Szegő's problem on curves*

Vilmos Totik†

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Abstract

For a system of smooth Jordan curves asymptotics for Christoffel functions is established almost everywhere for measures belonging to Szegő's class.

1 The result

Let μ be a finite Borel-measure on the plane with compact support consisting of infinitely many points. The Christoffel functions associated with μ are defined as

$$\lambda_n(z,\mu) = \inf_{P_n(z)=1} \int |P_n|^2 d\mu,$$

where the infimum is taken for all polynomials of degree at most n that take the value 1 at z.

Christoffel functions are closely related to orthogonal polynomials (for a survey see [15] by P. Nevai and [22] by B. Simon), to statistical physics (see e.g. [16] by L. Pastur), to universality in random matrix theory (see e.g. the recent breakthrough [11] by D. Lubinsky, as well as [3],[23],[29]), to spectral theory (see e.g. [24], [22] by B. Simon and [1] by Breuer, Last and Simon) and to several other fields in mathematics. For the role and various use of Christoffel functions see [5], [7], [24], and particularly [15] by P. Nevai and [22] by B. Simon.

Their asymptotics on the real line and on the unit circle has been thoroughly investigated (see e.g. [11], [12], [13], [24], [23], [21], [26], [28]), but until recently not much has been known on their asymptotic behavior on general curves. In this work we prove

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Theorem 1 *Let Γ be the union of finitely many C^2-smooth Jordan curves lying exterior to one another, and let μ be a Borel-measure on Γ such that its Radon-Nikodym derivative $w = d\mu/ds_\Gamma$ with respect to the arc measure s_Γ on Γ satisfies the Szegő condition $\log w \in L^1(s_\Gamma)$. Then for s_Γ-almost every $z \in \Gamma$ we have*

$$\lim_{n\to\infty} n\lambda_n(z,\mu) = \frac{d\mu(z)}{d\omega_\Gamma}, \tag{1}$$

where ω_Γ denotes the equilibrium measure of Γ, and on the right-hand side $d\mu(z)/d\omega_\Gamma$ is the Radon-Nikodym derivative of μ with respect to ω_Γ.

Recall that the equilibrium measure ω_Γ is the unique probability Borel-measure on Γ that minimizes the logarithmic energy

$$\int\int \log\frac{1}{|z-t|}d\omega(t)d\omega(z).$$

See e.g. [18] for the concepts from potential theory that are used in this paper.

In what follows, let

$$d\mu(x) = w(x)ds_\Gamma(x) + d\mu_{\rm sing}(x)$$

be the Lebesgue-Radon-Nikodym decomposition of μ into its absolutely continuous and singular part with respect to the arc measure s_Γ. With this notation we will actually show that (1) holds at every $z \in \Gamma$ which is a Lebesgue-point (with respect to s_Γ) for both μ and $\log w$ (c.f. (27)–(28)). The theorem can be written in the alternate form (c.f. [30, (3.4)])

$$\lim_{n\to\infty} n\lambda_n(z,\mu) = 2\pi w(z)\left(\frac{\partial g_\Omega(z_0,\infty)}{\partial\mathbf{n}}\right)^{-1} \tag{2}$$

s_Γ-almost everywhere, where $g_\Omega(z_0,\infty)$ denotes the Green's function with pole at infinity associated with the unbounded component Ω of $\overline{\mathbf{C}}\setminus\Gamma$, and $\partial(\cdot)/\partial\mathbf{n}$ denotes normal derivative in the direction of the inner normal to $\partial\Omega$. Note also that if σ_Γ is the density of the equilibrium measure ω_Γ with respect to arc measure, then the limit on the right-hand side of (1) is $w(z)/\sigma_\Gamma(z)$.

A feature of the limit in (1) is that the Christoffel functions "feel" the complete support of μ. This is through the condition $\log w \in L^1(s_\Gamma)$, and in a sense some global condition like that is necessary (just consider that it follows from the theorem itself that if we zero out μ on a component of Γ then the limit on other components will change even though locally there is no change there in the measure). For a much less restrictive global condition see Theorem 2 below.

A brief history of asymptotics of Christoffel functions is as follows. In 1915 G. Szegő proved that if $d\mu(t) = \mu'(t)dt$ is an absolutely continuous measure on the unit circle (identified with $[-\pi,\pi]$) then

$$\lim_{n\to\infty} \lambda_n(z,\mu) = (1-|z|^2)\exp\left(\Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}-z}{e^{it}+z}\log\mu'(t)dt\right), \qquad |z|<1,$$

provided $\log \mu'$ is integrable (otherwise the limit on the left is 0). This was later generalized by several authors (see e.g. [7], [9], [10]). On the boundary of the circle λ_n decreases as $1/n$, and Szegő ([27, Th. I', p. 461]) established that on the unit circle, i.e. on the support of the measure,

$$\lim_{n\to\infty} n\lambda_n(e^{i\theta}, \mu) = 2\pi\mu'(\theta) \tag{3}$$

under the condition that μ is absolutely continuous and $\mu' > 0$ is twice continuously differentiable. L. Golinskii [6] extended this to the arc case: if μ is a so-called Bernstein-Szegő weight on the arc $\{e^{i\theta} \,|\, \alpha \le \theta \le 2\pi - \alpha\}$, then

$$\lim_{n\to\infty} n\lambda_n(e^{i\theta}, \mu) = 2\pi\mu'(\theta)\frac{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\theta}{2}}}{\sin\frac{\theta}{2}} \tag{4}$$

for $e^{i\theta}$ in this arc.

The almost everywhere part of (3) was harder, it was proved only in 1991 by A. Máté, P. Nevai and V. Totik [13] that (3) is true almost everywhere provided $\log \mu'$ is integrable. This has a consequence for measures lying on an interval: if the support of μ is $[-1, 1]$ and $\log \mu' \in L^1_{\mathrm{loc}}(-1, 1)$ then

$$\lim_{n\to\infty} n\lambda_n(x, \mu) = \pi\sqrt{1 - x^2}\mu'(x) \tag{5}$$

for Lebesgue-almost every $x \in [-1, 1]$. On the proof in [13] (for the unit circle) Simon wrote in [22]: "The proof is clever but involved; it would be good to find a simpler proof". The proof we give for Theorem 1 provides such a new proof.

In [30] the Szegő asymptotics (3) (the case when w is continuous) was shown to be true on C^2 curves, namely it was proved that (1) is true if w is continuous and $\mu_{\mathrm{sing}} = 0$. M. Findley [4] verified the almost everywhere result: if Γ consists of a single smooth Jordan curve and $\log w \in L^1(s_\Gamma)$, then (1) is true s_Γ-almost everywhere. His method was a nontrivial refinement of the original proof in [13] (which was for the circle case) by mixing in the original argument conformal maps and Faber polynomials. This approach does not work when Γ has more than one components, and the general case remained open and requires different ideas. In this paper we present a new approach which not only solves this problem, but in a certain sense gives more than the proof in [13] even when Γ is the unit circle. Basically, we shall show that the almost everywhere result follows from the continuous one with the help of sharp estimates on harmonic measures. In a nutshell the proof is based on the new type inequality

$$|P_n(z)|^2 \le M e^{M\sqrt{n|z-z_0|}}\, n \int_\Gamma |P_n|^2 w \, ds_\Gamma, \qquad z \in \Gamma,\ \deg(P_n) \le n, \tag{6}$$

provided $\log w \in L^1(s_\Gamma)$ and $z_0 \in \Gamma$ is a Lebesgue-point for $\log w$. The other ingredient is the use of fast decreasing polynomials: there are polynomials R_m

of degree at most m such that $R_m(z_0) = 1$ and with some constants C_0, c_0

$$|R_m(z)| \le C_0 \exp\Big(-c_0(m|z-z_0|)^{2/3}\Big), \qquad z \in \Gamma, \tag{7}$$

i.e. these polynomials decrease very fast as we move away from z_0. The point is that even if m is small compared to n, say $m = \varepsilon n$, the factor $e^{-c_0(m|z-z_0|)^{2/3}}$ in (7) kills the factor $e^{M\sqrt{n|z-z_0|}}$ in (6).

Finally, we note that Theorem 1 has a local form. To formulate it let μ be an arbitrary Borel-measure with compact support on $\overline{\mathbf{C}}$ and let $K = \mathrm{supp}(\mu)$ be the support of μ. We assume that Ω, the unbounded component of $\overline{\mathbf{C}} \setminus K$, is regular with respect to solving Dirichlet problems. μ is called to be in the **Reg** class (see [25, Theorem 3.2.3]) if the $L^2(\mu)$-norms and the $L^\infty(\mu)$ norms of polynomials are asymptotically the same in n-th root sense, i.e. if

$$\lim_{n\to\infty} \sup_{P_n} \left(\frac{\|P_n\|_{L^\infty(\mu)}}{\|P_n\|_{L^2(\mu)}}\right)^{1/n} \to 1, \tag{8}$$

where the supremum is taken for all (nonzero) polynomials of degree at most n. This is a fairly weak condition on μ—see [25] for general regularity criteria and different equivalent formulations of $\mu \in \mathbf{Reg}$. For example, in the scenario of Theorem 1 if $w(t) = d\mu(t)/ds_\Gamma > 0$ is true s_Γ-almost everywhere, then $\mu \in \mathbf{Reg}$, so Theorem 1 is a special case of the following one, in which $\mathrm{cap}(K)$ stands for the logarithmic capacity of K, ω_K for its equilibrium measure, and $\mathrm{Pc}(K) = \overline{\mathbf{C}} \setminus \Omega$ is the so called polynomial convex hull of K (this is the union of K with the bounded components of $\mathbf{C} \setminus K$).

Theorem 2 *Assume that μ is in the* **Reg** *class and its support K satisfies* $\mathrm{cap}(K) = \mathrm{cap}(\mathrm{Int}(\mathrm{Pc}(K)))$, *where* Int *means two dimensional interior. Suppose that for some open disk D with center on $\partial\Omega$ the intersection $D \cap K$ is a C^2 Jordan arc J, and on J the Radon-Nikodym derivative $w = d\mu/ds_J$ of μ with respect to arc length s_J on J satisfies* $\log w \in L^1(s_J)$. *Then*

$$\lim_{n\to\infty} n\lambda_n(z,\mu) = \frac{d\mu(z)}{d\omega_K} \tag{9}$$

for s_J-almost every $z \in J$.

This again has the equivalent form (2) (see [30, (3.4)]).

We shall not prove Theorem 2, for the additional difficulties compared with Theorem 1 has already been dealt with in [30] (see particularly the difference in between the proofs of Theorems 1.1 and 1.2 in [30]).

2 Preliminaries for the proof

First we make some notations (see Figure 1). For some $0 < \alpha < 1$ let γ be a positively oriented $C^{1+\alpha}$-smooth Jordan curve, $\Omega^* = \Omega^*(\gamma)$ resp. $\Omega = \Omega(\gamma)$

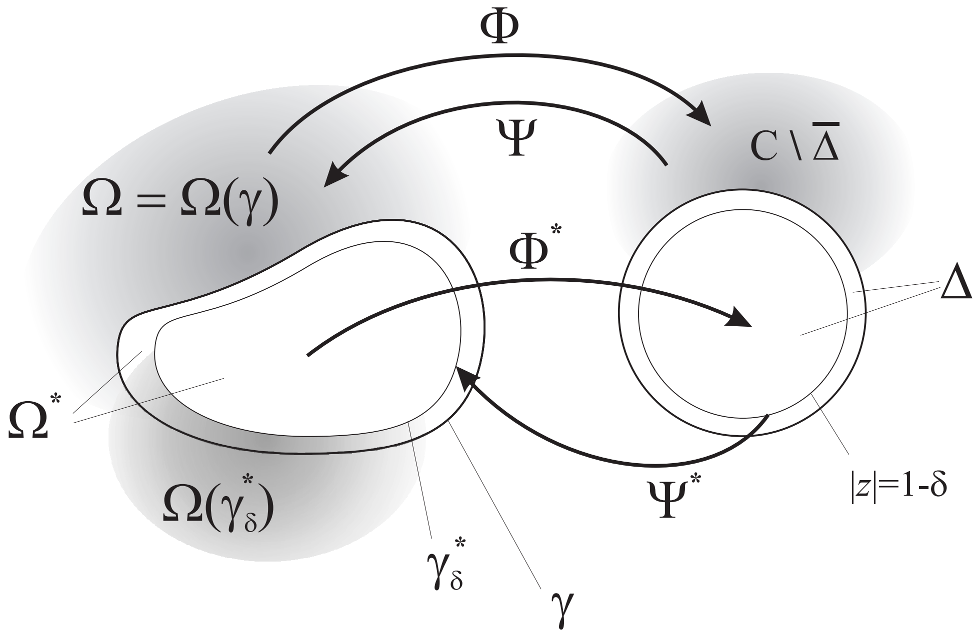

Figure 1:

its inner resp. outer domains. Fix a conformal map $\Phi = \Phi_\gamma$ from Ω onto the exterior of the unit circle Δ, and let Ψ be the inverse of Φ. In a similar manner, let Φ^* be a conformal map from Ω^* onto the unit disk Δ, and let Ψ^* be its inverse. We shall frequently use the Kellogg-Warschawski theorem (see [17, Theorems 3.5, 3.6]: Φ, Ψ, Φ^*, Ψ^* are $C^{1+\alpha}$ up to the boundary. Furthermore, their derivatives vanish nowhere (including the boundary).

Let Γ be a system of curves consisting of finitely many such C^2-smooth γ's lying exterior to one another. We shall denote by $s = s_\Gamma$ the arc length on Γ. Let μ be a measure on Γ such that its Radon-Nikodym derivative (with respect to arc length) $w = d\mu/ds$ satisfies $\log w \in L^1(s)$. It is enough to prove Theorem 1 on an arbitrary component of Γ, which we shall denote by γ. With this γ and with w on γ we shall consider the associated Szegő function D^* in Ω^*. Its definition is

$$D^*(\Psi^*(z)) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{it}-z}{e^{it}+z}\log w(\Psi^*(t))dt\right), \qquad |z|<1, \tag{10}$$

so on γ the function $D^*(z)$ has nontangential boundary limit $D^*(\zeta)$ s_γ-almost everywhere, and $|D^*(\zeta)|^2 = w(\zeta)$ for s_γ-almost every $\zeta \in \gamma$.

The proof of Theorem 1 is based on the next lemma. As usual, we say that $\zeta_0 \in \gamma$ is a Lebesgue-point for w (with respect to s) if

$$\lim_{s(J)\to 0}\frac{1}{s(J)}\int_J |w(\zeta)-w(\zeta_0)|ds(\zeta) = 0,$$

where the limit is taken for subarcs J of γ that contain ζ_0, the arc length $s(J)$ of which tends to 0.

Lemma 3 *Let γ be a $C^{1+\alpha}$ Jordan curve, $w \geq 0$ a (s_γ-measurable) function on γ such that $w, \log w \in L^1(s_\gamma)$, and let $\zeta_0 \in \gamma$ be a Lebesgue-point for* $\log w$. *Then there is a constant M such that for $z \in \gamma$ we have*

$$|P_n(z)|^2 \leq Me^{M\sqrt{n|z-\zeta_0|}}\, n \int_\gamma |P_n|^2 w\, ds_\gamma \tag{11}$$

for any polynomials P_n of degree at most $n = 1, 2, \ldots$.

For later reference we mention that (11) is actually true on and inside γ. Indeed, to verify this let Ω^* be the inner domain of γ, and we may assume

$$n \int_\gamma |P_n|^2 w\, ds \leq 1. \tag{12}$$

By the subharmonicity of $\log |P_n(z)|$ we have for $z \in \Omega^*$

$$\log |P_n(z)|^2 \leq \int_\gamma \log |P_n(\zeta)|^2 d\tilde{\omega}(z, \zeta, \Omega^*),$$

where $\tilde{\omega}(z, \cdot, \Omega^*)$ is the harmonic measure of z on Ω^*. The conformal invariance of harmonic measures and [18, Table 4.1] show that if J_k is the part of γ for which $2^k|z - \zeta_0| \leq |\zeta - \zeta_0| \leq 2^{k+1}|z - \zeta_0|$, then $\tilde{\omega}(z, J_k, \Omega^*) \leq C/2^k$, so (11) applied with ζ instead of z gives

$$\int_\gamma \log |P_n(\zeta)|^2 d\tilde{\omega}(z, \zeta, \Omega^*) \leq \log M + \sum_{k\geq 0} M\sqrt{n2^{k+1}|z-\zeta_0|}\frac{C}{2^k} \leq C + C\sqrt{n|z-\zeta_0|}.$$

Note however, that outside γ nothing more than

$$|P_n(z)| \leq Me^{Mn|z-\zeta_0|} \quad \text{(more precisely } |P_n(z)| \leq Me^{Mn\,\mathrm{dist}(z,\gamma)}\text{)}$$

can be said (just think of the unit circle with Lebesgue-measure and $P_n(z) = z^n$).

Proof of Lemma 3. Without loss of generality we may assume $\zeta_0 = 1$ and the bound (12).

In what follows we shall denote by γ_δ^* the image of $|z| = 1 - \delta$ under the conformal map Ψ^* (see Figure 1). Then γ_δ^*, $0 < \delta \leq 1/2$, are all uniformly $C^{1+\alpha}$, and the corresponding conformal maps $\Phi_{\gamma_\delta^*}$ (mapping the unbounded component $\Omega(\gamma_\delta^*)$ of $\overline{\mathbf{C}} \setminus \gamma_\delta^*$ onto the exterior of the unit disk $\overline{\Delta}$) are uniformly $C^{1+\alpha}$ up to the boundary. Therefore, if $g_{\Omega(\gamma_\delta^*)}(z, \infty)$ denotes the Green's function of $\Omega(\gamma_\delta^*)$ with pole at infinity, then the functions $g_{\Omega(\gamma_\delta^*)}(z, \infty)$ are also uniformly $C^{1+\alpha}$ because $g_{\Omega(\gamma_\delta^*)}(z, \infty) = \log |\Phi_{\gamma_\delta^*}(z)|$.

For a Jordan domain D bounded by a rectifiable $C^{1+\alpha}$-smooth Jordan curve ∂D let $\omega(z, \zeta, D)ds_{\partial D}(\zeta)$ be the harmonic measure of $z \in D$, where $s_{\partial D}$ is the arc measure on ∂D (one can easily see that this harmonic measure is absolutely continuous with respect to $s_{\partial D}$, hence it can be written in the form $\omega(z,\zeta,D)ds_{\partial D}(\zeta)$ actually with a continuous $\omega(z,\zeta,D)$). This is a unit measure on ∂D.

We claim that

$$\omega(z,\zeta,\Omega^*) \sim \frac{d(z,\gamma)}{|\zeta - z|^2 + d(z,\gamma)^2}, \qquad z \in \Omega^*,\ \zeta \in \gamma, \tag{13}$$

and uniformly in $0 \le \delta \le 1/2$

$$\omega(z,\zeta,\Omega(\gamma_\delta^*)) \sim \frac{d(z,\gamma_\delta^*)}{|\zeta - z|^2 + d(z,\gamma_\delta^*)^2}, \qquad z \in \gamma,\ \zeta \in \gamma_\delta^*, \tag{14}$$

where $d(z,\gamma)$ denotes the distance from z to γ, and $\sim$ means that the ratio of the two sides lies in between two fixed constants. In fact, if $\Phi^*(\zeta) = t$ then $\zeta = \Psi^*(t)$ and $ds_\gamma(\zeta) = |d\zeta| = |(\Psi^*)'(t)||dt|$, hence, by the conformal invariance of harmonic measure,

$$\omega(z,\zeta,\Omega^*)ds_\gamma(\zeta) = \omega(\Phi^*(z),t,\Delta)|(\Psi^*)'(t)||dt|,$$

and here $|(\Psi^*)'(t)|$ is uniformly bounded away from 0 and ∞ (recall that $\Delta = \{|z| < 1\}$ denotes the unit disk). Furthermore, $|\zeta - z| \sim |\Phi^*(\zeta) - \Phi^*(z)| = |t - \Phi^*(z)|$ and $d(z,\gamma) \sim d(\Phi^*(z),\partial\Delta)$. Now the claim follows, since for the unit circle $\omega(\Phi^*(z),t,\Delta)$ is the Poisson kernel, and hence

$$\omega(\Phi^*(z),t,\Delta) = \frac{1}{2\pi}\frac{1 - |\Phi^*(z)|^2}{|t - \Phi^*(z)|^2} \sim \frac{d(\Phi^*(z),\partial\Delta)}{|t - \Phi^*(z)|^2 + d(\Phi^*(z),\partial\Delta)^2}.$$

The proof of (14) is the same if we use the uniform $C^{1+\alpha}$-smoothness of γ_δ^* and the associated mappings (recall that the harmonic measure in the exterior of the unit disk is given again by the Poisson kernel).

We shall also use that (13) is true with Ω^* replaced by $\Omega^*(\gamma_\delta^*)$ (this is the inner domain enclosed by the curve γ_δ^*, and actually it is the image of $|z| < 1-\delta$ under the mapping Ψ^* of Δ onto Ω^*), i.e. uniformly in $0 < \delta \le 1/2$

$$\omega(z,\zeta,\Omega^*(\gamma_\delta^*)) \sim \frac{d(z,\gamma_\delta^*)}{|\zeta - z|^2 + d(z,\gamma_\delta^*)^2}, \qquad z \in \Omega^*(\gamma_\delta^*),\ \zeta \in \gamma_\delta^*. \tag{15}$$

Indeed, this is immediate from the proof of (13) just given.

Next, note that for $z \in \gamma$ we have $d(z,\gamma_\delta^*) \sim \delta$ and for $\theta \in \gamma_\delta^*$ we have $d(\theta,\gamma) \sim \delta$. Now we claim that for $\zeta, z \in \gamma$

$$\int_{\gamma_\delta^*} \omega(\theta,\zeta,\Omega^*)\omega(z,\theta,\Omega(\gamma_\delta^*))ds_{\gamma_\delta^*}(\theta) \le C\frac{\delta}{|\zeta - z|^2 + \delta^2} \tag{16}$$

with a constant C independent of $\zeta, z \in \gamma$ and $0 < \delta \leq 1/2$. In fact, if $\theta \in \gamma_\delta^*$ and $|\theta - \zeta| \geq |\zeta - z|/2$, then (13) (with z replaced by θ) gives

$$\omega(\theta, \zeta, \Omega^*) \leq C \frac{\delta}{|\zeta - z|^2 + \delta^2}$$

(note that $d(\theta, \gamma) \sim \delta$). Hence the integral over that part of γ_δ^* which is of distance $\geq |\zeta - z|/2$ from ζ has this bound. On the other hand, if $|\theta - \zeta| \leq |\zeta - z|/2$ then necessarily $|\theta - z| \geq |\zeta - z|/2$, and then (14) (with ζ replaced by θ) gives

$$\omega(z, \theta, \Omega(\gamma_\delta^*)) \leq C \frac{\delta}{|\zeta - z|^2 + \delta^2}.$$

Therefore, the integral over the rest of γ_δ^* is

$$\leq C \frac{\delta}{|\zeta - z|^2 + \delta^2} \int_{\gamma_\delta^*} \omega(\theta, \zeta, \Omega^*) ds_{\gamma_\delta^*}(\theta). \tag{17}$$

But $\omega(\theta, \zeta, \Omega^*)$ is a harmonic function of θ in Ω^* and on γ_δ^* the measure $ds_{\gamma_\delta^*}(\theta)$ is less than a constant times the harmonic measure

$$\omega(\Psi^*(0), \theta, \Omega^*(\gamma_\delta^*)) ds_{\gamma_\delta^*}(\theta)$$

(c.f. (15)), therefore the integral in (17) is at most $C\omega(\Psi^*(0), \zeta, \Omega^*)$, which is bounded for $\zeta \in \gamma$ according to (13). With this the proof of (16) is complete.

After these we turn to the statement in the lemma. For $z \in \gamma$ set in the formulas above

$$\delta = \begin{cases} 1/n & \text{if } |z-1| \leq 1/n \\ \sqrt{|z-1|/n} & \text{if } |z-1| \geq 1/n. \end{cases} \tag{18}$$

Recall now the Szegő function D^* from (10). For $\theta \in \gamma_\delta^*$ we have from (13) with some constant C_1

$$\begin{aligned} |P_n(\theta) D^*(\theta)|^2 &= \left| \int_\gamma P_n(\zeta)^2 D^*(\zeta)^2 \omega(\theta, \zeta, \Omega^*) ds_\gamma(\zeta) \right| \\ &\leq C_1 \frac{1}{\delta} \int_\gamma |P_n(\zeta)|^2 |D^*(\zeta)|^2 ds_\gamma(\zeta) \leq C_1 \frac{1}{\delta} \frac{1}{n} \leq C_1, \end{aligned} \tag{19}$$

where we used that $|D^*(\zeta)|^2 = w(\zeta)$ s_γ-almost everywhere, and we also used the bound (12) for the integral. Here the equality needs some explanation, since the analytic (and hence harmonic) function $(P_n D^*)^2$ is represented in Ω^* by the Poisson integral (relative to Ω^*, i.e. when the Poisson kernel is $\omega(\theta, \zeta, \Omega^*) ds_\gamma(\zeta)$) involving its nontangential limit (denoted again by $(P_n D^*)^2(\theta)$) on γ. However, everything is conformal invariant, so the equality needs to be verified only when γ is the unit circle, in which case the formula follows from the fact that $(P_n D^*)^2$

is in H^1, and therefore it is represented in the unit disk as the Poisson integral of it boundary values.

Let now h be the solution of the Dirichlet problem in $\Omega(\gamma_\delta^*)$ (the outer domain of γ_δ^*) with boundary data $\log|D^*(\theta)|$ on γ_δ^*, and $g_{\Omega(\gamma_\delta^*)}(z,\infty)$ the Green's function of $\Omega(\gamma_\delta^*)$ with pole at infinity. The function

$$u(z) = \log|P_n(z)| + h(z) - ng_{\Omega(\gamma_\delta^*)}(z,\infty)$$

is subharmonic in $\Omega(\gamma_\delta^*)$, is harmonic around ∞ and, as we have seen in (19), is $\le \log C_1^{1/2}$ on the boundary $\partial\Omega(\gamma_\delta^*) = \gamma_\delta^*$, so it is $\le \log C_1^{1/2}$ everywhere in $\Omega(\gamma_\delta^*)$. Thus,

$$|P_n(z)| \le C_1^{1/2} \exp\Big(-h(z) + ng_{\Omega(\gamma_\delta^*)}(z,\infty)\Big), \qquad z \in \gamma. \tag{20}$$

What we have said about the uniform $C^{1+\alpha}$ property of the Green's functions $g_{\Omega(\gamma_\delta^*)}(z,\infty)$ implies that

$$ng_{\Omega(\gamma_\delta^*)}(z,\infty) \le Cn\delta \le \begin{cases} C & \text{if } |z-1| \le 1/n \\ C\sqrt{n|z-1|} & \text{otherwise} \end{cases} \tag{21}$$

by the choice of δ in (18). For h we have the representation

$$h(z) = \int_{\gamma_\delta^*} \big(\log|D^*(\theta)|\big)\omega(z,\theta,\Omega(\gamma_\delta^*))ds_{\gamma_\delta^*}(\theta),$$

and here

$$\log|D^*(\theta)| = \int_\gamma \frac{1}{2}\big(\log w(\zeta)\big)\omega(\theta,\zeta,\Omega^*)ds_\gamma(\zeta)$$

(this latter one follows from the definition of D^* if we apply the conformal map Φ^* of Ω^* onto the unit disk). Substitute this into the previous formula, switch the order of integration and use (16) to conclude

$$|h(z)| \le C\int_\gamma |\log w(\zeta)| \frac{\delta}{|\zeta - z|^2 + \delta^2} ds_\gamma(\zeta).$$

Now let first $|z-1| \ge 1/n$. By the Lebesgue-point property of $\log|w(\zeta)|$ at $\zeta = 1$

$$\int_{|\zeta-1|\le 2|z-1|} |\log w(\zeta)| \frac{\delta}{|\zeta-z|^2+\delta^2} ds_\gamma(\zeta)$$

$$\le \frac{1}{\delta}\int_{|\zeta-1|\le 2|z-1|} |\log w(\zeta)| ds_\gamma(\zeta) \le C\frac{|z-1|}{\delta}$$

with a constant C that may depend on the value $w(1)$, but is independent of $z \in \gamma$. Similarly, for every $k = 1, 2, \dots$

$$\int_{2^k|z-1| \le |\zeta-1| \le 2^{k+1}|z-1|} |\log w(\zeta)| \frac{\delta}{|\zeta - z|^2 + \delta^2} ds_\gamma(\zeta)$$

$$\le \frac{\delta}{(2^k|z-1|/2)^2} \int_{|\zeta-1| \le 2^{k+1}|z-1|} |\log w(\zeta)| ds_\gamma(\zeta)$$

$$\le \frac{\delta}{(2^k|z-1|/2)^2} C 2^{k+1}|z-1| \le C \frac{\delta}{2^k|z-1|}.$$

Adding these together for all k we obtain (cf. also the choice of δ in (18))

$$|h(z)| \le C \left(\frac{|z-1|}{\delta} + \frac{\delta}{|z-1|} \right) \le C\sqrt{n|z-1|}. \tag{22}$$

When $|z-1| \le 1/n$ we get similarly

$$\int_{|\zeta-1| \le 2/n} |\log w(\zeta)| \frac{\delta}{|\zeta - z|^2 + \delta^2} ds_\gamma(\zeta)$$

$$\le \frac{1}{\delta} \int_{|\zeta-1| \le 2/n} |\log w(\zeta)| ds_\gamma(\zeta) \le \frac{1}{\delta} C(2/n) \le C,$$

and for every $k = 1, 2, \dots$

$$\int_{2^k/n \le |\zeta-1| \le 2^{k+1}/n} |\log w(\zeta)| \frac{\delta}{|\zeta - z|^2 + \delta^2} ds_\gamma(\zeta)$$

$$\le \frac{1/n}{(2^k/2n)^2} \int_{|\zeta-1| \le 2^{k+1}/n} |\log w(\zeta)| ds_\gamma(\zeta) \le C \frac{1}{2^k},$$

which yield $|h(z)| \le C$.

Now this, (22), (21) and (20) prove the lemma.

■

We shall also use

Lemma 4 *Let K be a compact subset of the plane, Ω the unbounded component of its complement, and $Z \in \partial\Omega$ a point on the outer boundary of K. Assume that there is a disk in Ω that contains Z on its boundary. Then for every $\beta < 1$ there are constants $c_\beta, C_\beta > 0$ and for every $n = 1, 2, \dots$ polynomials P_n of degree at most n such that $P_n(Z) = 1$, $|P_n(z)| \le 1$ for $z \in K$ and*

$$|P_n(z)| \le C_\beta e^{-c_\beta (n|z-Z|)^\beta}, \qquad z \in K. \tag{23}$$

We mention that this lemma is optimal in the sense that $\beta = 1$ is not possible in it.

The polynomials P_n allow good localization, for they decrease fast on K as we move away from Z.

A similar statement was proved in [30, Theorem 4.1], but there $(n|z - Z|)^\beta$ was replaced by $n|z - Z|^\gamma$ with some $\gamma > 1$. Nevertheless, we shall follow the argument in [30, Theorem 4.1].

Proof of Lemma 4. First we prove the claim for the closed unit disk, thus let first $K = \overline{\Delta}$. Without loss of generality we may assume $Z = 1$. It was proved in [8] that for every $\beta < 1$ there are constants $d_\beta, D_\beta > 0$ and for every $n = 1, 2, \ldots$ polynomials R_n of degree at most n such that $R_n(0) = 1$, $|R_n(x)| \le 1$ for $x \in [-1, 1]$ and

$$|R_n(x)| \le D_\beta e^{-d_\beta (n|x|)^\beta}, \qquad x \in [-1, 1]. \tag{24}$$

By replacing $R_n(x)$ with $(R_n(x) + R_n(-x))/2$ if necessary, we may assume that R_n is even. Then $R_n(\sin(t/2))$ is a trigonometric polynomial of degree at most $n/2$, hence $e^{i[n/2]t} R_n(\sin(t/2))$ coincides with some $P_n^*(e^{it})$, where P_n^* is an algebraic polynomial of degree at most n. It is clear that $P_n^*(1) = 1$, $|P_n^*(e^{it})| \le 1$, and for $t \in [-\pi, \pi]$ (see (24))

$$|P_n^*(e^{it})| \le D_\beta e^{-d_\beta (n|\sin(t/2)|)^\beta} \le D_\beta e^{-d_\beta (n|t|)^\beta / \pi^\beta}, \tag{25}$$

where we used that $|\sin(t/2)| \ge |t|/\pi$ for $t \in [-\pi, \pi]$. We claim that this is enough to prove the statement for the unit disk, i.e. P_n^* also satisfies (23) with $Z = 1$. By the maximum principle we certainly have $|P_n^*(z)| \le 1$ in the closed unit disk $\overline{\Delta}$.

Let

$$J_z = \{e^{it} \,|\, t \in [-\pi, -|1 - z|] \cup [|1 - z|, \pi]\}$$

be the arc of the unit circle consisting of points with arc length distance $\ge |1 - z|$ from the point 1. Let $\tilde{\omega}(z; J, \Delta)$ denote the harmonic measure of $J \subset \partial\Delta$ at z with respect to the unit disk Δ. It is clear (use that

$$\tilde{\omega}(z; J_z, \Delta) = \frac{1}{2\pi} \int_J \frac{1 - |z|^2}{|\zeta - z|^2} d|\zeta|$$

or apply a conformal map onto the upper half plane and note that on the upper half plane the harmonic measure is nothing else (see [18, Table 4.1] or [2]) than $1/\pi$-times the angle the set is seen from the point z) that there is a constant $\gamma > 0$ such that $\tilde{\omega}(z; J_z, \Delta) \ge \gamma$ for all $z \in \Delta$. Since on J_z we have by (25) the estimate

$$|P_n^*(e^{it})| \le D_\beta e^{-d_\beta (n|1-z|)^\beta / \pi^\beta}$$

while $|P_n^*(z)| \le 1$ everywhere in the closed unit disk, it follows from a comparison of the subharmonic function $\log|P_n^*(w)|$ with the harmonic function

$$\tilde{\omega}(w; J_z, \Delta)\left(\log D_\beta - d_\beta(n|1-z|)^\beta/\pi^\beta\right)$$

that

$$|P_n^*(w)| \le D_\beta^\gamma e^{-\gamma d_\beta(n|1-w|)^\beta/\pi^\beta}, \tag{26}$$

as was claimed.

After these we complete the proof for arbitrary sets. Since it is assumed that there is a disk in Ω that contains Z on its boundary, it is easy to construct a simply connected domain G with C^2 boundary containing $K \setminus Z$ in its interior such that $Z \in \partial G$. Choose a lemniscate (a level set of a polynomial) σ such that σ is a Jordan curve, its interior contains $G \setminus Z$ and $Z \in \sigma$. The existence of σ immediately follows from [14, Theorem 1.1]. Let T_N be a polynomial for which $\sigma = \{z||T_N(z)| = 1\}$, and without loss of generality we may assume that $T_N(Z) = 1$. In the rest of the proof this T_N (and hence N) is fixed, and it is also true that $T_N'(Z) \ne 0$ since σ is a Jordan curve. We claim that if P_n^* are the polynomials from (23) for the closed unit disk (i.e. P_n^* is actually the polynomials from (26)), then $P_n(z) = P_{[n/N]}^*(T_N(z))$ satisfy (23) with some constants C_β, c_β. In fact, if $z \in K$ is in a small neighborhood of Z then $T_N(z)$ is in a small neighborhood of 1, and $|z - Z| \sim |T_N(z) - 1|$, so (23) follows from (26) (applied with $w = T_N(z)$), for, say, $|z - Z| \le \delta$. On the other hand, if $z \in K$ and $|z - Z| \ge \delta$, then $|T_N(z) - 1| \ge \delta_1$ for some δ_1 (note that $K \setminus \{Z\}$ lies strictly inside σ, so for $z \in K$, $|z - Z| \ge \delta$ the value $T_N(z)$ cannot be close even to the boundary of the unit circle). Hence (23) follows again from (26) applied with $w = T_N(z)$.

■

3 Proof of Theorem 1

Assume, as in the theorem, that the system of curves Γ is C^2 and μ is a finite measure on Γ such that $\log w \in L^1(s_\Gamma)$, where w is the Radon-Nikodym derivative of μ with respect to the arc length measure s_Γ on Γ. We need to prove the theorem on each component of Γ, so let γ be one of the components of Γ.

Let $d\mu(x) = w(x)ds(x) + d\mu_{\mathrm{sing}}(x)$ be, as before, the decomposition of μ into its absolutely continuous and singular part with respect to the arc measure $s = s_\Gamma$ on Γ. The Lebesgue-point property of μ at a point ζ_0, say at $\zeta_0 = 1$, means that for every $\varepsilon > 0$ there is a $\rho > 0$ such that if $0 \le \tau \le \rho$ then

$$\int_{|\zeta-1|\le\tau} |w(\zeta) - w(1)|ds(\zeta) \le \varepsilon\tau \tag{27}$$

$$\mu_{\mathrm{sing}}(\{\zeta \,|\, |\zeta - 1| \le \tau\}) \le \varepsilon\tau. \tag{28}$$

Since the derivative of $\mu_{\rm sing}$ with respect to s_Γ is 0 s_Γ-almost everywhere (see [19, Theorem 7.13]), standard proof shows that s_Γ-almost every point is a Lebesgue-point for μ. So the theorem follows if we can prove (1) at every z which is a Lebesgue-point for both μ and $\log w$.

Let $1 \in \gamma$ be a Lebesgue-point for μ. We define the measure ν as $d\nu(\zeta) = w(1)ds_\gamma(\zeta)$ on γ and $d\nu = d\mu$ on other components of Γ, and we shall compare the values $\lambda_n(1,\mu)$ and $\lambda_n(1,\nu)$ of the Christoffel functions associated with μ and ν, respectively. The theorem will follow, since the measure ν has continuous Radon-Nikodym derivative ($\equiv w(1)$) with respect to s_Γ on γ and on other components of Γ this Radon-Nikodym derivative w is positive s_Γ-almost everywhere (recall that we have assumed $\log w \in L^1(s_\Gamma)$), and hence, by [30, Theorem 1.1], for it we have (36) below.

Denote the derivative $d\omega_\Gamma/ds_\Gamma$ by σ_Γ (to be more precise, let

$$\sigma_\Gamma(z) = \lim_{s_\Gamma(J)\to 0} \frac{\omega_\Gamma(J)}{s_\Gamma(J)}$$

where the limit, which is taken for subarcs J of Γ containing z, exists). Since Γ is assumed to be C^2, this σ_Γ, which is the density of the equilibrium measure of Γ with respect to arc length, is easily seen to be continuous.

Since $d\mu/d\omega_\Gamma = (d\mu/ds_\Gamma)(ds_\Gamma/d\omega_\Gamma) = w/\sigma_\Gamma$, we need to prove that under the assumption that the point 1 is a Lebesgue-point for both μ and $\log w$ we have

$$\limsup_{n\to\infty} n\lambda_n(1,\mu) \le \frac{w(1)}{\sigma_\Gamma(1)}, \tag{29}$$

and

$$\liminf_{n\to\infty} n\lambda_n(1,\mu) \ge \frac{w(1)}{\sigma_\Gamma(1)}. \tag{30}$$

Proof of (29). This part of the proof uses only the Lebesgue-point property for μ.

It was proven in [30, Theorem 1.1] that there are polynomials Q_n of degree at most n such that $Q_n(1) = 1$, $|Q_n(z)| \le 1$ for all $z \in \Gamma$ and

$$\lim_{n\to\infty} n \int |Q_n|^2 d\nu = \frac{w(1)}{\sigma_\Gamma(1)}. \tag{31}$$

With $\beta = 2/3$ and some $\delta > 0$ consider the polynomials $P_{\delta n}$ of degree δn from Lemma 4 for the point $Z = 1$ and for the set Γ, and set $R_n(z) = Q_n(z)P_{\delta n}(z)$. This is a polynomial of degree at most $n(1+\delta)$ with $R_n(1) = 1$, $|R_n(\zeta)| \le |Q_n(\zeta)| \le 1$ ($\zeta \in \Gamma$), and this will be our test polynomial to get an upper bound for $\lambda_{n(1+\delta)}(1,\mu)$.

We estimate the integral of $|R_n|^2$ against μ first on γ, using the Lebesgue-point properties (27)–(28). Since

$$|R_n(\zeta)| \le C_0 \exp\Big(-c_0(n\delta|\zeta-1|)^{2/3}\Big)$$

with some c_0, C_0, it follows for $2^k/n\delta < \rho/2$, $k = 1, 2, \ldots$ (see (27)) that (the next three integrals are taken on γ)

$$\int_{2^k/n\delta \le |\zeta-1| \le 2^{k+1}/n\delta} |R_n(\zeta)|^2 |w(\zeta) - w(1)| ds(\zeta) \le C_0 \varepsilon \frac{2^{k+1}}{n\delta} \exp\Big(-c_0 2^{2k/3}\Big),$$

and also

$$\int_{|\zeta-1| \le 2/n\delta} |R_n(\zeta)|^2 |w(\zeta) - w(1)| ds(\zeta) \le \varepsilon \frac{2}{n\delta}.$$

On the other hand, for the integral over $|\zeta - 1| \ge \rho/2$, we just write

$$\int_{\rho/2 \le |\zeta-1|} |R_n(\zeta)|^2 |w(\zeta) - w(1)| ds(\zeta) \le C \exp\Big(-c_0 (n\delta\rho/2)^{2/3}\Big).$$

Summing these up we obtain

$$\int_\gamma |R_n|^2 w ds - \int_\gamma |R_n|^2 d\nu \le C \frac{\varepsilon}{\delta n} + o(1/n).$$

Similar reasoning based on (28) rather than (27) gives

$$\int_\gamma |R_n|^2 d\mu_{\mathrm{sing}} \le C \frac{\varepsilon}{\delta n} + o(1/n).$$

On other components of Γ the measures μ and ν coincide, therefore

$$\int |R_n|^2 d\mu \le \int |R_n|^2 d\nu + C_2 \frac{\varepsilon}{\delta n} + o(1/n)$$

follows. Hence, in view of $|R_n(\zeta)| \le |Q_n(\zeta)|$, we obtain from (31)

$$\begin{aligned}
\limsup_{n\to\infty} n(1+\delta)\lambda_{n(1+\delta)}(1, \mu) &\le \limsup_{n\to\infty} n(1+\delta) \int |R_n|^2 d\mu \\
&\le \limsup_{n\to\infty} n(1+\delta) \int |Q_n|^2 d\nu + C_2 \frac{\varepsilon}{\delta}(1+\delta) \\
&\le (1+\delta) \frac{w(1)}{\sigma_\Gamma(1)} + C_2 \frac{\varepsilon}{\delta}(1+\delta)
\end{aligned}$$

with some fixed constant C_2. Now the monotonicity of λ_n in n implies that then for the whole sequence of natural numbers

$$\limsup_{n\to\infty} n\lambda_n(1, \mu) \le (1+\delta) \frac{w(1)}{\sigma_\Gamma(1)} + C_2 \frac{\varepsilon}{\delta}(1+\delta).$$

On letting $\varepsilon \to 0$ and then $\delta \to 0$ we obtain

$$\limsup_{n\to\infty} n\lambda_n(1, \mu) \le \frac{w(1)}{\sigma_\Gamma(1)},$$

what was needed to be proven.

■

Proof of (30). Let $d\mu(x) = w(x)ds(x) + d\mu_{\rm sing}(x)$ be as before, and recall that the assumption of Theorem 1 is that $\log w \in L^1(s_\Gamma)$. Assume now, as in the beginning of the proof, that $1 \in \gamma$ is a Lebesgue-point for both μ (see (27)–(28)) and $\log w$, and select ρ so that (27)–(28) is true for all $\tau \le \rho$.

Assume to the contrary that there is an $\alpha < 1$ and an infinite sequence $\mathcal{N} \subseteq \mathbf{N}$ such that for every $n \in \mathcal{N}$ there are polynomials Q_n of degree at most n with the properties $Q_n(1) = 1$

$$\int |Q_n|^2 d\mu \le \alpha \frac{w(1)}{\sigma_\Gamma(1)} \frac{1}{n}. \tag{32}$$

In particular,

$$\int_\gamma |Q_n|^2 w ds \le \alpha \frac{w(1)}{\sigma_\Gamma(1)} \frac{1}{n}, \tag{33}$$

and then Lemma 3 gives

$$|Q_n(\zeta)| \le M \exp(M\sqrt{n|\zeta - 1|}), \qquad \zeta \in \gamma, \tag{34}$$

with some constant M (recall that 1 is a Lebesgue-point for $\log w$, so Lemma 3 is applicable).

With $\beta = 2/3$ and some $\delta > 0$ consider again the polynomials $P_{\delta n}$ of degree δn from Lemma 4 for the point $Z = 1$ and for the set Γ, and set $R_n(z) = Q_n(z)P_{\delta n}(z)$. This is a polynomial of degree at most $n(1+\delta)$ with $R_n(1) = 1$, $|R_n(\zeta)| \le |Q_n(\zeta)|$ $(\zeta \in \Gamma)$, and this will be our test polynomial to get an upper bound for $\lambda_{n(1+\delta)}(1, \nu)$, $n \in \mathcal{N}$. Since

$$|P_n(\zeta)| \le C_0 \exp\Big(-c_0(n\delta|\zeta - 1|)^{2/3}\Big), \qquad \zeta \in \Gamma,$$

it immediately follows that

$$|R_n(\zeta)| \le MC_0 \exp\Big(M\sqrt{n|\zeta - 1|} - c_0(n\delta|\zeta - 1|)^{2/3}\Big), \qquad \zeta \in \gamma,$$

and hence

$$|R_n(\zeta)| \le M_\delta \exp\Big(-(c_0/2)(n\delta|\zeta - 1|)^{2/3}\Big), \qquad \zeta \in \gamma \tag{35}$$

with an M_δ depending on δ. In particular, $|R_n(\zeta)| \le M_\delta$ for all $\zeta \in \gamma$.

It follows from (27) and (35) for $2^k/n\delta < \rho/2$, $k = 1, 2, \ldots$ that (the next three integrals being taken on γ)

$$\int_{2^k/n\delta \le |\zeta - 1| \le 2^{k+1}/n\delta} |R_n(\zeta)|^2 |w(\zeta) - w(1)| ds(\zeta) \le M_\delta^2 \varepsilon \frac{2^{k+1}}{n\delta} \exp\Big(-\frac{c_0}{2} 2^{2k/3}\Big),$$

and also

$$\int_{|\zeta-1|\le 2/n\delta} |R_n(\zeta)|^2 |w(\zeta)-w(1)| ds(\zeta) \le M_\delta^2 \varepsilon \frac{2}{n\delta}.$$

For the integral over $|z-1|\ge \rho/2$, we write

$$\int_{\rho/2\le|\zeta-1|} |R_n(\zeta)|^2 |w(\zeta)-w(1)| ds(\zeta) \le CM_\delta^2 \exp\Big(-\frac{c_0}{2}(n\delta\rho/2)^{2/3}\Big).$$

Summing these up we obtain

$$\int_\gamma |R_n|^2 d\nu - \int_\gamma |R_n|^2 w ds \le CM_\delta^2 \frac{\varepsilon}{\delta n} + o(1/n).$$

These yield again (as $\nu=\mu$ on other components of Γ)

$$\int |R_n|^2 d\nu \le \int |R_n|^2 d\mu + CM_\delta^2 \frac{\varepsilon}{\delta n} + o(1/n).$$

Hence, in view of $|R_n(\zeta)|\le|Q_n(\zeta)|$, it follows from (32)

$$\begin{aligned}
\limsup_{n\in\mathcal{N}} n(1+\delta)\lambda_{n(1+\delta)}(1,\nu) &\le \limsup_{n\in\mathcal{N}} n(1+\delta)\int |R_n|^2 d\nu \\
&\le \limsup_{n\in\mathcal{N}} n(1+\delta)\int |R_n|^2 d\mu + CM_\delta^2 \frac{\varepsilon}{\delta}(1+\delta) \\
&\le (1+\delta)\alpha\frac{w(1)}{\sigma_\Gamma(1)} + CM_\delta^2 \frac{\varepsilon}{\delta}(1+\delta)
\end{aligned}$$

with some fixed constant C. But for $(1+\delta)\alpha<1$ (and we can make this happen by selecting a small δ) and small ε this contradicts the fact that

$$\lim_{m\to\infty} m\lambda_m(1,\nu) = \frac{w(1)}{\sigma_\Gamma(1)}, \tag{36}$$

which was proved in [30, Theorem 1.1] (note again that ν has continuous (actually constant) density with respect to s_γ on γ, so [30, Theorem 1.1] can be applied). This contradiction proves the lower estimate in (30) and the proof is complete.

■

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Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Ave, PHY 114
Tampa, FL 33620-5700, USA

and

Bolyai Institute
Analysis and Stochastics Research Group of the Hungarian Academy of Sciences
University of Szeged
Szeged
Aradi v. tere 1, 6720, Hungary

totik@mail.usf.edu