WAVELET-GALERKIN SOLUTION OF SOME ORDINARY DIFFERENTIAL EQUATIONS

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ETAI 2013

Ohrid, September 2013
- Spaces of functions
- Galerkin method for ODE
- Wavelets and MRA
- Wavelet-Galerkin method for ODE
- Application of W-G method
Spaces of functions

- $L^2(R)$

  $\langle f, g \rangle = \int_R f(t)\overline{g(t)}\,dt$

- $L^2([0, 1])$

- $C^2([0, 1])$
Galerkin method for ODE

**Sturm-Liouville equation**

\[ Lu(t) \equiv -\frac{d}{dt} \left( a(t) \frac{du}{dt} \right) + b(t) u(t) = f(t), \quad 0 \leq t \leq 1 \]  \hspace{1cm} (1)

with DBC

\[ u(0) = u(1) = 0. \] \hspace{1cm} (2)

1) \( \{v_j\} \) - complete orthonormal system for \( L^2([0, 1]) \)
2) every \( v_j \in C^2([0, 1]) \)
3) \( v_j(0) = v_j(1) = 0. \)

Approximation \( u_s \) of the exact solution \( u \)

\[ u_s = \sum_{k \in \Lambda} x_k v_k \] \hspace{1cm} (3)
Criterion for coefficients $x_k$

$$< Lu_s, v_j > = < f, v_j >, \forall j \in \Lambda. \quad (4)$$

If we substitute the equation (3) in (4) we obtain

$$\sum_{k \in \Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle, \forall j \in \Lambda. \quad (5)$$

$$A = [a_{j,k}]_{j,k \in \Lambda}, \quad a_{j,k} = \langle Lv_k, v_j \rangle;$$

$$X = (x_k)_{k \in \Lambda}; \quad Y = (y_k)_{k \in \Lambda}, \quad y_k = \langle f, v_k \rangle$$

$$AX = Y. \quad (6)$$

Wavelet-Galerkin method: functions $v_j$ are wavelets
Wavelets

- wavelet $\psi$: $L^2$ function which satisfy the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty.$$  (7)

The condition (7) implies that

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t) dt = 0.$$

- wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi \left( \frac{t - b}{a} \right), \quad a > 0, \ b \in \mathbb{R}.$$
Multiresolution analysis (MRA)

Multiresolution analysis of the space $L^2(\mathbb{R})$ consist of a sequence of closed subspace $\{V_j\}_{j=-\infty}^{\infty}$ with the following properties:

1. $V_j \subset V_{j+1}$
2. $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$
3. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
4. $f(t) \in V_j \iff f(2t) \in V_{j+1}$
5. $f(t) \in V_j \iff f(t - k) \in V_j, \forall k \in \mathbb{Z}$
6. there exists a function $\phi$ (called scaling function or father wavelet) such that $\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k), k \in \mathbb{Z}$ constitute orthonormal basis for corresponding subspace $V_j$. 
Let $\phi \in L^2(\mathbb{R})$ be compactly supported scaling function of MRA. Then

1) $\int_{-\infty}^{\infty} \phi(t) \, dt \neq 0$ \hspace{1cm} (8)

2) $\phi(t) = \sum_{k \in \mathbb{Z}} a_k \phi(2t - k)$ \hspace{1cm} (9)

where $a_k$ are real coefficients and $a_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ (the number of nonzero coefficients $a_k$ is denoted by $L$).

3) $\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k)$, $j, k \in \mathbb{Z}$ are orthonormal in $L^2(\mathbb{R})$ i.e.

$$\int_{-\infty}^{\infty} \phi(t - n) \phi(t - k) \, dt = \delta_{k,n},$$ \hspace{1cm} (10)

where

$$\delta_{n,k} = \begin{cases} 
0, & n \neq k \\
1, & n = k \end{cases} \hspace{1cm} (11)$$
One can construct wavelet $\psi$ such that

$$
\psi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), \ j, k \in \mathbb{Z}
$$

constitute an orthonormal basis for $L^2(\mathbb{R})$.

Daubechies scaling function

$$
\phi(t) = \sum_{k=0}^{L-1} a_k \phi(2t - k) \tag{12}
$$

Daubechies wavelet function

$$
\psi(t) = \sum_{k=2-L}^{1} (-1)^k a_{1-k} \phi(2t - k) \tag{13}
$$

where $L$ is a positive even integer and denotes the genus of the Daubechies wavelet.
Wavelet-Galerkin method for ODE

First type of Sturm-Liouville differential equation

\[ u''(t) + \alpha u(t) = f(t), \quad t \in [0, 1], \]  

(14)

with DBC

\[ u(0) = u(1) = 0 \]  

(15)

- Application: electromagnetic radiation, seismology and acoustics.
- Approximate solution

\[ u_j(t) = \sum_{k=1-L}^{2^j} c_k \phi_{j,k}(t), \quad k \in \mathbb{Z}, \]  

(16)

where \( \phi \) is the scaling function of MRA.
Remark

- There are no closed-form formulas for the Daubechies wavelets and scaling functions.
- W-G method with Daubechies scaling functions: homogeneous differential equations.
- Application of this method to nonhomogeneous differential equations
\[ \phi(t) = \begin{cases} 
\frac{1}{6}(2 + t)^3, & t \in (-2, -1) \\
\frac{1}{6}(4 - 6t^2 - 3t^3), & t \in (-1, 0) \\
\frac{1}{6}(4 - 6t^2 + 3t^3), & t \in (0, 1) \\
\frac{1}{6}(2 - t)^3, & t \in (1, 2) \\
0, & t \notin (-2, 2) 
\end{cases} \] (17)

satisfies
\[ \phi(t) = \frac{1}{8}\phi(2t + 2) + \frac{1}{2}\phi(2t + 1) + \frac{3}{4}\phi(2t) + \frac{1}{2}\phi(2t - 1) + \frac{1}{8}\phi(2t - 2), \]

so \( L = 5 \).
For $j = 0$

$$u_0(t) = \sum_{k=-4}^{1} c_k \phi(t - k), \ t \in [0, 1].$$  \hfill (18)

$$\frac{d^2}{dt^2} \sum_{k=-4}^{1} c_k \phi(t - k) + \alpha \sum_{k=-4}^{1} c_k \phi(t - k) = f(t).$$  \hfill (19)

$$\alpha = -1$$
Taking inner product with $\phi(t - n)$, $n \in \{-4, -3, -2, -1, 0, 1\}$, we obtain

$$\sum_{k=-4}^{1} c_k \Omega_{n-k} - \sum_{k=-4}^{1} c_k a_{n,k} = b_n,$$

(20)

where

$$\Omega_{n-k} = \int_{-4}^{5} \phi''(t - k)\phi(t - n)dt,$$

(21)

$$a_{n,k} = \int_{-4}^{5} \phi(t - k)\phi(t - n)dt,$$

(22)

$$b_n = \int_{-4}^{5} \phi(t - n)f(t)dt.$$

(23)
By using DBC (15) we obtain

\[ u_0(0) = \sum_{k=-4}^{1} c_k \phi(-k) = 0 \]  \hspace{1cm} (24)

and

\[ u_0(1) = \sum_{k=-4}^{1} c_k \phi(1 - k) = 0 \]  \hspace{1cm} (25)

We replace the first and the last equation of system (20) by (24) and (25) respectively and obtain the matrix equation

\[ TC = B \]  \hspace{1cm} (26)
Wavelet-Galerkin method for ordinary differential equations

\[ T = \begin{bmatrix}
\phi(4) & \phi(3) & \phi(2) \\
\Omega_0 - a_{-4,-4} & \Omega_1 - a_{-4,-3} & \Omega_2 - a_{-4,-2} \\
\Omega_1 - a_{-3,-4} & \Omega_0 - a_{-3,-3} & \Omega_1 - a_{-3,-2} \\
\Omega_2 - a_{-2,-4} & \Omega_1 - a_{-2,-3} & \Omega_0 - a_{-2,-2} \\
\Omega_3 - a_{-1,-4} & \Omega_2 - a_{-1,-3} & \Omega_1 - a_{-1,-2} \\
\Omega_4 - a_{0,-4} & \Omega_3 - a_{0,-3} & \Omega_2 - a_{0,-2} \\
\phi(5) & \phi(4) & \phi(3) \\
\end{bmatrix} \]
Wavelet-Galerkin method for ordinary differential equations

\[
C = \begin{bmatrix}
  c_{-4} \\
  c_{-3} \\
  c_{-2} \\
  c_{-1} \\
  c_0 \\
  c_1
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  b_{-3} \\
  b_{-2} \\
  b_{-1} \\
  b_0 \\
  0
\end{bmatrix}.
\]
Example.

\[ u''(t) - u(t) = t - 1, \quad 0 \leq t \leq 1, \quad (27) \]

with DBC \( u(0) = u(1) = 0. \)

- **Exact solution**
  \[ u(t) = -\frac{1}{1 - e^2} e^t + \frac{e^2}{1 - e^2} e^{-t} - t + 1. \]

- **Approximate solution obtained by W-G method is**
  \[ u_0(t) = c_{-1} \phi(t + 1) + c_0 \phi(t) + c_1 \phi(t - 1), \quad t \in [0, 1]. \]
## Comparison of results

<table>
<thead>
<tr>
<th>Case</th>
<th>numerical solution (u_0)</th>
<th>exact solution (u)</th>
<th>absolute error</th>
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Wavelet-Galerkin method for ordinary differential equations

Legend

Numerical solution

Exact solution
Second type of Sturm-Liouville differential equation

\[ u''(t) + g(t)u'(t) = f(t), \quad t \in [0, 1], \quad (28) \]

with DBC

\[ u(0) = u(1) = 0, \quad (29) \]

\[ \sum_{k=-4}^{1} c_k \Omega_{n-k} + \sum_{k=-4}^{1} c_k d_{n,k} = b_n, \quad (30) \]

\[ n \in \{-4, -3, -2, -1, 0, 1\}, \text{ where} \]
\begin{equation}
\Omega_{n-k} = \int_{-4}^{5} \phi''(t - k)\phi(t - n)dt,
\end{equation}

\begin{equation}
d_{n,k} = \int_{-4}^{5} g(t)\phi'(t - k)\phi(t - n)dt,
\end{equation}

\begin{equation}
b_{n} = \int_{-4}^{5} \phi(t - n)f(t)dt.
\end{equation}
Example. Van Der Pol equation

\[ u''(t) + \mu(t^2 - 1)u'(t) = -t, \quad 0 \leq t \leq 1, \quad (34) \]

with DBC

\[ u(0) = u(1) = 0. \quad (35) \]

- Stable oscillations (relaxation-oscillations/ limit cycles) in electrical circuits employing vacuum tubes
- Physical and biological sciences, seismology
- No exact analytic solution
Numerical solution $u_0$ of the Van Der Pol equation for different values of parameter $\mu$

<table>
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<th>$\mu=0.1$</th>
<th>$\mu=1$</th>
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