Some random matrix results with application to the multiple access channel

Marija Miteva #, Tome Eftimov* and Zoran Utkovski #*
* - Laboratory for Complex Systems and Networks, Macedonian Academy of Sciences and Arts
# - Faculty of Computer Science, University Goce Delcev Stip, R. Macedonia

The study of the capacity bounds for the K-user multiple access channel (MAC), reveals that certain random matrix theory results are of practical relevance to the problem. The result form the point-to-point block Rayleigh fading channel indicate that isotropically distributed input signals are capacity achieving in the high-SNR regime. The derivation of the mutual information obtained with these input signals in the MAC case requires analysis of the eigenvalues of Gramm matrices of the type $V V^H$, where $V \in \mathbb{C}^{N \times T}$ and the rows of $V$ are random vectors which are uniformly distributed on a unit sphere in $\mathbb{C}^T$.

**System model**

- K single-antenna users;
- one receiver with $N \geq K$ receive antennas;

The system model is the following:

$$Y = SX + W,$$

where $X \in \mathbb{C}^{K \times T}$, $Y \in \mathbb{C}^{N \times T}$, $S \in \mathbb{C}^{N \times K}$ with i.i.d. $CN(0,1)$ entries and $W \in \mathbb{C}^{N \times T}$ with i.i.d. $CN(0,1)$ entries.

**Non-coherent communication**

- The channel $S$ is unknown;
- Communication based on subspaces;
- Conjecture: The capacity achieving inputs are of the form:

$$x_i = \sqrt{T/K} v_i,$$

where the vectors $v_i$ are independent and uniformly distributed on the unit sphere $\mathbb{S}^{T-1}$.

- $r$ - SNR per receive antenna.

**Computation of mutual information**

We are interested in the mutual information of non-coherent MAC:

$$I(X, Y) = h(Y) - h(Y | X),$$

in high-SNR regime.

**1. Derivation of $h(Y)$**

In the high-SNR approximation, for $h(Y)$ we obtain:

$$h(Y) \geq (TK - K^2 + NK) \log_2 \left( \frac{T}{K} \right) + \log_2 |G(T, K)|$$

$$+ (T - K) \mathbb{E} \left[ \log_2 \det (S^H S) \right]$$

$$+ (T + N - KT) \log_2 (|S^H S|),$$

where $V = \mathbf{U} \Sigma \mathbf{V}^H$ is the SVD decomposition.

**2. Derivation of $h(Y | X)$**

For $h(Y | X)$ we obtain:

$$h(Y | X) = NK \log_2 \left( \frac{T}{K} \right) + N \mathbb{E} \left[ \log_2 \det (\Sigma^H) \right]$$

$$+ TN \log_2 (|S^H S|).$$

**Mutual information**

For $I(X, Y)$ we have:

$$I(X, Y) \geq K \left[ 1 - \frac{K}{T} \right] \log_2 \frac{T}{K} + \frac{1}{T} \log_2 |G(T, K)|$$

$$- K \left[ 2 - \frac{K}{T} \right] \log_2 (|\mathbf{v} e |) + \left( 1 - \frac{K}{T} \right) \mathbb{E} \left[ \log_2 \det (\Sigma^H) \right]$$

$$+ \left( 1 - \frac{K}{T} \right) \mathbb{E} \left[ \log_2 \det (\Sigma^H) \right].$$

Of interest is to evaluate

$$\Delta = \mathbb{E} \left[ \log_2 \det (\Sigma^H) \right],$$

where $\mathbf{v} = \mathbf{U} \Sigma \mathbf{V}^H$ is the SVD decomposition.

First, we note that $\det (\Sigma^H) = \det (\Sigma)$.

By using the classical QR decomposition of $\mathbf{V}$, we get

$$\det (\Sigma) = \prod_{k=1}^{N} R_{kk}^2.$$

In the Wishart case, when the row vectors of $\mathbf{V}$ are choosen independently from normal distribution, the variables $R_{kk}^2$ are independent and chi-square distributed with respective parameters $\Gamma(T - k + 1), k = 1, \ldots, K$. In the Gram case, when the row vectors of $\mathbf{V}$ are choosen independently and uniformly distributed on the unite sphere in $\mathbb{S}^{T-1}$, the variables $R_{kk}^2$ are independent and beta distributed with respective parameters $(b'(T - k + 1), b'(k - 1)), k = 2, \ldots, K$, $b' = b/2$, where $b = 2, 4$, which corresponds to the classical matrix models (real, complex and quaternionic).

Let we denote

$$G_{T,k} = \ln \det (\Sigma^H) = \ln \prod_{k=1}^{N} R_{kk}^2 = \sum_{k=1}^{N} \ln R_{kk}^2.$$

Our case corresponds to the Gram case. Using the result from the Gram case, we obtain

$$\mathbb{E} [G_{T,k}] = \sum_{k=1}^{N} \frac{\Gamma(T)}{\Gamma(T - k + 1) \Gamma(k - 1)} \frac{b(k - 1 + i, N - k + 1)}{i},$$

where $\mathbb{B}(_) \_i$ is beta function.

When $N \to \infty$, and the ratio $K / N$ is fixed, $K / N = c, c \in (0, 1)$.

$$\lim_{k \to \infty} \mathbb{E} [G_{T,k}] = \frac{1}{T} \mathbb{E} [G_{T,k}] + J \left( 1 - \frac{K}{T} \right) \log_2 e^i = 0,$$

where

$$J(u) = u \log_2 u - u + 1, \text{ for } u > 0$$

$$J(u) = 1, \text{ for } u = 0$$

$$J(u) = +\infty, \text{ for } u < 0$$

Finally we have

$$\Delta = \log_2 e \mathbb{E} [G_{T,k}]$$

We derive some results for the eigenvalues of Gram matrices of the type $VV^H$ where $V \in \mathbb{C}^{K \times T}$ and the rows of $V$ are random vectors which are uniformly distributed on a unit sphere in $\mathbb{S}^{T-1}$. These matrices are of relevance to the derivation of the mutual informan of the K-user MAC, obtained with isotropically distributed unitary input signals. As result, we pave the way for the capacity characterization of the non-coherent K-user MAC in the high-SNR regime.