied to each one of the components of \( \Gamma \)) we have

\[
|Q_n'(z)| \leq C_1 M n^2, \quad \text{for } \text{dist}(z, \Gamma) \leq 1/n^2
\]

with some constant \( C_1 \geq 1 \). Therefore, if \( z_0 \in \Gamma \) is a place with \( |Q_n(z_0)| = M \), then for \( |z - z_0| \leq 1/2 C_1 n^2 \) we have \( |Q_n(z)| \geq M/2 \). The \( sr \)-measure of the set of these \( z \)'s is at least \( 1/2 C_1 n^2 \), hence

\[
\int |Q_n|^2 ds_\Gamma \geq (M/2)^2 / 2 C_1 n^2,
\]

(35)
from which the claim follows.

\[ \limsup_{n \to \infty} n \lambda_n(z, \nu) \leq \frac{w(z_0)}{\omega_{\Gamma}(z_0)}, \quad (4.1) \]

was proven in Theorem 2.1.

In particular, if \( w(z_0) = 0 \), then (1.4) is true, so in establishing the matching lower bound to (4.1) we may assume that \( w(z_0) > 0 \). Let \( \Sigma \) be the set of zeros of \( w \), and for a small \( \tau > 0 \) let \( \Sigma_\tau \) be the \( \tau \)-neighborhood of \( \Sigma \). The set \( \Gamma_\tau := \Gamma \setminus \Sigma_\tau \) consists of finitely many Jordan curves and arcs, some of which may be degenerated (may consist of a single point), which we discard from \( \Gamma_\tau \). If \( \tau \) is sufficiently small, then \( z_0 \) is a point on \( \Gamma_\tau \) which is not an endpoint of any of \( \Gamma_\tau \)'s components. Now on \( \Gamma_\tau \) the measure \( \nu_\tau := \nu|_{\Gamma_\tau} \) has already a strictly positive density \( w \) with respect to the arc measure, so we can apply the already proven case to it to conclude that

\[ \liminf_{n \to \infty} n \lambda_n(z_0, \nu) \geq \liminf_{n \to \infty} n \lambda_n(z_0, \nu_\tau) \geq \frac{w(z_0)}{\omega_{\Gamma_\tau}(z_0)}, \]

so it has remained to show that on the right-hand side \( \omega_{\Gamma_\tau}(z_0) \) tends to \( \omega_{\Gamma}(z_0) \) as \( \tau \to 0 \).

To this end first we prove that the logarithmic capacity \( \operatorname{cap}(\Gamma_\tau) \) tends to the capacity \( \operatorname{cap}(\Gamma) \) of \( \Gamma \), and in doing so we may assume that \( \Gamma \) lies inside the disk \( \{|z| \leq 1/2\} \) (apply a homothetic transformation). The equilibrium measure \( \mu_\Gamma \) is absolutely continuous with respect to the arc measure \( s_\Gamma \) (see Proposition 2.2), hence \( \alpha_n := \mu_\Gamma(\Gamma_\tau) \) tends to 1 as \( \tau \to 0 \). Now the measure

\[ \mu_n := \frac{1}{\alpha_n} \mu_\Gamma|_{\Gamma_\tau} \]

is a positive unit measure on \( \Gamma_\tau \) for which the logarithmic energy

\[ I(\mu_n) := \int \int \frac{1}{|z-t|} d\mu_n(z) d\mu_n(t) \leq \frac{1}{\alpha_n^2} \int \int \frac{1}{|z-t|} d\mu_\Gamma(z) d\mu_\Gamma(t) = \frac{1}{\alpha_n^2} I(\mu_\Gamma). \]

4 Proof of Theorems 1.1 and 1.2

So far we have established Theorem 1.1 for the case when \( w \) is strictly positive. Now we can easily complete the

**Proof of Theorem 1.1.** That

\[ \limsup_{n \to \infty} n \lambda_n(z, \nu) \leq \frac{w(z_0)}{\omega_{\Gamma}(z_0)}, \quad (4.1) \]

was proven in Theorem 2.1.

In particular, if \( w(z_0) = 0 \), then (1.4) is true, so in establishing the matching lower bound to (4.1) we may assume that \( w(z_0) > 0 \). Let \( \Sigma \) be the set of zeros of \( w \), and for a small \( \tau > 0 \) let \( \Sigma_\tau \) be the \( \tau \)-neighborhood of \( \Sigma \). The set \( \Gamma_\tau := \Gamma \setminus \Sigma_\tau \) consists of finitely many Jordan curves and arcs, some of which may be degenerated (may consist of a single point), which we discard from \( \Gamma_\tau \). If \( \tau \) is sufficiently small, then \( z_0 \) is a point on \( \Gamma_\tau \) which is not an endpoint of any of \( \Gamma_\tau \)'s components. Now on \( \Gamma_\tau \) the measure \( \nu_\tau := \nu|_{\Gamma_\tau} \) has already a strictly positive density \( w \) with respect to the arc measure, so we can apply the already proven case to it to conclude that

\[ \liminf_{n \to \infty} n \lambda_n(z_0, \nu) \geq \liminf_{n \to \infty} n \lambda_n(z_0, \nu_\tau) \geq \frac{w(z_0)}{\omega_{\Gamma_\tau}(z_0)}, \]

so it has remained to show that on the right-hand side \( \omega_{\Gamma_\tau}(z_0) \) tends to \( \omega_{\Gamma}(z_0) \) as \( \tau \to 0 \).

To this end first we prove that the logarithmic capacity \( \operatorname{cap}(\Gamma_\tau) \) tends to the capacity \( \operatorname{cap}(\Gamma) \) of \( \Gamma \), and in doing so we may assume that \( \Gamma \) lies inside the disk \( \{|z| \leq 1/2\} \) (apply a homothetic transformation). The equilibrium measure \( \mu_\Gamma \) is absolutely continuous with respect to the arc measure \( s_\Gamma \) (see Proposition 2.2), hence \( \alpha_n := \mu_\Gamma(\Gamma_\tau) \) tends to 1 as \( \tau \to 0 \). Now the measure

\[ \mu_n := \frac{1}{\alpha_n} \mu_\Gamma|_{\Gamma_\tau} \]

is a positive unit measure on \( \Gamma_\tau \) for which the logarithmic energy

\[ I(\mu_n) := \int \int \frac{1}{|z-t|} d\mu_n(z) d\mu_n(t) \leq \frac{1}{\alpha_n^2} \int \int \frac{1}{|z-t|} d\mu_\Gamma(z) d\mu_\Gamma(t) = \frac{1}{\alpha_n^2} I(\mu_\Gamma). \]

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Therefore,

$$I(\mu_{\Gamma_\tau}) \leq \frac{1}{\alpha_n^2} I(\mu_{\Gamma})$$

because $\mu_{\Gamma_\tau}$ minimizes the logarithmic energy. Hence

$$\text{cap}(\Gamma_\tau) = \exp(-I(\mu_{\Gamma_\tau})) \geq \exp(-I(\mu_{\Gamma}))^{1/\alpha_n^2} = \text{cap}(\Gamma)^{1/\alpha_n^2},$$

from which $\text{cap}(\Gamma_\tau) \to \text{cap}(\Gamma)$ follows (note that $\Gamma_\tau \subseteq \Gamma$ implies $\text{cap}(\Gamma_\tau) \leq \text{cap}(\Gamma)$).

The function $g_{\Gamma_\tau}(z) - g_{\Gamma}(z)$ is nonnegative and harmonic in $\Omega$ (the exterior of $\Gamma$) including infinity, and at infinity it takes the value (see [19, (1.4.8)] or [17, p. 107]) $\log(\text{cap}(\Gamma)/\text{cap}(\Gamma_\tau))$, which tends to 0 as $\tau \to 0$ by what we have just established. Hence, by Harnack’s principle, this function tends 0 (as $\tau \to 0$) locally uniformly in $\Omega$. From this it follows via the maximum principle that $g_{\Gamma_\tau}(z) - g_{\Gamma}(z)$ tends to 0 locally uniformly inside any connected bounded component of $C \setminus \Gamma$, as well (these are the interiors of those components of $\Gamma$ that are Jordan curves). This and the fact that for sufficiently small $\tau > 0$ we have $g_{\Gamma_\tau}(z) - g_{\Gamma}(z) = 0$ on any small fixed arc $J \subset \Gamma$ about $z_0$ (so small that on $J$ the function $w$ is strictly positive) implies, by [12, Lemma 7.1], that if $n$ is either of the normals to $\Gamma$ at $z_0$, then

$$\frac{\partial g_{\Gamma_\tau}(z_0)}{\partial n} \to \frac{\partial g_{\Gamma}(z_0)}{\partial n} \quad \text{as} \quad \tau \to 0.$$

Now the claim $\omega_{\Gamma_\tau}(z_0) \to \omega_{\Gamma}(z_0)$ as $\tau \to 0$ follows from here and from formula (5.3) below.

Finally, we give the

**Proof of Theorem 1.2.** The upper estimate (2.1) was given in Theorem 2.1, and that theorem holds under the assumptions of Theorem 1.2, so (2.1) is true.

In the proof of the lower estimate (3.1) the only place where we used the strict positivity of $w$ was (3.10) (proved in Lemma 3.2), and it is clear from the proof that (3.10) can be replaced by

$$\|Q_n\|_{\text{Reg}} = e^{o(n)}. \quad (4.2)$$

But this is true in our case, since $\nu \in \text{Reg}$ and

$$\int |Q_n|^2 d\nu \leq \frac{C}{n}$$

imply (4.2) (see (1.3)). Thus, (3.1) is also true under the conditions of Theorem 1.2, and hence Theorem 1.2 follows.
5 Appendix

Proof of Proposition 2.2. In short, the proof is that $\omega_\Gamma$ is given by the normal derivative of the Green’s function (see formula (5.3) below) $g_{C\setminus \Gamma}$, and away from the endpoints of the arc components of $\Gamma$, this Green’s function is $C^{1+\alpha}$ smooth on $\Gamma$ due to the $C^{1+\alpha}$ smoothness of $\Gamma$. We shall use a standard localization technique. The details are as follows.

As has already been said, the $\alpha > 0$ in the $C^{1+\alpha}$ smoothness assumption is assumed to be less than 1. First of all, note that the Green’s function $g_{C\setminus \Gamma}$ is continuous on $C$ by Wiener’s criterion [17, Theorem 5.4.1].

First, let $J$ be a closed arc on $\Gamma$ not containing an endpoint of an arc-component of $\Gamma$. Let $G$ be a simply connected domain with $C^{1+\alpha}$ boundary that lies in the unbounded component $\Omega$ of $C \setminus \Gamma$ such that $J$ lies on the boundary of $G$, and let $\Phi$ be a conformal map from the unit disk $\Delta$ onto $G$. By the Kellogg-Warschawski theorem (see [16, Theorem 3.6]) this $\Phi$ is $C^{1+\alpha}$ on the closed unit disk and it has a nonzero derivative there. The function $h(z) = g_{C\setminus \Gamma}(\Phi(z))$ is harmonic in $\Delta$ and continuous on the closed unit disk, so we have Poisson’s formula for it:

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} h(e^{it})dt.$$  

(5.1)

If $J'$ is the arc of the unit circle that is mapped by $\Phi$ into $J$, then $h(e^{it}) = 0$ on $J'$, so it follows from (5.1) that $h$ (considered as a function on the closed unit disk) is $C^\infty$ on any closed subarc of the interior of $J'$. Hence $g_{C\setminus \Gamma}(z) = h(\Phi^{-1}(z))$ is $C^{1+\alpha}$-smooth on any closed subarc of the interior of $J$. Furthermore, (5.1) gives also that

$$h(re^{it}) \geq \frac{1 - r}{1 + r} \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{it})dt = \frac{1 - r}{1 + r} h(0) > 0,$$

which gives via the mapping $\Phi$

$$g_{C\setminus \Gamma}(z + tn) \geq ct$$

for any $z \in J$ with a positive constant $c > 0$ depending only on $G$, where $n$ is the normal to $\Gamma$ at $z$ in the direction of $G$. As a consequence,

$$\frac{g_{C\setminus \Gamma}}{\partial n}(z) \geq c, \quad z \in J.$$  

(5.2)

Now all we need to do is to cite that in the interior of $J$ we have (see e.g. [14, II.(4.1)] or [19, Theorem IV.2.3] and [19, (I.4.8)])

$$\omega_\Gamma(z) = \frac{1}{2\pi} \left( \frac{g_{C\setminus \Gamma}}{\partial n_+}(z) + \frac{g_{C\setminus \Gamma}}{\partial n_-}(z) \right),$$  

(5.3)
where \( \mathbf{n}_\pm \) are the two normals to \( \Gamma \) at \( z \). The \( C^\alpha \) smoothness of \( \omega_T \) on \( J \) follows from the \( C^{1+\alpha} \)-smoothness of \( g_{C\setminus\Gamma} \) there, while the positivity is a consequence of \((5.2)\) (where \( \mathbf{n} \) is one of \( \mathbf{n}_\pm \) pointing to \( \Omega \) and note also that both normal derivatives in \((5.3)\) are nonnegative).

Next, let the arc \( J \) contain an endpoint of an arc-component of \( \Gamma \). We may assume that this endpoint is 0, and the positive semi-axis is a tangent to \( J \) at 0. Then in some small neighborhood of 0 the arc \( J \) has parametrization \( t + i\gamma(t) \), \( 0 \leq t \leq t_0 \) where \( \gamma(0) = \gamma'(0) = 0 \) and \( \gamma' \) is Lip \( \alpha \) continuous on \([0, t_0]\). Hence \( |\gamma'(t)| \leq C t^\alpha \) and \( |\gamma(t)| \leq C t^{1+\alpha} \). Consider a small disk \( D_\rho \) with center at 0 and of radius \( \rho \), and in \( D_\rho \setminus J \) take the branch of \( \sqrt{z} \) for which \( \sqrt{\rho} = \sqrt{t}(1 + i)/\sqrt{2} \) for \( t > 0 \). Then \( w = \sqrt{z} \) maps \( D_\rho \setminus J \) into a set \( D^* \) which is a subset of \( D_{\sqrt{\rho}}^* \) the boundary of which consists of two parts: a half-circle of \( D_{\sqrt{\rho}}^* \) and an arc \( J^* \), which is the union of the two images \( \pm J^* \) of \( J \) under this map (the two images are symmetric with respect to the origin). One of these images, say \( J^* \) has representation \( \theta + i\sigma(\theta) \) (and the symmetric part has then the representation \( -\theta - i\sigma(\theta) \)) with \( 0 \leq \theta \leq \theta_0 \) where \( (\theta + i\sigma(\theta))^2 = t + i\gamma(t) \). Straightforward calculation gives that then \( \sigma(0) = \sigma'(0) = 0 \) and \( \sigma' \) is in the class Lip \( \alpha \). As a consequence, \( J^* \cup (-J^*) \) is again \( C^{1+\alpha} \) smooth. Now the argument that we used above gives that then \( g^*(z) := g_{C\setminus\Gamma}(z^2) \) defined on \( D^* \) is of class \( C^{1+\alpha} \) on the (one dimensional) interior of \( J^* \cup (-J^*) \) with positive and \( C^\alpha \)-smooth normal derivatives there. Now if \( z_0 \in D_\rho \cap J \) is any point, then the normal vector \( \mathbf{n}^* \) at \( \sqrt{z_0} \in J^* \) to \( J^* \) in the direction of \( D^* \) and (one of the) normal vector \( \mathbf{n} \) at \( z_0 \) to \( J \) is related by \( \mathbf{n} = (2\sqrt{z_0}/2\sqrt{|z_0|}) \mathbf{n}^* \) since around \( \sqrt{z_0} \) the mapping \( z \rightarrow z^2 \) is like multiplication by \( 2\sqrt{z_0} \). Hence

\[
\frac{\partial g^*}{\partial \mathbf{n}^*}(\sqrt{z_0}) = \lim_{t \rightarrow 0^+} \frac{g((\sqrt{z_0} + t\mathbf{n}^*)^2) - g((\sqrt{z_0})^2)}{t} = \lim_{t \rightarrow 0^+} \frac{g(z_0 + 2\sqrt{z_0}t\mathbf{n}^* + O(t^2)) - g(z_0)2\sqrt{|z_0|}}{2\sqrt{|z_0|}t} = 2 \sqrt{|z_0|} \frac{\partial g_{C\setminus\Gamma}}{\partial \mathbf{n}}(z_0).
\]

This implies, in view of the fact that \( \partial g^*/\partial \mathbf{n}^* \) is positive and Lip \( \alpha \) around 0, that

\[
\frac{\partial g_{C\setminus\Gamma}}{\partial \mathbf{n}}(z_0) \sim 1/\sqrt{|z_0|}. \tag{5.4}
\]

A similar formula is true for the normal derivative with respect to the other normal to \( J \) at \( z_0 \). Now \( \omega_T(z_0) \sim 1/\sqrt{|z_0|} \) follows from these and from formula \((5.3)\).

An alternative proof of \((5.4)\) is to use \([16, \text{Theorem 3.9}]\), which implies \((5.4)\) for \( g_{C\setminus\Gamma_k} \) (the Green’s function of the complement of the arc component \( \Gamma_k \) in question), and use the comparison \( g_{C\setminus\Gamma} \leq g_{C\setminus\Gamma_k} \leq C g_{C\setminus\Gamma} \) valid in some neighborhood of \( \Gamma_k \) (apply the maximum principle in that neighborhood to the difference \( g_{C\setminus\Gamma_k} - g_{C\setminus\Gamma} \)).
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