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# The vanishing of the log term of the Szegő kernel and Tian–Yau–Zelditch expansion.

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## Abstract

This thesis consists in two results.

In [Z. Lu, G. Tian, *The log term of Szegő kernel*, Duke Math. J. 125, N 2 (2004), 351-387], the authors conjectured that *given a Kähler form* $\omega$ *on* $\mathbb{CP}^n$ *in the same cohomology class of the Fubini–Study form* $\omega_{FS}$ *and considering the hyperplane bundle* $(L, h)$ *with* $\mathrm{Ric}(h) = \omega$, *if the log–term of the Szegő kernel of the unit disk bundle* $D_h \subset L^*$ *vanishes, then there is an automorphism* $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ *such that* $\varphi^*\omega = \omega_{FS}$.

The first result of this thesis consists in showing a particular family of rotation invariant forms on $\mathbb{CP}^2$ that confirms this conjecture.

In the second part of this thesis we find explicitly the Szegő kernel of the Cartan–Hartogs domain and we show that this non–compact manifold has vanishing log–term. This result confirms the conjecture of Z. Lu for which *if the coefficients* $a_j$ *of the TYZ expansion of the Kempf distortion function of a* $n$*–dimensional non–compact manifold* $M$ *vanish for* $j > n$*, then the log–term of the disk bundle associated to* $M$ *is zero.*

# Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

# Indice

# Introduction

In this thesis we analyze how the vanishing of some coefficients of the asymptotic expansion of the Kempf distortion function of a Kähler manifold affects the geometry of the manifold.

Given a polarized compact Kähler manifold $(M, \omega)$, S. Zelditch [59] proved the existence of a complete asymptotic expansion of the Kempf distortion function $T_m$ associated to $\omega$:

$$T_m(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j},$$

where $a_j$, $j = 0, 1, \ldots$, are smooth coefficients with $a_0(x) = 1$. In [40], Z. Lu showed that each of the coefficients $a_j(x)$ is a polynomial of the curvature and its covariant derivatives at $x$ of the metric $g$ associated to $\omega$, which can be computed by finitely many algebraic operations. Z. Lu and G. Tian pointed out that these coefficients are strictly related to the geometry of $(M, \omega)$. Consider the reproducing kernel of the Hilbert space consisting of the closure in $L^2(X)$ of the restriction to $X$ of the continuous functions in $\bar{D}$ that are holomorphic in all $D$, where $D = \{v \in M |\, \rho(v) > 0\}$ is the disk bundle of the dual bundle of the polarization of $(M, \omega)$ and $X = \partial D$. This kernel is called the Szegő kernel of the unit disk bundle $D$ over $M$. A direct computation of the Szegő kernel could be in general very complicated. Although, when $D \subset M$ is a strictly pseudoconvex domain with smooth boundary, the following celebrated formula due to Fefferman (see [22], [7] and also [6]) shows that there exist functions $a$ and $b$ continuous on

$\bar{D}$ and with $a \neq 0$ on $X$, such that:

$$\mathcal{S}(v) = \frac{a(v)}{\rho(v)^{n+1}} + b(v) \log \rho(v).$$

In particular, in [41], Lu and Tian proved the following results:

1. *If one considers a Kähler form $\omega$ on $\mathbb{CP}^n$ in the same cohomology class of the Fubini–Study metric which is "close" to $\omega_{FS}$ (in the sense expressed in* (3.22)*) and such that the log–term of the Szegő kernel vanishes, then there is an automorphism $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $\varphi^* \omega = \omega_{FS}$.*

2. *If the log–term of the Szegö kernel of the unit disk bundle over $M$ vanishes then $a_k = 0$, for all $k > n$.*

(We refer the reader to Section 3.2 for more details).

It is rather natural to ask the following:

**Question 1:** *Does result (1) above holds true when the hypothesis to be "close" is removed?*

**Question 2:** *Is it also true the converse of result (2)? In other words, if the coefficients $a_k$ of the expansion given by Zelditch vanish for all $k > n$, does the log–term of the Szegö kernel of the unit disk bundle over $M$ vanish?*

A positive answer to Question 1 has been conjectured by Lu and Tian in [41], while a positive answer to Question 2 was conjectured by Lu in a private communication.

In this thesis we give a positive answer to Question 1 and 2 in particular cases. For the first one, we consider for each $a > 0$, the one parameter family of Kähler forms on $\mathbb{CP}^2$ given by

$$\omega_a = \Phi^* \omega_{FS}$$

where $a = |\alpha|^2, \alpha \in \mathbb{C}^*$ and $\Phi$ is a holomorphic Veronese–type embedding given by:

$$\mathbb{CP}^2 \xrightarrow{\Phi} \mathbb{CP}^5$$

$$[Z_0, Z_1, Z_2] \longmapsto [Z_0^2, Z_1^2, Z_2^2, \alpha Z_0 Z_1, \alpha Z_0 Z_2, \alpha Z_1 Z_2],$$

where $Z_0, Z_1, Z_2$ are homogeneous coordinates on $\mathbb{CP}^2$.

For the second one, in the non–compact situation, we consider the case when $(M, \omega)$ is a Cartan–Hartogs domain (see Section 4.4 for the definition). In particular, we prove the following:

**Theorem A.** *The log–term of the Szegő kernel of the disk bundle over a Cartan–Hartogs domain vanishes.*

The proof is based on the fact that the disk bundle of a Cartan–Hartogs domain $M_\Omega^{d_0}(\mu)$ is the Cartan–Hartogs domain $M_\Omega^{d_0+1}(\mu)$. Observe that since the boundary is not smooth, we cannot apply Fefferman's result. However, we say that *the log–term of the Szegő kernel of the disk bundle vanishes* if there exists a continuous function $a$ on $\bar{D}$ with $a \neq 0$ on $X$ such that $\mathcal{S}(v) = \frac{a(v)}{\rho(v)^{n+1}}$ (see Definition 4.10).
In [21], Z. Feng and Z. Tu proved that the coefficients $a_k$ of the TYZ expansion of the Kempf distortion function of the Cartan–Hartogs domain $M_\Omega^{d_0}(\mu)$ vanish for $k > d + d_0$, where $d + d_0$ is the dimension of $M_\Omega^{d_0}(\mu)$. Combining this result with Theorem A, we show that Cartan–Hartogs domains are an example of non–compact manifolds for which the Lu's conjecture holds true.

The thesis is organized in four Chapters as follows:

In Chapter 1, we recall the basic notions on Kähler geometry and on the theory of fiber bundles used in the thesis, with particular attention to the case of $\mathbb{CP}^n$.

In Chapter 2, we summarize useful results of complex analysis. In particular, in Sections 2.1 and 2.2 we define, respectively, the Bergman and the Szegő kernel for domains of $\mathbb{C}^n$, investigate some of their properties and recall the results of C. Fefferman [22] and L. Boutet de Monvel and J. Sjöstrand [6]. In Section 2.3 we extend the definition of the Szegő kernel to domains of manifolds.

In Chapter 3, we describe the first result of this thesis. In Section 3.1, we introduce the Kempf distortion function for a compact Kähler manifold $M$ explaining what TYZ asymptotic expansion of this function means and what coefficients $a_j$ of TYZ expansion are, recalling the results due to S. Zelditch [59], Z. Lu [40] and Z. Lu and G. Tian [41]. In Subsection 3.1.1 we define the disk bundle of a polarized Kähler manifold $(M, \omega)$ and we prove that it is a strictly pseudoconvex domain. In Subsection 3.1.2 we remake the construction of the Szegő kernel of the disk bundle over $M$ as in [59] and in [41], using the natural volume form induced by the contact form on the boundary of the disk bundle. Moreover we illustrate the relation showed by Lu and Tian in [41], between this Szegő kernel and the coefficients of the TYZ expansion of the Kempf distortion function. In the last section we show that the family of Kähler forms $\omega_a$ is a particular family of metrics on $\mathbb{CP}^2$, which gives a positive answer to Lu and Tian's conjecture.

In the last chapter, we describe the second result of this thesis. In Section 4.1 we generalize the definition of the asymptotic expansion of the Kempf distortion function to the non–compact case giving a necessary condition to the existence of such an expansion. In sections 4.2 and 4.3 we introduce Hartogs domains and Cartan domains that are used to give, in Section 4.4, the definition of Cartan–Hartogs domains. In the same Section 4.4, we also prove that the disk bundle of a Cartan–Hartogs domain of dimension $d$ is a Cartan–Hartogs domain of dimension $d + 1$. Finally, in Section 4.5 we find explicitly the Szegő kernel of a Cartan–Hartogs domain and prove Theorem A.

# Capitolo 1

# Preliminaries

In this chapter we illustrate the notations used in this thesis and recall some notions on complex and Kähler geometry.

## 1.1 Kähler metrics

Recall that if $(M, g)$ is a hermitian manifold, with $g$ a hermitian metric, we define the fundamental form $\omega \in \Omega^{(1,1)}(M, \mathbb{C})$ of $g$ as

$$\omega(X, Y) = g(X, JY), \tag{1.1}$$

for all smooth fields $X$, $Y$ on $M$, where $J$ is the almost complex structure on $M$ and denoting by $\Omega^{(1,1)}(M, \mathbb{C})$ the space of all $(1,1)$–forms on $M$.

**Definition 1.1.** *A hermitian manifold $(M, \omega)$ is Kähler if and only if for all $p \in M$ there exist a neighborhood $U$ and a plurisubharmonic* [1] *function $\Phi : U \to \mathbb{C}$ such that*

$$\omega_{|U} = \frac{i}{2}\partial\bar{\partial}\Phi.$$

The function $\Phi$ is called a *Kähler potential* for the metric $g$ and it is univocally determined up to the addition of the real part of a holomorphic function. Observe

[1] In our contest, we consider $\Phi$ of class $C^2$ and being plurisubharmonic means that the matrix $[\partial\partial\Phi(x)]$ of second derivatives is positive semi–definite, for all $x \in U$.

that in local coordinates one has

$$\omega = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\ dz_\alpha \wedge d\bar{z}_\beta,$$

where

$$g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

Let Ric be the Ricci curvature of $(M, g)$ and let $\rho$ be the Ricci form associated to Ric, i.e.

$$\rho(X, Y) = \mathrm{Ric}(JX, Y),$$

for all smooth fields $X$, $Y$ on $M$. The nice feature of the Kähler metrics is that the Ricci form has a very simple expression in terms of the metric tensor, i.e.

$$\rho = -i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}),$$

(we refer the reader to [53]).

### 1.1.1 The complex projective space

In this section we recall some aspects of the Kähler geometry of the complex projective space which will be useful throughout the thesis.

The complex projective space $\mathbb{CP}^n$ is the set of all complex lines in $\mathbb{C}^n$. If we consider on $\mathbb{C}^{n+1}$ the equivalence relation where $x$, $y \in \mathbb{C}^n$ are equivalent, $x \sim y$, if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}^*$, then the complex projective space can be described as the quotient space

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1}}{\sim}.$$

Denote with $[Z_0, \ldots, Z_n]$ the equivalence class of $(Z_0, \ldots, Z_n) \in \mathbb{C}^{n+1}$. Consider in $\mathbb{CP}^n$ the canonical atlas $(U_\alpha, \varphi_\alpha)$ with $U_\alpha = \{[Z_0, \ldots, Z_n] \in \mathbb{CP}^n \mid Z_\alpha \neq 0\}$, $\alpha = 1, \ldots, n$ and

$$\begin{aligned} \phi_\alpha : \quad & U_\alpha \longrightarrow \mathbb{C}^n \\ [Z_0, \ldots, Z_n] & \mapsto \Big(\frac{Z_0}{Z_\alpha}, \ldots, \frac{Z_{\alpha-1}}{Z_\alpha}, \frac{Z_{\alpha+1}}{Z_\alpha}, \ldots, \frac{Z_n}{Z_\alpha}\Big), \end{aligned}$$

and inverse map

$$\begin{aligned}\phi_\alpha^{-1}: \qquad & \mathbb{C}^n \longrightarrow U_\alpha \\ & (W_1, \dots, W_n) \mapsto [W_1, \dots, W_{\alpha-1}, 1, W_{\alpha+1}, \dots, W_n].\end{aligned}$$

Observe that when $U_\alpha \cap U_\beta \neq \emptyset$, the composition

$$\begin{aligned}&\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta) \\ &\qquad (Z_1, \dots, Z_n) \mapsto \Big(\frac{Z_1}{Z_\alpha}, \dots, \frac{Z_{\alpha-1}}{Z_\alpha}, \frac{Z_{\alpha+1}}{Z_\alpha}, \dots, \frac{Z_{\beta-1}}{Z_\alpha}, \frac{1}{Z_\alpha}, \frac{Z_{\beta+1}}{Z_\alpha}, \dots, \frac{Z_n}{Z_\alpha}\Big),\end{aligned}$$

is clearly holomorphic.
Set for convenience $(z_1, \dots, z_n) = \Big(\frac{Z_0}{Z_\alpha}, \dots, \frac{Z_{\alpha-1}}{Z_\alpha}, \frac{Z_{\alpha+1}}{Z_\alpha}, \dots, \frac{Z_n}{Z_\alpha}\Big)$ and define on each $U_\alpha$ the $(1,1)$–form

$$\omega_{FS} = \omega_{|U_\alpha} = \frac{i}{2}\partial\bar\partial \log\Big(1 + |z_1|^2 + \dots + |z_n|^2\Big), \tag{1.2}$$

the so–called *Fubini–Study form* on $\mathbb{CP}^n$ and let $\Phi_{FS} = \log\Big(1+|z_1|^2+\dots+|z_n|^2\Big)$ be the Kähler potential associated to $\omega_{FS}$ on $U_\alpha$.

## 1.2 Holomorphic vector bundles

A holomorphic vector bundle over $M$ of rank $r$ is a complex manifold $E$ together with a holomorphic function $\pi : E \to M$ such that

- $\pi$ is surjective,
- for any point $p \in M$ the fiber $E_p = \pi^{-1}(p)$ is a complex vector space of dimension $r$,
- for every $p \in M$ there exist an open set $U_p \subset M$, $p \in U_p$, and a biholomorphism $\varphi$ such that the diagram:

$$\begin{tikzcd} \pi^{-1}(U_p) \arrow[rr, "\varphi"] \arrow[dr, "\pi_|"'] & & U_p \times \mathbb{C}^r \arrow[dl, "pr_1"] \\ & U_p & \end{tikzcd}$$

commutes, where $\pi_|$ is the restriction of $\pi$ to $\pi^{-1}(U_p)$ and $pr_1$ is the standard projection on the first factor.

Observe that, denoting with $pr_2$ the standard projection on the second factor, the map $pr_2 \cdot \varphi_{|E_p} : E_p \to \mathbb{C}^r$ is an isomorphism of vector spaces.
Given a holomorphic vector bundle $\pi : E \to M$, the pair $(U_p, \varphi_p)$ is a *local trivialization.*

For each $U_\alpha$, $U_\beta$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$
$$(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$$

is holomorphic and induces the maps, $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to GL_r(\mathbb{C})$ which satisfy

$(i)$ $g_{\alpha\beta} = g_{\beta\alpha}$,

$(ii)$ $g_{\alpha\alpha} = \mathrm{Id}_{\mathbb{C}^r}$,

$(iii)$ $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{Id}_{\mathbb{C}^r}$.

The maps $g_{\alpha\beta}$ are called *transition functions* of the vector bundle. Observe that prescribing maps $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to GL_r(\mathbb{C})$ on $M$ which satisfy the conditions $(i)$, $(ii)$ and $(iii)$ above, determines uniquely the bundle.

**Definition 1.2.** *Given a vector bundle $\pi : E \to M$ of rank $r$ on $M$, a global section $s$ of $E$ is a map $s : M \to E$ such that $\pi \circ s = id_M$.*

Note that if $s$ is a global section then $s(p) \in E_p$ for all $p \in M$. Denote with $\Gamma(E)$ the set of all smooth sections of $E$ and with $H^0(E)$ the set of all holomorphic sections on $E$. In particular, by the vector space structure of $E_p$ it is possible to endow $H^0(E)$ with the vector space structure, setting $(s+t)(p) = s(p)+t(p)$, for all $s, t \in H^0(E)$, for all $p \in M$ and $(\lambda s)(p) = \lambda s(p)$ with $\lambda \in \mathbb{C}$. If $\pi : E \to M$ is a vector bundle of rank $r$ with local trivialization $(\varphi_\alpha, U_\alpha)$ and $s \in H^0(E)$ is a holomorphic section, then

$$\begin{aligned}\varphi_\alpha \circ s_{|U_\alpha} = s_\alpha : U_\alpha &\to U_\alpha \times \mathbb{C}^r \\ p &\mapsto (p, \sigma_\alpha(p))\end{aligned}$$

and

$$\begin{aligned}\varphi_\beta \circ s_{|U_\beta} = s_\beta : U_\beta &\to U_\beta \times \mathbb{C}^r \\ p &\mapsto (p, \sigma_\beta(p))\end{aligned}$$

where $s_{|U_\alpha}$ and $s_{|U_\beta}$ are called *local trivializing sections*. Observe that if $p \in U_\alpha \cap U_\beta$, then $s_{|U_\alpha}(p) = s_{|U_\beta}(p)$ and we have

$$\begin{aligned}(p, \sigma_\alpha(p)) =& s_\alpha(p) = (\varphi_\alpha \circ s_{|U_\alpha})(p) = (\varphi_\alpha \circ s_{|U_\beta})(p) = (\varphi_\alpha \circ \varphi_\beta^{-1} \circ s_\beta)(p) \\ =& (\varphi_\alpha \circ \varphi_\beta^{-1})(s_\beta)(p) = \varphi_\alpha \circ \varphi_\beta^{-1}(p, \sigma_\beta(p)) = (p, g_{\alpha\beta}\sigma_\beta(p))\end{aligned} \tag{1.3}$$

that implies $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$.

**Remark 1.3.** Observe that the product and the direct sum of two (or more) vector bundles is still a vector bundle. In particular, if $\pi^1 : E^1 \to M$ and $\pi^2 : E^2 \to M$ are vector bundles of rank $k_1$ and $k_2$ respectively, with transition functions $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$, then $\pi^1 \otimes \pi^2 : E^1 \otimes E^2 \to M$ is a vector bundle of rank $k_1k_2$ with transition functions $g^{\otimes}_{\alpha\beta} = g^1_{\alpha\beta}g^2_{\alpha\beta}$. Analogously, $\pi^1 \oplus \pi^2 : E^1 \oplus E^2 \to M$ is a vector bundle of rank $k_1 + k_2$ with transition functions

$$g^{\oplus}_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix}.$$

### 1.2.1 Tangent bundle

Let $TM = \bigcup_{p\in M} T_pM$ be the classical tangent space of $M$ and let $(U_\alpha, h_\alpha)$ be an atlas for $M$. Let $\pi$ be the canonical projection defined by $\pi : TM \to M$, $\pi(v_p) = p$, for all $v_p \in T_pM$. Define $\varphi_\alpha = dh_\alpha$, where

$$\begin{aligned}(dh_\alpha)_p : T_pU_\alpha &\to T_{h_\alpha(p)}\mathbb{R}^n \simeq \mathbb{R}^n \\ v &\mapsto \sum_{j=1}^n a_j^\alpha \frac{\partial}{\partial x_j}\bigg|_{h(p)}.\end{aligned}$$

Then the diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi_|"'] & & U_\alpha \times \mathbb{R}^n \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes, with $\pi^{-1}(U_\alpha) = TU_\alpha = \{(p, v) \mid p \in U_\alpha, v \in T_pU_\alpha \simeq T_pM\}$. Thus $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ are local trivializations for $TM$.

### 1.2.2 Dual bundle

To any vector bundle $\pi : \sqcup_{p\in M} E_p \to M$, with local trivializations $(\varphi_\alpha, U_\alpha)$, we can associate a dual bundle $\pi^* : \sqcup_{p\in M} E_p^* \to M$, where $E_p^*$ is the dual space of $E_p$, i.e. $E_p^* = \{f_p \in C^0(E_p, \mathbb{C})\}$. A local trivialization $(U_\alpha^*, \varphi_\alpha^*)$ for $\pi^*$ is given by

$$\varphi_\alpha^* : ((\pi^*)^{-1}(U_\alpha^*)) \to U_\alpha^* \times (\mathbb{C}^r)^*$$
$$f_p \mapsto (p, f_p \circ \varphi_\alpha^{-1}).$$

Observe that since by definition the transition functions for $\pi^*$ are

$$\varphi_\alpha^* \circ (\varphi_\beta^*)^{-1} : (U_\alpha^* \cap U_\beta^*) \times (\mathbb{C}^r)^* \to (U_\alpha^* \cap U_\beta^*) \times (\mathbb{C}^r)^*$$
$$(p, f_p) \mapsto (p, f_p \circ \varphi_\beta \circ \varphi_\alpha^{-1})$$

we have $g_{\alpha\beta}^* = g_{\beta\alpha}^t = (g_{\alpha\beta}^t)^{-1}$. Note that if rank of $E$ is one then $g_{\alpha\beta}^* = g_{\beta\alpha}$.

### 1.2.3 Universal bundle on $\mathbb{CP}^n$

Now we are interested in constructing an important line bundle (a vector bundle where the rank is 1) on $\mathbb{CP}^n$, called the *universal bundle* or *tautological bundle*. Let $\mathcal{U}$ be the disjoint union of lines in $\mathbb{C}^{n+1}$ and consider the map $\pi : \mathcal{U} \to \mathbb{CP}^n$ where the fiber of a point $p = [Z_0, \dots, Z_n]$ is the complex line through $p$, i.e.

$$\pi^{-1}(p) = \{(p, \lambda(Z_0, \dots, Z_n)) \mid \lambda \in \mathbb{C} \text{ and } (Z_0, \dots, Z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

and clearly $\pi((p,\lambda(Z_0,\dots,Z_n))) = \pi((p,Z_0,\dots,Z_n)) = p = [Z_0,\dots,Z_n]$. Consider the open set $U_\alpha = \{[Z_0,\dots,Z_n] \in \mathbb{CP}^n \mid Z_\alpha \neq 0\}$ then

$$\begin{aligned}\pi^{-1}(U_\alpha) &= \{([Z_0,\dots,Z_n],\lambda(Z_0,\dots,Z_n)) \mid \lambda \in \mathbb{C}, Z_\alpha \neq 0\}\\ &= \left\{\left([Z_0,\dots,Z_n],\lambda_\alpha\left(\frac{Z_0}{Z_\alpha},\dots,1,\dots,\frac{Z_n}{Z_\alpha}\right)\right) \mid \lambda_\alpha := \lambda Z_\alpha\right\}.\end{aligned}$$

Define

$$\begin{aligned}\varphi_\alpha : \quad \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times \mathbb{C}\\ ([Z_0,\dots,Z_n],\lambda(Z_0,\dots,Z_n)) &\mapsto ([Z_0,\dots,Z_n],\lambda_\alpha)\end{aligned}$$

and the diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi_|"'] & & U_\alpha \times \mathbb{C} \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes. Observe that $\varphi_\alpha$ is bijective and $\mathbb{C}$-linear, with inverse map

$$\begin{aligned}\varphi_\alpha^{-1} : \quad U_\alpha \times \mathbb{C} &\longrightarrow \pi^{-1}(U_\alpha)\\ ([Z_0,\dots,Z_n],\lambda) &\mapsto \left([Z_0,\dots,Z_n],\lambda\Big(\frac{Z_0}{Z_\alpha},\dots,\frac{Z_n}{Z_\alpha}\Big)\right).\end{aligned}$$

The pair $(U_\alpha,\varphi_\alpha)$ is a local trivialization of the universal bundle $\mathcal{U}$ that is a subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}$. For the transition function observe that if $(\lambda(Z_0,\dots,Z_n)) \in \pi^{-1}(U_\alpha \cap U_\beta)$ then

$$\begin{aligned}\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}\\ ([Z_0,\dots,Z_n],\lambda) &\mapsto \Big([Z_0,\dots,Z_n],\lambda\frac{Z_\alpha}{Z_\beta}\Big),\end{aligned}$$

thus

$$\begin{aligned}g_{\alpha\beta} : (U_\alpha \cap U_\beta) &\longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*\\ [Z_0,\dots,Z_n] &\mapsto \frac{Z_\alpha}{Z_\beta}.\end{aligned}$$

The dual bundle of $\mathcal{U}$ is a linear bundle called the *hyperplane* bundle and is denoted by $\mathcal{O}(1)$, for this reason we will denote the universal bundle by $\mathcal{O}(-1)$. Other important bundles of $\mathbb{CP}^n$ are the tensor power of $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ (see

Remark 1.3), so sometimes we write $O(m)$ for $\mathcal{O}(1)^{\otimes m}$ and $O(-m)$ for $\mathcal{O}(-1)^{\otimes m}$ and $\mathcal{O}(0) = \mathbb{CP}^n \otimes \mathbb{C}$ by definition.

Now we investigate the set of holomorphic sections of $\mathcal{O}(1)$ and $\mathcal{O}(-1)$.

**Proposition 1.4.** *Let $\mathcal{O}(1)$ be the hyperplane bundle of $\mathbb{CP}^n$ and $m \in \mathbb{Z}$, then the following holds:*

$$H^0(\mathcal{O}(m)) = \begin{cases} \mathbb{C}, & \text{if } m = 0, \\ 0, & \text{if } m < 0, \\ \mathbb{C}[z_0, \dots, z_n] \text{ homogeneous of degree } m, & \text{if } m > 0, \end{cases}$$

*and*

$$\dim(H^0(\mathcal{O}(m))) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m < 0, \\ \binom{m+n}{m}, & \text{if } m > 0. \end{cases}$$

*Dimostrazione.* If $m = 0$, by definition $\mathcal{O}(0) = \mathbb{CP}^n \otimes \mathbb{C}$ and $H^0(\mathcal{O}(0))$ is the set of holomorphic sections $s$ from $\mathbb{CP}^n$ to $\mathbb{CP}^n \otimes \mathbb{C}$, which since $\mathbb{CP}^n$ is compact by the maximum principle are constant functions on $\mathbb{C}$, i.e. $H^0(\mathcal{O}(0)) \simeq \mathbb{C}$ and its dimension is 1.

If $m < 0$ then $\mathcal{O}(m)$ is a tensor power of the universal bundle $\mathcal{O}(-1)$ that has no holomorphic sections, (see for example [45, Th. 15.3] or [55, Ex.2.13, Ch.1]).

If $m > 0$, let $s$ be a holomorphic section, i.e. $s : \mathbb{CP}^n \to \mathcal{O}(m)$. Consider the canonical atlas $(U_i, \phi_i)$ for $\mathbb{CP}^n$, with $i = 0, \dots, n$, $U_i = \{[Z_0, \dots, Z_n] \in \mathbb{CP}^n | Z_i \neq 0\}$ and

$$\begin{aligned} \phi_i : \quad U_i &\longrightarrow \mathbb{C}^n \\ [Z_0, \dots, Z_n] &\mapsto \Big(\frac{Z_0}{Z_i}, \dots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \dots, \frac{Z_n}{Z_i}\Big), \end{aligned}$$

and consider a local trivialization $(U_i, \varphi_i)$ of $\mathcal{O}(m)$, with $i = \alpha$. If $\sigma_i$ and $\sigma_j$ are like in (1.3) for $p \in U_i \cap U_j$, $i < j$ we have that $\sigma_i(\cdot) = g_{ij}(\cdot)\sigma_j(\cdot)$, where in this case $g_{ij}([Z_0,\dots,Z_n]) = \left(\frac{Z_j}{Z_i}\right)^m$ (see Remark 1.3). Each of $\sigma_i \circ \phi_i^{-1}$, $\sigma_j \circ \phi_j^{-1}$ is a holomorphic function from $\mathbb{C}^n$ to $\mathbb{C}$, thus there exists a power expansion such that

$$(\sigma_i \circ \phi_i^{-1})(z_1,\dots,z_n) = \sum_{\alpha_1,\dots,\alpha_n=0}^{+\infty} a_{\alpha_1,\dots,\alpha_n} z_1^{\alpha_1}\cdots z_n^{\alpha_n} \tag{1.4}$$

and

$$(\sigma_j \circ \phi_j^{-1})(z_1,\dots,z_n) = \sum_{\beta_1,\dots,\beta_n=0}^{+\infty} b_{\beta_1,\dots,\beta_n} z_1^{\beta_1}\cdots z_n^{\beta_n}. \tag{1.5}$$

If the point $(z_1,\dots,z_n) \in \phi_i(U_i \cap U_j) = \phi_j(U_i \cap U_j)$ then

$$\sigma_i\left(\left[\frac{Z_0}{Z_i},\dots,\frac{Z_{i-1}}{Z_i},1,\frac{Z_{i+1}}{Z_i},\dots,\frac{Z_n}{Z_i}\right]\right) = g_{ij}\left([Z_0,\dots,Z_n]\right)\sigma_j\left(\left[\frac{Z_0}{Z_j},\dots,\frac{Z_{j-1}}{Z_j},1,\frac{Z_{j+1}}{Z_j},\dots,\frac{Z_n}{Z_j}\right]\right)$$

and using the power expansions (1.4) and (1.5) we have

$$\sum_{\alpha_1,\dots,\alpha_n=0}^{+\infty} a_\alpha\left(\frac{Z_0}{Z_i}\right)^{\alpha_1}\cdots\left(\frac{Z_{i+1}}{Z_i}\right)^{\alpha_{i+1}}\cdots\left(\frac{Z_n}{Z_i}\right)^{\alpha_n} = \left(\frac{Z_j}{Z_i}\right)^m \sum_{\beta_1,\dots,\beta_n=0}^{+\infty} b_\beta\left(\frac{Z_0}{Z_j}\right)^{\beta_1}\cdots\left(\frac{Z_{j+1}}{Z_j}\right)^{\beta_{j+1}}\cdots\left(\frac{Z_n}{Z_j}\right)^{\beta_n}$$

with $\alpha$ and $\beta$ the multi-indicies $\alpha = \alpha_1,\dots,\alpha_n$, $\beta = \beta_1,\dots,\beta_n$, which implies

$$Z_i^m \sum_{\alpha_1,\dots,\alpha_n=0}^{+\infty} a_\alpha\left(\frac{Z_0}{Z_i}\right)^{\alpha_1}\cdots\left(\frac{Z_{i+1}}{Z_i}\right)^{\alpha_{i+1}}\cdots\left(\frac{Z_n}{Z_i}\right)^{\alpha_n} = Z_j^m \sum_{\beta_1,\dots,\beta_n=0}^{+\infty} b_\beta\left(\frac{Z_0}{Z_j}\right)^{\beta_1}\cdots\left(\frac{Z_{j+1}}{Z_j}\right)^{\beta_{j+1}}\cdots\left(\frac{Z_n}{Z_j}\right)^{\beta_n}.$$

From this last equality it follows that $\alpha_1 + \cdots + \alpha_n \leq m$ and $\beta_1 + \cdots + \beta_n \leq m$ and the power series (1.4) and (1.5) become

$$\sum_{\alpha_1,\dots,\alpha_n=0}^{m} a_\alpha Z_0^{\alpha_1}\cdots Z_i^{m-\alpha_1-\cdots-\alpha_n} Z_{i+1}^{\alpha_{i+1}}\cdots Z_n^{\alpha_n} = \sum_{\beta_1,\dots,\beta_n=0}^{m} b_\beta Z_0^{\beta_1}\cdots Z_j^{m-\beta_1-\cdots-\beta_n} Z_{j+1}^{\beta_{j+1}}\cdots Z_n^{\beta_n}.$$

Finally, we get that an holomorphic section on $\mathcal{O}(m)$ can be identified with

$$\sum_{\gamma_0+\cdots+\gamma_n=m} c_{\gamma_0,\dots,\gamma_n} Z_0^{\gamma_0}\cdots Z_n^{\gamma_n}, \tag{1.6}$$

with $a_{m-\alpha_1-\cdots-\alpha_n,\alpha_1,\dots,\alpha_{n-1}} = b_{m-\beta_1-\cdots-\beta_n,\beta_1,\dots,\beta_{n-1}} = c_{\gamma_0,\dots,\gamma_n}$.
In other words, a holomorphic section of the form (1.6) can be viewed as a homogeneous polynomial of degree $m$ in the $n+1$ complex variables $Z_0,\dots,Z_n$.

Finally, by combinatory computation the dimension of the space of homogeneous polynomials of degree $m$ in $n+1$ variables is $\binom{m+n}{m}$ (see for example [27, p.166]). $\square$

### 1.2.4 Canonical bundle on $\mathbb{CP}^n$

Given a complex manifold $M$ of real dimension $2n$, the complex line bundle $K_M := \Lambda^{n,0}M$, called *canonical bundle of* $M$, is the holomorphic line bundle whose global holomorphic sections are the $n$–forms on $M$. In particular, for $\mathbb{CP}^n$ we have the following characterization:

**Proposition 1.5.** *The canonical bundle* $K_{\mathbb{CP}^n} := \Lambda^{n,0}\mathbb{CP}^n$ *of* $\mathbb{CP}^n$ *is isomorphic to* $\mathcal{O}(-n-1))$.

*Dimostrazione.* Consider on $\mathbb{CP}^n$ the canonical holomorphic atlas $(U_\alpha, \phi_\alpha)$, then $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ induces $\phi_\alpha^*$ :

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \subset K_{\mathbb{CP}^n} \arrow[rr, "(\phi_\alpha^*)^{-1}"] \arrow[dr, "\pi_|"'] & & U_\alpha \times \Lambda^{n,0}\mathbb{C}^n \arrow[dl, "pr_1"] \\ & U_\alpha \subset \mathbb{C}P^n \arrow[d, "\phi_\alpha"'] & \\ & \mathbb{C}^n & \end{tikzcd}$$

so a trivialization for $\pi : K_{\mathbb{CP}^n} \to \mathbb{CP}^n$ is given by $(\phi_\alpha^*)^{-1} : \pi^{-1}(U_\alpha) \to U_\alpha \times \Lambda^{n,0}\mathbb{C}^n$ and the transition functions are $h_{\alpha\beta} := (\phi_\alpha^*)^{-1}(\phi_\beta^*)$. Let $\omega := d\omega_1 \wedge \cdots \wedge d\omega_n$ be the canonical generator of $\Lambda^{n,0}\mathbb{C}^n$ and consider holomorphic coordinates on $U_\alpha \cap U_\beta$, $z_i = \frac{Z_i}{Z_\alpha}$, $i \neq \alpha$, and $w_j = \frac{Z_j}{Z_\beta}$, $j \neq \beta$. Then

$$\phi_\alpha^*(\omega) = dz_0 \wedge \cdots \wedge dz_{\alpha-1} \wedge dz_{\alpha+1} \wedge \cdots \wedge dz_n,$$

$$\phi_\beta^*(\omega) = dw_0 \wedge \cdots \wedge dw_{\beta-1} \wedge dw_{\beta+1} \wedge \cdots \wedge dw_n,$$

and in particular we can write

$$dw_0 \wedge \cdots \wedge dw_{\beta-1} \wedge dw_{\beta+1} \wedge \cdots \wedge dw_n = h_{\alpha\beta} dz_0 \wedge \cdots \wedge dz_{\alpha-1} \wedge dz_{\alpha+1} \wedge \cdots \wedge dz_n. \quad (1.7)$$

Observe that in $U_\alpha \cap U_\beta$, we have

$$z_\beta w_\alpha = 1, \qquad z_i = \frac{Z_i}{Z_\alpha} = \frac{w_i Z_\beta}{Z_\alpha} = w_i z_\beta$$

for $i \neq \alpha, \beta$ and

$$dz_i = z_\beta dw_i + w_i dz_\beta, \quad dz_\beta = -\frac{1}{w_\alpha^2} dw_\alpha = -z_\beta^2 dw_\alpha,$$

which gives

$$h_{\alpha\beta} = (-1)^{\alpha-\beta} z_\beta^{-n-1} = (-1)^{\alpha-\beta} \left(\frac{Z_\alpha}{Z_\beta}\right)^{n+1},$$

after a substitution in equation (1.7). □

### 1.2.5 Hermitian product on fiber bundles

Let $\pi : E \to M$ be a complex vector bundle over $M$. A *hermitian structure* or *hermitian metric* $h$ on $E$ is a $C^\infty$ field of hermitian inner products in the fibers of $E$. In other words for all $p \in M$ the following holds:

- $h_p(\lambda v_1 + v_2, w) = \lambda h_p(v_1, w) + h_p(v_2, w)$ $\forall v_1, v_2, w \in E_p$ and $\lambda \in \mathbb{C}$,
- $h_p(v, \lambda w_1 + w_2) = \bar{\lambda} h_p(v, w_1) + h_p(v, w_2)$ $\forall v, w_1, w_2 \in E_p$ and $\lambda \in \mathbb{C}$,
- $h_p(v, w) = \overline{h_p(w, v)}$ $\forall v, w \in E_p$.

and $h(s(\cdot), s(\cdot)) \in C^\infty(E \times E, \mathbb{C})$ for $s$ trivializing section of $\pi : E \to M$.
Observe that it is possible to construct a hermitian structure on every complex vector bundle of rank $r$. In fact, it suffices to consider a trivialization $(U_\alpha, \varphi_\alpha)$ on $E$ and a partition of the unity $\{f_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$ of $M$. For every point $p \in U_\alpha$ denote by $(H_\alpha)_p$ the pull-back of the hermitian metric on $\mathbb{C}^r$ through $\varphi_{\alpha_{|E_p}}$. Then $\sum f_\alpha H_\alpha$ is a well defined hermitian metric on $E$.
In the following examples we write explicitly the hermitian product on the universal bundle of $\mathbb{CP}^1$ and on $\mathbb{CP}^n$, which will be useful in Chapter 3.

**Example 1.6** (Hermitian product on the universal bundle of $\mathbb{CP}^1$)**.** Consider the hermitian metric $h^{-1}$ on $\mathcal{O}(-1)$. Given two points $v, w \in \pi^{-1}(U_0)$ in the same fiber, we have

$$h^{-1}_{|U_0}(v, w) = h^{-1}_{|U_0}\left(\left(\left[1, \frac{Z_1}{Z_0}\right], \lambda\left(1, \frac{Z_1}{Z_0}\right)\right), \left(\left[1, \frac{Z_1}{Z_0}\right], \mu\left(1, \frac{Z_1}{Z_0}\right)\right)\right) = \lambda\bar{\mu}\left(1 + \left|\frac{Z_1}{Z_0}\right|^2\right).$$

Clearly, $h^{-1}$ is well defined. In fact, if one consider another representation of $\left[1, \frac{Z_1}{Z_0}\right]$, e.g. $[aZ_0, aZ_1] \in U_0$ we have

$$h^{-1}_{|U_0}\left(\left([aZ_0, aZ_1], \frac{\lambda}{aZ_0}\left(aZ_0, aZ_1\right)\right), \left([aZ_0, aZ_1], \frac{\lambda}{aZ_0}\left(aZ_0, aZ_1\right)\right)\right) = \frac{\lambda\bar{\mu}}{|a|^2|Z_0|^2}|a|^2\left(|Z_0|^2 + |Z_1|^2\right).$$

**Example 1.7** (Hermitian product on the universal bundle of $\mathbb{CP}^n$)**.** Analogously to the one dimensional case, we can define hermitian products $h^{-1}$, $h$, $h^{-m}$, $h^m$ on $\mathcal{O}(-1)$, $\mathcal{O}(1)$, $\mathcal{O}(-m)$ and $\mathcal{O}(m)$, respectively. Let $p \in U_\alpha \subset \mathbb{CP}^n$, $p = [Z_0, \dots, Z_n]$, and consider two points in the same fiber $v = (p, \lambda p), w = (p, \mu p)$. Define

$$h^{-1}_{|U_\alpha}(v, w) = \lambda\bar{\mu}\left(\left|\frac{Z_0}{Z_\alpha}\right|^2 + \cdots + 1 + \cdots + \left|\frac{Z_n}{Z_\alpha}\right|^2\right),$$

and in affine coordinates $p = (z_1, \dots, z_n) = \left(\frac{Z_0}{Z_\alpha}, \dots, \frac{Z_n}{Z_\alpha}\right)$ we get

$$h^{-1}_{|U_\alpha}(v, w) = \lambda\bar{\mu}\Big(1 + |z_1|^2 + \cdots + |z_n|^2\Big),$$

that is a well defined hermitian product on $\mathcal{O}(-1)$. Observe finally that on $\mathcal{O}(1)$ we have $h = (h^{-1})^{-1}$

$$h_{|U_\alpha}(v, w) = \Big(\lambda\bar{\mu}\Big(1 + |z_1|^2 + \cdots + |z_n|^2\Big)\Big)^{-1}, \tag{1.8}$$

and in a similar way on $\mathcal{O}(m)$, $h^m = (h)^m$.

# Capitolo 2

# Complex Analysis of Bergman and Szegő kernels

In this chapter we summarize known results on complex analysis about reproducing kernels for domains in $\mathbb{C}^n$ and for domains of manifolds. In particular we analyze some properties of the Bergman and Szegő kernels. For this Chapter we refer to [33].

## 2.1 Bergman kernel for domains in $\mathbb{C}^n$

Let $\Omega \subset \mathbb{C}^n$ be a compact and bounded domain and let $L^2(\Omega)$ be the set of all holomorphic functions with finite norm on $\Omega$. Consider the set $A^2(\Omega) := L^2(\Omega) \cap \mathrm{Hol}(\Omega)$ of all holomorphic functions $f$ such that

$$\int_\Omega |f|^2 d\mu^{\frac{1}{2}} < \infty$$

where $d\mu$ is the restriction on $\Omega$ of the flat metric of $\mathbb{C}^n$. The space $A^2(\Omega)$ is called *the Bergman space* and it is a separable [1] Hilbert space with respect to the canonical inner product

$$< f, g >= \int_\Omega f \bar{g} d\mu.$$

[1] It is a subspace of $L^2(\Omega)$ that is a separable Hilbert space

Let $\varphi_0, \ldots, \varphi_j, \ldots$ be a complete orthonormal basis of $A^2(\Omega)$, then the *Bergman kernel* of $\Omega$ is the function

$$K(z, \bar{w}) = \sum_{j=0}^{+\infty} \varphi_j(z)\overline{\varphi_j(w)}.$$

It is easy to prove that $K(\cdot, \cdot)$ does not depend on the particular orthonormal basis chosen. Notice that if $z, w$ are two points of $\Omega$ then $K(z, \bar{w}) = \overline{K(w, \bar{z})}$ and holds the reproducing property

$$f(z) = \int_\Omega K(z, \bar{w}) f(w) d\mu$$

for all $f \in A^2(\Omega)$. Moreover, $K(z, \bar{w})$ is uniquely determined by these two last properties and it is an element of $A(\Omega)$ for each $w \in \Omega$. From the geometric point of view, the Bergman kernel of a domain $\Omega$ is very interesting because it is an invariant for biholomorphic maps in the following sense. Let $\Omega \subset \mathbb{C}^n$ be a domain and $f : \Omega \to \mathbb{C}^n$, $f = (f_1, \ldots, f_n)$ a holomorphic map, i.e. each of the $f_j$ is holomorphic. Given $z \in \Omega$ denote $w_j = f_j(z)$ and by

$$J_\mathbb{C} f = \frac{\partial(w_1, \ldots, w_n)}{\partial(z_1, \ldots, z_n)}$$

the $n \times n$ matrix that represents the *holomorphic Jacobian* of $f$. Recall that if $f : \Omega_1 \to \Omega_2$ is a holomorphic map, it is a biholomorphism from $\Omega_1 \subset \mathbb{C}^n$ to $\Omega_2 \subset \mathbb{C}^n$ if is invertible and its inverse is holomorphic, i.e. it is 1–1, onto and $\det J_\mathbb{C} f(z) \neq 0$ for all $z \in \Omega_1$.

**Proposition 2.1.** *Let $f : \Omega_1 \to \Omega_2$ be a biholomorphism and $\Omega_1$, $\Omega_2$ domains in $\mathbb{C}^n$. Then*

$$K_{\Omega_1}(z, \bar{w}) = \det J_\mathbb{C} f(z) K_{\Omega_2}(f(z), \overline{f(w)}) \det \overline{J_\mathbb{C} f(w)}$$

*for all $z$,$w \in \Omega_1$.*

When $\Omega$ is compact a characteristic of the Bergman kernel is that $K(z, \bar{z}) > 0$. In fact by definition we have

$$K(z, \bar{z}) = \sum_{j=0}^{+\infty} \varphi_j(z)\overline{\varphi_j(z)} = \sum_{j=0}^{+\infty} |\varphi_j(z)|^2 \geq 0$$

and $K(z,\bar{z}) = 0$ cannot be achieved due to the compactness of $\Omega$. When $\Omega$ is not compact, the first question that arises is when this reproducing kernel vanishes:

**Conjecture 2.2** (Lu Qi-Keng)**.** *Let $\Omega \subset \mathbb{C}^n$ be a simply connected domain with smooth boundary, then the Bergman kernel of $\Omega$ is non-vanishing.*

R. Greene and S. Krantz, in [25] and [26], proved that if $\Omega$ is $C^\infty$ sufficiently close to the ball in $\mathbb{C}^n$ then the Bergman kernel does not vanish.

Given a domain $\Omega \subset \mathbb{C}^n$ it is possible to define a hermitian metric on $\Omega$ using the Bergman kernel $K(\cdot,\cdot)$ of $\Omega$ in the following way,

$$g^\Omega_{\alpha\beta}(z) = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log(K(z,\bar{z})).$$

The metric $g^\Omega$ is called the *Bergman metric* of $\Omega$.

If the domain has good symmetric properties, we are able to calculate the Bergman kernel explicitly, for example, consider the unit ball $B^n \subset \mathbb{C}^n$ and let $z^j$ ($j$-multi-index) be a complete orthogonal basis of $A^2(B^n)$. With a bit of calculation we gets

$$K(z,\bar{w}) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \bar{w})^{n+1}} \tag{2.1}$$

where $z \cdot \bar{w} = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$, (see [33, p.60] for details). In general, it can be very difficult to find an explicit expression of the Bergman kernel (also to know if it is different from zero) for a given domain. In the case of strictly pseudoconvex domains, the Bergman kernel can be described by a celebrated formula due to C. Fefferman (see [22] and Theorem 2.4 below). Recall the definition of a strictly pseudoconvex domain:

**Definition 2.3.** *A domain $\Omega \subset \mathbb{C}^n$ with smooth boundary and with $\rho$ as defining function (i.e. $\Omega = \{z \in \mathbb{C}^n\} | \rho(z,z) > 0$) is* a strictly pseudoconvex *domain if*

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(w_j, \bar{w}_k) > 0 \tag{2.2}$$

*where $w_j$ are vectors of the boundary such that*

$$\frac{\partial \rho}{\partial z_j} w_j = 0.$$

Fefferman's results can be stated as follows:

**Theorem 2.4** (Fefferman's formula [22],[7]). *Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary and let $\rho : \Omega \to \mathbb{R}$ be the defining function of $\Omega$ i.e. $\Omega = \{z \in \mathbb{C}^n | \rho(z) > 0\}$, with boundary $\partial\Omega = \{z \in \mathbb{C}^n | \rho(z) = 0\}$. Then the Bergman kernel in the diagonal of $\Omega$ is of the form*

$$K(z, \bar{z}) = \frac{a(z)}{\rho(z)^{n+1}} + b(z) \log \rho(z) \tag{2.3}$$

*where $a$ and $b$ are continuous functions on $\overline{\Omega}$ and $a(z)_{|\partial\Omega} \neq 0$. For the points $(z, w) \in \Omega_\epsilon$, where $\Omega_\epsilon = \{|z - w| < \epsilon, dist(z, \partial\Omega) < \epsilon\}$ for sufficiently small $\epsilon > 0$, the Bergman kernel can be written as*

$$K(z, \bar{w}) = \frac{a(z, \bar{w})}{\rho(z, w)^{n+1}} + b(z, \bar{w}) \log \rho(z, w) \tag{2.4}$$

*where $a(z, \bar{w})$ $b(z, \bar{w})$ and $\rho(z, w)$ are extensions of $a(z)$, $b(z)$ and $\rho(z)$ in (2.3) such that*

- *$\rho(z, w)$ is almost analytic in $z$ and $w$ in the sense that $\bar{\partial}_z \rho(z, w)$ and $\partial_w \rho(z, w)$ vanish to infinite order [2] at $z = w$,*
- *$\rho(z, z) = \rho(z)$,*
- *the same holds for $a(z, \bar{w})$ and $b(z, \bar{w})$.*

This theorem tells us that although we are not able to calculate explicitly the Bergman kernel, we know that it can be represented in a elegant way. In particular, observe that the Bergman kernel of the $n$–dimensional ball is in Fefferman's form, without the logarithmic part (see eq. (2.1)). This characteristic is very important and we give the following definition

[2] i.e. A function $f : \Omega \to \mathbb{R}$ vanishes to infinite order if $\partial_z^k f(z, w)_{|z=w} = 0$ for all $k \in \mathbb{N}$.

**Definition 2.5.** *The log–term of the Bergman kernel $K(z, \bar{w})$ vanishes if the function $b$ in (2.4) is identically zero.*

There are several questions related to the previous observation, the first one is considered by the celebrated Ramadanov's Conjecture.

**Conjecture 2.6** (Ramadanov [47]). *Let $\Omega$ be a strictly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary. If the log–term of the Bergman kernel is zero then $\Omega$ is biholomorphically equivalent to the unit ball $B \subset \mathbb{C}^n$.*

Observe that when $\Omega$ is a complete Reinhardt domain in $\mathbb{C}^n$, Nakazawa, in [46], proved that the conjecture holds true. Moreover, it was proved to be true for any strictly pseudoconvex domain in $\mathbb{C}^2$ in [23] and for rotationally invariant domains in [28].

**Remark 2.7.** When we deal with complex manifolds instead of a complex domains in $\mathbb{C}^n$, we can still define the Bergman kernel and the Bergman metric. Throughout this thesis, we will not make any use of these concept. The interested reader is referred to [34].

## 2.2 Szegő kernel of domains in $\mathbb{C}^n$

In a similar way, we can define another reproducing kernel, (that is not invariant by biholomorphism) called the Szegő kernel.

Let $\Omega \subset \mathbb{C}^n$ be a compact bounded domain with smooth boundary $\partial\Omega$ (or such that it fails to be smooth in a set of points of measure zero) and consider the Hilbert space $H^2(\partial\Omega)$, that is the $L^2(\partial\Omega)$ closure of the set of all continuos functions defined on $\bar{\Omega}$ that are holomorphic on $\Omega$, restricted to $\partial\Omega$. The space $H^2(\partial\Omega)$ is called the Hardy space of $\partial\Omega$. The space $H^2(\partial\Omega)$ is a Hilbert space with respect to the inner product

$$< f, g >= \int_{\partial\Omega} f\bar{g}d\sigma,$$

where $d\sigma$ is the volume form on $\partial\Omega$ induced by $\Omega \subset \mathbb{C}^n$. Consider in $H^2(\Omega)$ a complete orthonormal basis $\psi_0, \dots, \psi_j, \dots$ with respect to $< \cdot, \cdot >$, then the *Szegő kernel* of $\Omega$ is the smooth function

$$\mathcal{S}(z, \bar{w}) = \sum_{j=0}^{+\infty} \psi_j(z)\overline{\psi_j(w)}.$$

It is easy to prove that the Szegő kernel does not depend on the particular orthonormal basis chosen. Moreover, given two points $z, w \in \Omega$, the reproducing property formula

$$f(z) = \int_{\partial\Omega} S(z, \bar{w})f(w)d\sigma$$

holds for all $f \in H^2(\partial\Omega)$.

The analogue of the Fefferman's formula holds true also for the Szegő kernel (see [22] and [6] for references):

**Theorem 2.8.** *Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary and let $\rho : \Omega \to \mathbb{R}$ be the defining function of $\Omega$ i.e. $\Omega = \{z \in \mathbb{C}^n | \rho(z) > 0\}$ with boundary $\partial\Omega = \{z \in \mathbb{C}^n | \rho(z) = 0\}$ such that $\frac{\partial\rho}{\partial z_j} \neq 0$ on $\partial\Omega$. Then the Szegő kernel in the diagonal of $\Omega$ is of the form*

$$\mathcal{S}(z, \bar{z}) = \frac{a(z)}{\rho(z)^n} + b(z) \log \rho(z) \tag{2.5}$$

*where $a$ and $b$ are continuous functions on $\overline{\Omega}$ and $a(z)_{|\partial\Omega} \neq 0$. For the points $(z, w) \in \Omega_\epsilon$, where $\Omega_\epsilon = \{|z - w| < \epsilon, dist(z, \partial\Omega) < \epsilon\}$ for sufficiently small $\epsilon > 0$, the Szegő kernel can be written as*

$$\mathcal{S}(z, \bar{w}) = \frac{a(z, \bar{w})}{\rho(z, w)^n} + b(z, \bar{w}) \log \rho(z, w) \tag{2.6}$$

*where $a(z, \bar{w})$, $b(z, \bar{w})$ and $\rho(z, w)$ are extensions of $a(z)$ $b(z)$ and $\rho(z)$ in (2.5) such that*

- *$\rho(z,w)$ is almost analytic in $z$ and $\bar{w}$ in the sense that $\bar{\partial}_z\rho(z,w)$ and $\partial_w\rho(z,w)$ vanish to infinite order at $z=w$,*

- *$\rho(z,z)=\rho(z)$,*

- *the same holds for $a(z,\bar{w})$ and $b(z,\bar{w})$.*

Analogously to the Bergman kernel case, we have the following definition:

**Definition 2.9.** *The log–term of the Szegő kernel $\mathcal{S}(z,\bar{w})$ is said to vanish if the function $b$ in* (2.6) *is identically zero.*

## 2.3 Szegő kernel of domains on manifolds

In this section we define the Szegő kernel of a domain $\Omega \subset M$.

Let $\Omega$ be a relatively compact domain on a complex manifold $M$ of dimension $n$. Consider a contact form $\alpha$ on $\partial\Omega$ [3] (we are assuming that $\partial\Omega$ is smooth, or fails to be smooth in a set of points of measure zero) and consider the induced volume form $\alpha \wedge (d\alpha)^{n-1}$. Let $H^2(\partial\Omega,\alpha)$ be the Hardy space obtained from the closure in $L^2(\partial\Omega)$ of the restricted functions $f$ that are holomorphic on $\Omega$. The space $H^2(\partial\Omega,\alpha)$ is a Hilbert space with respect to

$$< f, f >= \int_{\partial\Omega} |f|^2 \alpha \wedge (d\alpha)^{n-1}.$$

Pick in $H^2(\partial\Omega,\alpha)$ an orthonormal basis $\psi_0,\dots,\psi_j,\dots$ with respect to $<\cdot,\cdot>$. Then the *Szegő kernel* of $\Omega$ is the function:

$$S(z,\bar{w}) = \sum_{j=0}^{+\infty} \psi_j(z)\overline{\psi_j(w)}.$$

It is easy to prove that the Szegő kernel does not depend on the particular orthonormal basis chosen and as in the $\mathbb{C}^n$ case, it is the reproducing kernel of $H^2(\partial\Omega,\alpha)$. If $\Omega$ is a strictly pseudoconvex domain and $\rho$ is its defining function, then holds the Fefferman's formula (2.5) (see [22] and [6]).

[3]Recall that a contact form $\alpha$ on $\partial\Omega$ is a differential 1–form such that $\alpha \wedge (d\alpha)^{n-1} \neq 0$.

# Capitolo 3

# On a conjecture of Z. Lu and G. Tian

In this chapter we introduce the Kempf distortion function for a compact manifold $M$ and its complete asymptotic expansion. Then we analyze the close relationship between this expansion and the Szegő kernel of a particular domain constructed on the line bundle $L^*$ of $M$.

## 3.1 The TYZ expansion on compact manifolds

**Definition 3.1.** *Given a holomorphic line bundle $\pi : L \to M$ over a complex manifold $M$ we say that $L$ is a* positive *line bundle if the first Chern class of $L$ is exactly the class of a Kähler form $\omega$ on $M$, that is $c_1(L) = [\omega]$.*

Let $(L, h)$ be a holomorphic line bundle on $M$, we can associate to $h$ a $(1,1)$–form on $M$ that locally reads

$$\mathrm{Ric}(h)_{|U} := -\frac{i}{2}\partial\bar{\partial}\log(h(\sigma(x), \sigma(x))),$$

for a trivializing section $\sigma : U \to L \setminus \{0\}$. In general $\omega = \mathrm{Ric}(h)$ is a closed form which is also Kähler when $L$ is positive.

Given a positive line bundle $(L, h)$ with $h$ a hermitian product on $L$, we say that $(L, h)$ is a *polarization* for $(M, \omega)$ if $\mathrm{Ric}(h) = \omega$ and $\omega$ is a Kähler form. Observe that the existence of such a hermitian product on $L$ is guaranteed by the positivity of $L$ and is also equivalent to requiring that $\omega$ is an integral form.

**Remark 3.2.** The Kähler form $\omega_{FS}$ on $\mathbb{C}P^n$ defined in (1.2) is exactly $\mathrm{Ric}(h_{FS})$ where is $h_{FS} := h$ defined in (1.8).

Let $L \to M$ be a positive holomorphic line bundle over a compact Kähler manifold $(M, \omega)$ of dimension $n$ and let $s = \{s_0, \ldots, s_N\}$ be a basis of $H^0(M, L)$, the space of global holomorphic sections of $L$. Compactness of $M$ ensure that $\dim H^0(M, L) = N + 1$.

Let $i_{\underline{s}} : M \to \mathbb{C}P^N$ be the Kodaira map associated to the basis $s$ (see, e.g. [27]), namely $i_s : M \to \mathbb{C}P^N$ for a trivializing section $\sigma : U \to L \setminus \{0\}$ is given by:

$$i_s(x) = \begin{bmatrix} s_0(x) \\ \vdots \\ s_N(x) \end{bmatrix}, \ x \in M, \tag{3.1}$$

where $s_j = f_j \sigma$, $j = 0, \ldots, N$. Here the square bracket denotes the equivalence class in $\mathbb{C}P^N$. Note that, if we consider another trivializing section, say $\tau : V \to L$, then $\sigma = h \cdot \tau$ with $h : U \cap V \to \mathbb{C}$ holomorphic function. For each section $s_j \in H^0(L)$ we have $s_j = f_j \sigma = f_j h \cdot \tau = g_j \tau$, so $g_j = f_j \cdot h$ and represents the same point $i_s(x)$ in $\mathbb{C}P^N$. This map is induced by

$$\begin{aligned} \varphi_\sigma : U &\longrightarrow \mathbb{C}^N \\ x &\mapsto (f_0(x), \ldots, f_N(x)). \end{aligned} \tag{3.2}$$

It is clear that if we consider another trivializing section, for example $\varphi_\tau$, this map is different from $\varphi_\sigma$ but it induces the same Kodaira map.

The well known Kodaira Theorem can be summarized as follows:

**Theorem 3.3** (Kodaira)**.** *A compact complex manifold $M$ admits a positive line bundle $L \to M$ if and only if there exists a sufficiently large $m$ such that $i_s$ is a holomorphic embedding of $M$ in $\mathbb{CP}^{\dim H^0(L^m)-1}$.*

*Dimostrazione.* For a complete proof we refer to [50, Ch.VI, p.234]. □

The holomorphic embedding $i_s$ of Theorem 3.3 is called *Kodaira embedding.*

**Remark 3.4.** The universal bundle $\mathcal{O}(-1)$ for $\mathbb{CP}^n$ is a negative line bundle (it is the dual of a positive line bundle $\mathcal{O}(1)$). From Proposition 1.4 we know that the universal bundle has no holomorphic sections and then it is not possible to apply Theorem 3.3. On the contrary, if one consider the line bundle $\mathcal{O}(m)$ for $\mathbb{CP}^n$ we have that $\dim H^0(\mathcal{O}(m)) = \binom{m+n}{n}$ and in this case the Kodaira embedding is

$$\begin{aligned} i_{\underline{s}} : \mathbb{CP}^n &\longrightarrow \mathbb{CP}^{\binom{m+n}{n}-1} \\ [Z_0, \dots, Z_n] &\mapsto [Z_0^{j_0} \cdots Z_n^{j_n}, \dots, Z_0^{j_0} \cdots Z_n^{j_n}] \end{aligned}$$

with $j_0 + \cdots + j_n = m$. This map is called the *Veronese map* and the pull back of $\omega_{FS}$ on $\mathbb{CP}^{\binom{m+n}{n}-1}$ with the Veronese map is exactly the Fubini-Study form of $\mathbb{CP}^n$.

In general, given a complex manifold $(M, \omega)$ with a polarization $(L, h)$, the pull back of the Fubini–Study form through the Kodaira embedding is not equal to the form $\omega = \mathrm{Ric}(h)$, but it is in the same cohomology class as $\omega_{FS}$, i.e. $i_s^*(\omega_{FS}) \sim \omega$. More precisely, if $(L^m, h_m)$ is the holomorphic line bundle over the compact manifold $(M, \omega)$ where $h_m(\cdot) = h(\cdot)^m$ and $\mathrm{Ric}(h_m) = m\omega$, define in $H^0(L^m)$ the hermitian product

$$< s, t >_m = \int_M h_m(s(x), t(x)) \frac{\omega^n}{n!}(x)$$

for $s, t \in H^0(L^m)$. Let $\dim H^0(L^m) = N_m + 1$ be the dimension of $H^0(L^m)$ and let $s^m = (s_0^m, \dots, s_{N_m}^m)$ be an orthonormal basis of $H^0(L^m)$ with respect to $< \cdot, \cdot >_m$.

Then the *Kempf distortion function* is the smooth function $T_m(x) \in C^\infty(M, \mathbb{R}^+)$ defined by

$$T_m(x) := \sum_{j=0}^{N_m} h_m(s_j^m(x), s_j^m(x)). \tag{3.3}$$

Observe that in general

$$\int_M T_m(x) \frac{\omega^n}{n!} = N_m + 1. \tag{3.4}$$

The function $T_m$ is known in the literature by different names, for examples in [48] it's called $\eta-$function by Rawnsley, and later renamed $\theta-$function in [8]. In [30] Kempf called $T_m$ as distortion function and it is also called distortion function by Ji [29] for abelian varieties and by Zhang in [63] for complex projective varieties. It coincides with the diagonal of the Bergman kernel on $L^m$ associated to $h_m$ and thus is also frequently called *Bergman kernel* in the literature (see, for example, [42]).

The Kodaira embedding constructed using the orthonormal basis $s^m$, (crf. with (3.1) where the basis is not necessarily orthonormal) is given by

$$i_{s^m}(x) = \begin{bmatrix} s_0^m(x) \\ \vdots \\ s_{N_m}^m(x) \end{bmatrix}, \ x \in M, \tag{3.5}$$

and it is called *coherent states map.* Moreover, we have that

$$i_{s^m}^*(\omega_{FS}) = m\omega + \frac{i}{2}\partial\bar{\partial} \log T_m,$$

as it can be easily seen recalling that the Kodaira embedding $i_{s^m}$ is induced by

$\varphi_\sigma$ as in (3.2) (with $\sigma : U \to L \setminus \{0\}$ trivializing section) and thus locally we have

$$\begin{aligned} i^*_{s^m}(\omega_{FS})_{|U} =& \frac{i}{2}\partial\bar\partial \log(|f_0|^2 + \cdots + |f_{N_m}|^2) \\ =& m\omega - m\omega + \frac{i}{2}\partial\bar\partial \log(|f_0|^2 + \cdots + |f_{N_m}|^2) \\ =& m\omega + \frac{i}{2}\partial\bar\partial \log(h_m(\sigma(x), \sigma(x))) + \frac{i}{2}\partial\bar\partial \log(|f_0|^2 + \cdots + |f_{N_m}|^2) \\ =& m\omega + \frac{i}{2}\partial\bar\partial \log(h_m(\sigma(x), \sigma(x))(|f_0|^2 + \cdots + |f_{N_m}|^2)) \\ =& m\omega + \frac{i}{2}\partial\bar\partial \log T_m(x). \end{aligned} \tag{3.6}$$

Here we are using the fact that in the trivializing open set $U$ the Kempf distortion function reads

$$T_m(x) = \sum_{j=0}^{N_m} h_m(s^m_j(x), s^m_j(x)) = \sum_{j=0}^{N_m} h_m(\sigma(x), \sigma(x))(f_j(x), f_j(x)) =$$

$$= h_m(\sigma(x), \sigma(x))(|f_0|^2 + \cdots + |f_{N_m}|^2).$$

We say that $m\omega$ is projectively induced via the coherent states map if and only if $\partial\bar\partial T_m$ is zero, i.e. if and only if $T_m$ is constant, since $M$ is compact.

**Definition 3.5.** *Let $(L, h)$ be a polarization of a Kähler manifold $(M, \omega)$ with $\mathrm{Ric}(h) = \omega$. We say that $\omega$ is balanced if and only if the Kempf distortion function $T_1$ of $M$ is constant.*

Thanks to this definition, we can say that a compact manifold is projectively induced via the coherent state map if and only if it is balanced.

**Definition 3.6.** *A bundle $(L, h)$ of a manifold $(M, \omega)$ with $\mathrm{Ric}(h) = \omega$ is called a regular quantization if $T_m$ is constant for all $m > 0$.*

If $(L, h)$ is a regular quantization of the compact Kähler manifold $(M, \omega)$ we have the following result

**Proposition 3.7.** *If $(L, h) \to (M, \omega)$ is a positive line bundle and $(L, h)$ a regular quantization then*

$$T_m(x) = \frac{\dim H^0(L^m)}{\mathrm{Vol(M)}}.$$

*Dimostrazione.* Consider the integral

$$\int_M T_m(x)\frac{\omega^n}{n!},$$

that from (3.4) is equal to $\dim H^0(L^m)$. Since $T_m(x)$ is constant for all $m > 0$, then

$$\dim H^0(L^m) = \int_M T_m(x)\frac{\omega^n}{n!} = T_m(x)\int_M \frac{\omega^n}{n!} = T_m(x)\mathrm{Vol(M)}$$

that is the assertion. □

In the following example apply Prop. 3.7 to compute $T_m(x)$ for $\mathbb{CP}^n$.

**Example 3.8.** *Consider* $(\mathbb{CP}^n, \omega_{FS})$ *with* $(\mathcal{O}(m), h_{FS}^m)$ *(see Remark 3.2 and equation* (1.8)*). By Prop.1.4 we have* $\dim H^0(\mathcal{O}(m)) = \binom{m+n}{m}$. *Further* $\mathrm{Vol}(\mathbb{CP}^\mathrm{n}) = \int_{\mathbb{CP}^\mathrm{n}} \frac{\omega_{\mathrm{FS}}^\mathrm{n}}{\mathrm{n}!} = \frac{4\pi^\mathrm{n}}{\mathrm{n}!}$, *thus*

$$T_m(x) = \frac{\binom{m+n}{m}}{4\pi^n}.$$

In general, there are Kähler metrics that are not projectively induced via the coherent state map, but when $M$ is compact, Tian [52] and Ruan [51] proved that any polarized metric ($\omega = \mathrm{Ric}(h)$) is the $C^\infty$–limit of a sequence of normalized and projectively induced Kähler metrics. Zelditch in [59] generalized Tian and Ruan theorem proving the existence of a complete asymptotic expansion.

**Theorem 3.9** (Zelditch's)**.** *There is a complete asymptotic expansion*

$$T_m(x) \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2}\dots \tag{3.7}$$

*with* $a_j(x)$ *smooth and* $a_0(x) = 1$. *Asymptotic expansion means that for* $m \to +\infty$

$$\left\| T_m(x) - \sum_{j=0}^{k} a_j(x)m^{n-j} \right\|_{C^r} \leq C_{k,r}m^{n-k-1},$$

*where* $C_{k,r}$ *is a constant depending on* $k$ *and* $r$, *and on the Kähler form* $\omega$. *Moreover* $||\cdot||_{C^r}$ *is the* $C^r$ *norm in local coordinates.*

This asymptotic expansion is called *Tian–Yau–Zelditch expansion* or briefly TYZ expansion.

Another important result is due to Zhiqin Lu [40] who through Tian's peak method proved that each $a_j$ in (3.7) can be found with finitely many steps of algebraic computations, more precisely

**Theorem 3.10** (Lu). *Each of the coefficients $a_j$ of the TYZ expansion is a polynomial of the curvature and its covariant derivatives at $x$ of the metric $g$ of the manifold. In particular, the first three coefficients read*

$$a_1 = \frac{1}{2}\mathrm{Scal},$$

$$a_2 = \frac{1}{3}\Delta\mathrm{Scal} + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\mathrm{Scal}^2),$$

$$\begin{aligned} a_3 =& \frac{1}{8}\Delta\Delta\mathrm{Scal} + \frac{1}{24}divdiv(R, \mathrm{Ric}) - \frac{1}{6}divdiv(\mathrm{Scal}, \mathrm{Ric}) + \frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\mathrm{Scal}^2) \\ &+ \frac{1}{48}\mathrm{Scal}(\mathrm{Scal}^2 - 4|\mathrm{Ric}|^2 + |R|^2) + \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) - R(\mathrm{Ric}, \mathrm{Ric})),\end{aligned}$$

*where $R$,* Ric *and* Scal *represent the curvature tensor, the Ricci curvature and the scalar curvature of the metric of $M$, and $\Delta$ is the Laplacian of $M$.*

See examples 3.11 and 3.12 below for the definition of each element in the previous expressions of $a_1$, $a_2$ and $a_3$.

Observe that such coefficients can be computed also using a recursive formula written in terms of Calabi's diastasis function (see [36], [37] and references therein for details). In the following examples, we calculate explicitly the first coefficients of the TYZ expansion for $(\mathcal{O}(1), h_{FS})$ over $(\mathbb{CP}^1, \omega_{FS})$ and over $(\mathbb{CP}^2, \omega_{FS})$:

**Example 3.11.** *Consider $(\mathbb{CP}^1, \omega_{FS})$ with polarization $(\mathcal{O}(1), h_{FS})$ such that $\omega_{FS} = \mathrm{Ric}(h_{FS})$, (see Remark 3.2). By* (1.2) *we have locally*

$$\omega_{FS|U_0} = \frac{i}{2}\partial\bar{\partial}log(1 + |z|^2),$$

*and in particular*

$$g_{1\bar{1}} = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1}\log(1 + |z|^2) = \frac{1}{(1 + |z|^2)^2},$$

$$g^{1\bar{1}} := g^{-1}_{1\bar{1}} = (1+|z|^2)^2.$$

*The only Christoffel symbol for* $\mathbb{CP}^1$ *is* $\Gamma^1_{11} = g^{1\bar{1}}\frac{\partial g^{1\bar{1}}}{\partial z}$ *that reads*

$$\Gamma^1_{11} = -\frac{2\bar{z}}{1+|z|^2}.$$

*The curvature tensor for a Kähler manifold of dimension* 1 *is*

$$R_{1\bar{1}1\bar{1}} = \frac{\partial^2 g_{1\bar{1}}}{\partial z \partial \bar{z}} - g^{1\bar{1}}\frac{\partial g_{1\bar{1}}}{\partial z}\frac{\partial g_{1\bar{1}}}{\partial \bar{z}},$$

*that in our case gives*

$$R_{1\bar{1}1\bar{1}} = -\frac{2}{(1+|z|^2)^4}.$$

*The Ricci curvature tensor is*

$$\mathrm{Ric}_{1\bar{1}} = -g^{1\bar{1}}R_{1\bar{1}1\bar{1}} = \frac{2}{(1+|z|^2)^2}$$

*and the scalar curvature as the trace of the Ricci curvature is*

$$\mathrm{Scal} = g^{1\bar{1}}\mathrm{Ric}_{1\bar{1}} = 2.$$

*In order to compute* $a_1$ *and* $a_2$ *we also need* $|R|^2$ *and* $|\mathrm{Ric}|^2$ *that must be the same for all points of* $\mathbb{CP}^1$ *so we calculate them in* $(1, z) = (1, 0)$ *that reads*

$$|R|^2 = 4 \textit{ and } |\mathrm{Ric}|^2 = 4,$$

*while* $\Delta\mathrm{Scal} = 0$. *Then we have*

$$a_1 = \frac{1}{2}\mathrm{Scal} = 1,$$

$$a_2 = \frac{1}{3}\Delta\mathrm{Scal} + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\mathrm{Scal}^2) = \frac{1}{24}(4 - 16 + 12) = 0.$$

**Example 3.12.** *Consider* $(\mathbb{CP}^2, \omega_{FS})$ *with* $(\mathcal{O}(1), h_{FS})$ *(see Remark 3.2). By equation* (1.2)*) (note that in this case* $\omega_{FS} = \mathrm{Ric}(h_{FS})$ *is the Fubini-Study form on* $\mathbb{CP}^2$*) we have locally* $\omega_{FS|U_0} = \frac{i}{2}\partial\bar{\partial}log(1+|z_1|^2+|z_2|^2)$.

*The matrix of the metric* $g$ *associated to* $\omega_{FS}$ *reads*

$$(g_{\alpha\bar{\beta}}) = \frac{1}{(1+|z|^2)^2}\begin{pmatrix} 1+|z_2|^2 & -\bar{z}_1 z_2 \\ -z_1\bar{z}_2 & 1+|z_1|^2 \end{pmatrix},$$

*with inverse*

$$\left(g^{\alpha\overline{\beta}}\right) = (1+|z|^2)\begin{pmatrix} 1+|z_1|^2 & \overline{z}_1 z_2 \\ z_1\overline{z}_2 & 1+|z_2|^2 \end{pmatrix}.$$

*In our notation the Christoffel symbols are* $\Gamma^i_{jk} = \sum_{s=1}^n g^{s\bar{i}}\frac{\partial g^{k\bar{s}}}{\partial z_j}$ *and in particular for* $\mathbb{CP}^2$*, the only symbols different from* $0$ *are:*

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{\overline{z}_2}{1+|z|^2},\ \ \Gamma^2_{12} = \Gamma^2_{21} = -\frac{\overline{z}_1}{1+|z|^2},\ \ \Gamma^1_{11} = -\frac{2\overline{z}_1}{1+|z|^2},\ \ \Gamma^2_{22} = -\frac{2\overline{z}_2}{1+|z|^2}.$$

*The curvature tensor is defined as*

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial\overline{z}_l} - \sum_{p=1}^n\sum_{q=1}^n g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial\overline{z}_l},$$

*or*

$$R_{i\bar{j}k\bar{l}} = \sum_{s=1}^n g_{s\bar{l}}R^s_{i\bar{j}k},$$

*where* $R^l_{i\bar{j}k} = -\frac{\partial\Gamma^l_{ik}}{\partial\bar{z}_j}$*. In* $\mathbb{CP}^2$ *the only not zero elements are the following*

$$R_{1\bar{1}1\bar{1}} = -\frac{2(1+|z_2|^2)^2}{(1+|z|^2)^4}, \qquad R_{2\bar{2}2\bar{2}} = -\frac{2(1+|z_1|^2)^2}{(1+|z|^2)^4},$$

$$R_{1\bar{2}1\bar{1}} = R_{1\bar{1}1\bar{2}} = \frac{2\bar{z}_1 z_2(1+|z_2|^2)}{(1+|z|^2)^4}, \qquad R_{1\bar{1}2\bar{1}} = R_{2\bar{1}1\bar{1}} = \frac{2z_1\bar{z}_2(1+|z_2|^2)}{(1+|z|^2)^4},$$

$$R_{1\bar{2}2\bar{1}} = R_{2\bar{1}1\bar{2}} = R_{2\bar{2}1\bar{1}} = R_{1\bar{1}2\bar{2}} = -\frac{|z_1|^2|z_2|^2 + (1+|z_1|^2)(1+|z_2|^2)}{(1+|z|^2)^4}$$

$$R_{1\bar{2}2\bar{2}} = R_{2\bar{2}1\bar{2}} = \frac{2\bar{z}_1 z_2(1+|z_1|^2)}{(1+|z|^2)^4}, \qquad R_{2\bar{1}2\bar{2}} = R_{2\bar{2}2\bar{1}} = \frac{2z_1\bar{z}_2(1+|z_1|^2)}{(1+|z|^2)^4},$$

$$R_{1\bar{2}1\bar{2}} = -\frac{2\bar{z}_1^2 z_2^2}{(1+|z|^2)^4}, \qquad R_{2\bar{1}2\bar{1}} = -\frac{2z_1^2\bar{z}_2^2}{(1+|z|^2)^4}.$$

*The Ricci curvature tensor reads*

$$Ric_{i\bar{j}} = -\sum_{k,l=1}^n g^{k\bar{l}}R_{i\bar{j}k\bar{l}}$$

*and the scalar curvature (i.e. the trace of the Ricci curvature) is*

$$\mathrm{Scal} = \sum_{i,j=1}^n g^{i\bar{j}}\mathrm{Ric}_{i\bar{j}} = 6.$$

*In order to compute $a_1$, $a_2$ and $a_3$ we also need*

$$|R|^2 = \sum_{i,j,k,l,p,q,r,s=1}^{n} \overline{g^{i\bar{p}}} g^{j\bar{q}} \overline{g^{k\bar{r}}} g^{l\bar{s}} R_{i\bar{j}k\bar{l}} \overline{R_{p\bar{q}r\bar{s}}}$$

$$\text{and} \qquad |\mathrm{Ric}|^2 = \sum_{i,j,k,l=1}^{n} \overline{g^{i\bar{k}}} g^{j\bar{l}} \mathrm{Ric}_{i\bar{j}} \overline{\mathrm{Ric}_{k\bar{l}}},$$

*which in $\mathbb{CP}^2$ are constant and respectively equal to* 12 *and* 18*,*

$$|D'\mathrm{Scal}|^2 = \sum_{i,j=1^n} g^{i\bar{j}} \frac{\partial \mathrm{Scal}}{\partial z_i} \frac{\partial \mathrm{Scal}}{\partial \bar{z}_j},$$

$$|D'\mathrm{Ric}|^2 = \sum_{i,j,k,l,m=1}^{n} \overline{g^{i\bar{k}}} g^{j\bar{l}} \mathrm{Ric}_{i\bar{j},m} \overline{\mathrm{Ric}_{k\bar{l},m}},$$

*where* $\mathrm{Ric}_{i\bar{j},k} = \frac{\partial \mathrm{Ric}_{i\bar{j}}}{\partial z_k} - \sum_{s=1}^{n} \Gamma_{ik}^{s} \mathrm{Ric}_{s\bar{j}}$,

$$|D'R|^2 = \sum_{i,j,k,l,p,q,r,s,t=1}^{n} \overline{g^{i\bar{p}}} g^{j\bar{q}} \overline{g^{k\bar{r}}} g^{l\bar{s}} R_{i\bar{j}k\bar{l},t} \overline{R_{p\bar{q}r\bar{s},t}},$$

*where* $R_{i\bar{j}k\bar{l},p} = \frac{\partial R_{i\bar{j}k\bar{l}}}{\partial z_p} - \sum_{s=1}^{n} \Gamma_{ip}^{s} R_{s\bar{j}k\bar{l}} - \sum_{s=1}^{n} \Gamma_{kp}^{s} R_{i\bar{j}s\bar{l}}$ *that in $\mathbb{CP}^2$ are all equal to* 0*. Recall that the Laplacian is defined as* $\Delta = \sum_{i=1}^{n} \sum_{j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$.

*We need to compute also*

$$divdiv(\mathrm{Scal}, \mathrm{Ric}) = 2|D'\mathrm{Scal}|^2 + \sum_{i,j=1}^{n} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2 \mathrm{Scal}}{\partial z_i \partial \bar{z}_j} + \mathrm{Scal}\Delta\mathrm{Scal},$$

$$divdiv(R, \mathrm{Ric}) = -\sum_{i,j=1}^{n} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2 \mathrm{Scal}}{\partial z_i \partial \bar{z}_j} - 2|D'\mathrm{Ric}|^2 + \sum_{i,j,k,l,p,q,r=1}^{n} g^{i\bar{p}} R_{p\bar{i}k\bar{q}} g^{q\bar{k}} g^{i\bar{r}} \mathrm{Ric}_{r\bar{j},k\bar{l}} - R(\mathrm{Ric}, \mathrm{Ric}) - \sigma_3(\mathrm{Ric}),$$

*which for $\mathbb{CP}^2$ are $divdiv(\mathrm{Scal}, \mathrm{Ric}) = 0$ and $divdiv(R, \mathrm{Ric}) = 0$.*

*Finally*

$$\mathrm{Ric}(R, R) = \sum_{i,j,k,l,p,q,r,s,t,u=1}^{2} g^{il} \mathrm{Ric}_{lj} g^{jr} R_{rkps} g^{sq} g^{kt} R_{tiqu} g^{up},$$

$$R(\mathrm{Ric}, \mathrm{Ric}) = \sum_{i,j,k,l,p,q,r,s=1}^{2} g^{ip} R_{pjkq} g^{ql} g^{jr} \mathrm{Ric}_{ri} g^{sk} \mathrm{Ric}_{ls},$$

$$\sigma_3(\mathrm{Ric}) = \sum_{a,b,c,i,j,k=1}^{n} g^{i\bar{a}}\mathrm{Ric}_{a\bar{j}}g^{j\bar{b}}\mathrm{Ric}_{b\bar{k}}g^{k\bar{c}}\mathrm{Ric}_{c\bar{i}},$$

*which in* $\mathbb{CP}^2$ *are* $\mathrm{Ric}(R,R) = 36$, $R(\mathrm{Ric},\mathrm{Ric}) = -54$ *e* $\sigma_3(\mathrm{Ric}) = 54$.

*Concludingly, the coefficients are*

$$a_0 = 1,$$

$$a_1 = \frac{1}{2}\mathrm{Scal} = 3,$$

$$a_2 = \frac{1}{3}\Delta\mathrm{Scal} + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\mathrm{Scal}^2) = 2,$$

$$\begin{aligned}
a_3 =& \frac{1}{8}\Delta\Delta\mathrm{Scal} + \frac{1}{24}divdiv(R,\mathrm{Ric}) - \frac{1}{6}divdiv(\mathrm{Scal},\mathrm{Ric}) + \\
&+\frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\mathrm{Scal}^2) + \frac{1}{48}\mathrm{Scal}(\mathrm{Scal}^2 - 4|\mathrm{Ric}|^2 + |R|^2) \\
&+\frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R,R) - R(\mathrm{Ric},\mathrm{Ric})) \\
=& \frac{1}{48}6(36 - 72 + 12) + \frac{1}{24}(54 - 36 + 54) = \frac{-24}{8} + \frac{36}{12} = 0.
\end{aligned}$$

Observe that in these two examples, we find that the coefficient $a_{n+1}$ of the TYZ expansion is zero, where $n$ is the dimension of the complex projective space. We will illustrate in the following that this is not a particular case, but can be generalized to any compact Kähler manifold $M$ that admits a polarization and for which the log–term of the disk bundle of the dual of the positive line bundle is zero.

Before proving this result, we introduce the notion of disk bundle and its relationship with the Kempf distortion function of the manifold $M$.

### 3.1.1 Disk bundle

In this section, we want to define the Szegő kernel for a particular domain of the holomorphic vector bundle on the manifold $M$.

Let $(L^*, h^*)$ be the dual bundle of $(L, h)$ over $(M, \omega)$ such that $\mathrm{Ric}(h^*) = -\omega$. A disk bundle is a subset $D_h = \{v \in L^* | h^*(v,v) < 1\}$ of $L^*$ and $X_h = \partial D_h = \{v \in L^* | h^*(v,v) = 1\}$ is a unit circle bundle.

If the line bundle $(L, h)$ is also a polarization of $(M, \omega)$ (i.e $\mathrm{Ric}(h) = \omega$ with $\omega$ Kähler) we have the following

**Theorem 3.13.** *Let $D_h \subset L^*$ be the disk bundle of $(L^*, h^*)$, dual bundle of a polarization $(L, h)$ of $(M, \omega)$. Then $D_h$ is a strictly pseudoconvex domain.*

*Dimostrazione.* Clearly the defining function of $D_h$ is $\rho(v, v) = 1 - h^*(v, v) = 1 - h^{-1}(v, v)$, but if $v$ is a vector on $(\pi^*)^{-1}(z_1, \dots, z_n)$ then it is of the form $v = (v_1, \dots, v_{n+1}) = (z_1, \dots, z_n, \alpha)$ $(\alpha \in \mathbb{C}^*)$, so $\rho(v, v) = (1 - |\alpha|^2 h^{-1}(z, z))$. From Definition 2.3 we need to prove that $\frac{\partial^2 \rho}{\partial v_j \partial \bar{v}_k}((w_j, \beta), (\bar{w}_k, \bar{\beta})) > 0$ holds for boundary points $v' = (w_1, \dots, w_n, \beta)$ such that $\frac{\partial \rho}{\partial v_j}(w, \beta) = 0$. Recall that for a trivializing section $\sigma : U \to L^* \setminus 0$ we have

$$\mathrm{Ric}(h)_{|U} = \omega = \frac{i}{2} \partial \bar{\partial} \log h^{-1}(\sigma(z), \sigma(z))$$

that is positive defined so

$$0 > \partial \bar{\partial} \log h^{-1} = h^2 \left( \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} \cdot h^{-1} - \frac{\partial h^{-1}}{\partial z_j} \frac{\partial h^{-1}}{\partial \bar{z}_k} \right) = h \Big( \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} - h \frac{\partial h^{-1}}{\partial z_j} \frac{\partial h^{-1}}{\partial \bar{z}_k} \Big). \tag{3.8}$$

The quantity $\frac{\partial^2 \rho}{\partial v_j \partial \bar{v}_k}((w_j, \beta), (\bar{w}_k, \bar{\beta}))$ locally reads

$$\begin{pmatrix} w & \beta \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} & \frac{\partial^2 \rho}{\partial z_j \partial \bar{\alpha}} \\ \frac{\partial^2 \rho}{\partial \alpha \partial \bar{z}_k} & \frac{\partial^2 \rho}{\partial \alpha \partial \bar{\alpha}} \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{\beta} \end{pmatrix},$$

that is

$$\begin{pmatrix} w & \beta \end{pmatrix} \begin{pmatrix} -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} & -\alpha \frac{\partial h^{-1}}{\partial z_j} \\ -\bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} & -h^{-1}, \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{\beta} \end{pmatrix}.$$

Now expanding

$$\begin{pmatrix} w & \beta \end{pmatrix} \begin{pmatrix} -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} \bar{w}_k - \alpha \frac{\partial h^{-1}}{\partial z_j} \bar{\beta} \\ -\bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k - h^{-1} \bar{\beta} \end{pmatrix},$$

we need to evaluate if

$$\frac{\partial^2 \rho}{\partial v_j \partial \bar{v}_k}((w_j, \beta), (\bar{w}_k, \bar{\beta})) = -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k - \alpha \frac{\partial h^{-1}}{\partial z_j} w_j \bar{\beta} - \bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k \beta - h^{-1} \beta \bar{\beta} \tag{3.9}$$

is positive. Using the condition on the boundary points we have locally

$$\frac{\partial \rho}{\partial z_j} w_j + \frac{\partial \rho}{\partial \alpha} \beta = 0 \implies -|\alpha|^2 \frac{\partial h^{-1}}{\partial z_j} w_j - \bar{\alpha} h^{-1} \beta = 0,$$

which implies the two relations

$$\alpha \frac{\partial h^{-1}}{\partial z_j} w_j + h^{-1} \beta = 0,$$
$$\bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k + h^{-1} \bar{\beta} = 0.$$

Thus

$$\begin{aligned} h^{-1}\beta &= -\alpha \frac{\partial h^{-1}}{\partial z_j} w_j, \\ h^{-1}\bar{\beta} &= -\bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k, \end{aligned} \tag{3.10}$$

and multiplying the two equations we have

$$h^{-2} |\beta|^2 = |\alpha|^2 \frac{\partial h^{-1}}{\partial z_j} w_j \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k. \tag{3.11}$$

Now using both of (3.10) in (3.9) we get that

$$\begin{aligned} \frac{\partial^2 \rho}{\partial v_j \partial \bar{v}_k}((w_j, \beta), (\bar{w}_k, \bar{\beta})) &= -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k - \alpha \frac{\partial h^{-1}}{\partial z_j} w_j \bar{\beta} - \bar{\alpha} \frac{\partial h^{-1}}{\partial \bar{z}_k} \bar{w}_k \beta - h^{-1}|\beta|^2 \\ &= -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + h^{-1}|\beta|^2 + h^{-1}|\beta|^2 - h^{-1}|\beta|^2 \\ &= -|\alpha|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + h^{-1}|\beta|^2, \end{aligned} \tag{3.12}$$

and multiplying by $h|\beta|^2$ and its inverse and using (3.11)

$$\begin{aligned} &h\Big(-h^{-1} |\alpha|^2 |\beta|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + h^{-2} |\beta|^4\Big) |\beta|^{-2} \\ &= h\Big(-h^{-1}|\alpha|^2|\beta|^2 \frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + |\alpha|^2|\beta|^2 \frac{\partial h^{-1}}{\partial z_j} w_j \frac{\partial h^{-1}}{\partial \bar{z}_j} \bar{w}_k\Big)|\beta|^{-2} \\ &= |\alpha|^2\Big(-\frac{\partial^2 h^{-1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + h \frac{\partial h^{-1}}{\partial z_j} w_j \frac{\partial h^{-1}}{\partial \bar{z}_j} \bar{w}_k\Big), \end{aligned} \tag{3.13}$$

which is positive, because equation (3.8) is true and $(L, h)$ is a positive line bundle. $\square$

### 3.1.2 Szegő kernel of the disk bundle

Now we can define the Szegő kernel of the disk bundle on a manifold $(M, \omega)$ and by Theorem 3.13 we ensure that this kernel is of the form given by Fefferman's formula in (2.5), (see also [6]).

Consider the separable Hilbert space $H^2(X_h)$ defined as the closure in $L^2(X_h)$ of the set given by the restrictions to $X_h$ of the continuous functions in $D_h$ that are holomorphic in $D_h$ (see [6] and [59] for references). Let $d\mu = \alpha \wedge (d\alpha)^n$ be the natural measure on $X_h$ where $\alpha = -i\partial\rho_{|X_h} = i\bar{\partial}\rho_{|X_h}$ is the contact form on $X_h$ associated to the strictly pseudoconvex domain $D_h$ (observe that $D_h \subset L^*$ is a domain of dimension $n+1$). Let $\psi_0, \ldots, \psi_j, \ldots$ be an orthonormal basis of $H^2(X_h)$ with respect to

$$< \psi, \psi >= \int_{X_h} |\psi|^2 d\mu.$$

Then on the diagonal of $D_h \times D_h$, the Szegő kernel of $D_h$ is the function

$$\mathcal{S}(v, \bar{v}) = \sum_{j=0}^{+\infty} \psi_j(v)\overline{\psi_j(v)}.$$

From Theorem 3.13 we know that the disk bundle $D_h \subset L^*$ is a strictly pseudoconvex domain with smooth boundary and by Theorem 2.8, Fefferman's formula (2.5) holds for $D_h$, i.e. there exist functions $a$ and $b$ continuous on $\bar{D}_h$ and with $a \neq 0$ on $X_h$ such that:

$$\mathcal{S}(v) = \frac{a(v)}{\rho(v)^{n+1}} + b(v) \log \rho(v), \tag{3.14}$$

where $n+1$ is the dimension of $D_h \subset L^*$ and we write $\mathcal{S}(v) := \mathcal{S}(v, \bar{v})$.

We have seen that the study of the log–term of the Bergman kernel is related to an important conjecture (for example Ramadanov's Conjecture 2.6). A corresponding conjecture for the Szegő kernel of the disk bundle was formulated by M. Engliš and G. Zhang in [17], inspired by the paper [41] of G. Tian and Z. Lu. More precisely, they asked if the vanishing of the log–term of the Szegő kernel

of the disk bundle of a simply connected Kähler manifold implies that the circle bundle ($X_h$ in our notation) is diffeomorphic to the sphere or at least locally CR equivalent to the sphere. In [17], Engliš and Zhang showed with a counterexample that the conjecture is false for both the diffeomorphic and the CR equivalent case. In the first case, they considered the tensor power of the tautological bundle $\mathcal{O}(-1)$ over the complex projective space, namely the line bundle $L^* = \mathcal{O}(-m)$ over $\mathbb{CP}^n$: in fact in this case the Szegő kernel of the disk bundle $D_{h_{FS}} \subset \mathcal{O}(-m)$ has no log-term (cfr. (3.21) below), but $X_{h_{FS}}$, being the lens space $\mathbb{S}^{2n+1}/\mathbb{Z}_m$, is not diffeomorphic to $\mathbb{S}^{2n+1}$ for $m > 1$, but it is CR equivalent to $\mathbb{S}^{2n+1}$ (see [17] for details). For the locally CR equivalence case, they considered compact symmetric spaces of higher rank whose disk bundles have vanishing log–terms, but they are not locally spherical at any point (nor diffeomorphic to $\mathbb{S}^{2n+1}$).
In a recent paper [4], C. Arezzo, A. Loi and F. Zuddas generalized these results by showing that the disk bundles over homogeneous Hodge manifolds form an infinite family of strictly pseudoconvex domains (also smoothly bounded) for which the log–term vanishes but are not locally CR equivalent to the sphere.

Now we want to show the reason for considering this particular Szegő kernel and the relationship with the Kempf distortion function. Consider the disk bundle $D_h$ of the dual $(L^*, h^*)$ of a positive hermitian line bundle $(L, h)$ and let $H^2(X_h)$ be the Hardy space of holomorphic functions on $D_h$. It is possible to prove that the volume form $d\mu$ can be written as $d\mu = d\theta \wedge \pi^*(\omega^n)$, where $\pi : L^* \to M$, $\pi^*$ is the pull back through $\pi$ and $d\theta$ is the canonical $\mathbb{S}^1$–invariant volume form on $X_h$ (see [59, p.6] for more details). By the $\mathbb{S}^1$–action, the space $H^2(\partial D)$ splits into several parts with finite dimension. First of all, there exists a function

$$\begin{aligned} \wedge : H^0(L^m) &\to H^2(X_h) \\ s &\mapsto \hat{s}(v) := v^{\otimes m} s(x) \end{aligned} \tag{3.15}$$

where $x = \pi(v)$ and clearly if $\lambda \in \mathbb{C}^*$ we have $\hat{s}(\lambda v) = \lambda^m v^{\otimes m} s(x) = \lambda^m \hat{s}(v)$.

Let $H^2_m(X_h) := \{f \in H^2(X_h) \mid f(\lambda v) = \lambda^m f(v)\}$. Then $\hat{s} \in H^2_m(X_h)$ and the restriction of (3.15) to $H^2_m(X_h)$ becomes an isometry with respect to

$$\int_{X_h} \hat{s}(v)\hat{t}(v)d\mu = \int_M h_m(s(x), t(x))\frac{\omega^n}{n!}.$$

Thanks to the isometry, if $M$ is compact, the space $H^2_m(X_h)$ has finite dimension, in particular $\dim H^2_m(X_h) = \dim H^0(L^m) = N_m + 1$. Moreover, to an orthonormal basis $s^m = s^m_0, \ldots, s^m_{N_m}$ of $H^0(L^m)$ there corresponds an orthonormal basis $\wedge s^m = \hat{s}^m_0, \ldots, \hat{s}^m_{N_m}$ of $H^2_m(X_h)$. From the Fourier decomposition of $H^2(X_h)$ into irreducible factors, we have

$$H^2(X_h) = \bigoplus_{m=0}^{+\infty} H^2_m(X_h)$$

by the $\mathbb{S}^1$–action on $X_h$.

We need the following lemma

**Lemma 3.14.** *Let $s, t \in H^0(L^m)$. Then*

$$\hat{s}(v)\overline{\hat{t}(v)} = (h^*(v,v))^m h_m(s(x), t(x))$$

*where $x = \pi(v)$.*

*Dimostrazione.* If $v \in L^*$ then $v = \alpha\sigma^*(x)$ where $\sigma^* : M \to L^*$ is a global holomorphic frame of $H^0(L^*)$. A section on $H^0(L^m)$ can be written as $s(x) = f^m_s \sigma(x)^m$, $t(x) = f^m_t \sigma(x)^m$ where $\sigma : M \to L$ is a global holomorphic frame of $H^0(L)$.

The product $\hat{s}(v)\overline{\hat{t}(v)}$ reads

$$\begin{aligned}
\hat{s}(v)\overline{\hat{t}(v)} =& \hat{s}(\alpha\sigma^*(x))\overline{\hat{t}(\alpha\sigma^*(x))} = |\alpha|^{2m}\hat{s}(\sigma^*(x))\overline{\hat{t}(\sigma^*(x))} \\
\overset{def}{=}& |\alpha|^{2m}\sigma^*(x)^{\otimes m}s(x)\overline{\sigma^*(x)^{\otimes m}}\,\overline{t(x)} \\
=& |\alpha|^{2m}\sigma^*(x)^{\otimes m}\overline{\sigma^*(x)^{\otimes m}} f^m_s\sigma(x)^m \overline{f^m_t}\,\overline{\sigma(x)^m} \\
=& |\alpha|^{2m} f^m_s\overline{f^m_t} h^*(\sigma^*(x), \sigma^*(x))^m h(\sigma(x), \sigma(x))^m \\
=& (h^*(v,v))^m h_m(s(x), t(x)),
\end{aligned}$$

as wished. □

If one considers the orthogonal projections of the Szegő kernel onto each $H^2_m(X_h)$ we have that the Szegő kernel for $H^2(X_h)$ is the sum of all projections of the Szegő kernels onto $H^2_m(X_h)$, that is

$$\mathcal{S}(v) = \sum_{m=0}^{+\infty} \mathcal{S}_m(v) = \sum_{m=0}^{+\infty}\sum_{j=0}^{N_m} \hat{s}_j^m(v)\overline{\hat{s}_j^m(v)} \tag{3.16}$$

where $\hat{s}_0^m, \dots, \hat{s}_{N_m}^m$ is an orthonormal basis for $H^2_m(X_h)$, $x = \pi(v)$, $v \in L^*$ and $\mathcal{S}_m(v)$ is the projection on of $\mathcal{S}(v)$ on $H^2_m$. Using Lemma 3.14 we have

$$\begin{aligned} \mathcal{S}(v) &= \sum_{m=0}^{+\infty}\sum_{j=0}^{N_m} \hat{s}_j^m(v)\overline{\hat{s}_j^m(v)} = \sum_{m=0}^{+\infty}\sum_{j=0}^{N_m}(h^*(v,v))^m h_m(s_j^m(x), s_j^m(x)) \\ &= \sum_{m=0}^{+\infty}(h^*(v,v))^m T_m(x), \end{aligned} \tag{3.17}$$

and comparing with (3.16) gives

$$\mathcal{S}_m = (h^*(v,v))^m T_m(x), \tag{3.18}$$

where $T_m(x)$ is the Kempf distortion function on $M$ defined in (4.2).

**Remark 3.15.** Observe that from (3.18) it follows that if the Kempf distortion function $T_m$ of $M$ admits an asymptotic expansion as in (3.7) and if for example $h^*(v,v) < 1$ then also the projection $\mathcal{S}_m(x)$ does.

If $(L,h)$ is a regular quantization for $(M,\omega)$ (i.e for all $m > 0$ the function $T_m(x)$ is constant) then we have already seen (Prop. 3.7) that

$$T_m(x) = \frac{\dim H^0(L^m)}{\mathrm{Vol(M)}}.$$

In this case the Szegő kernel becomes

$$\mathcal{S}(v) = \sum_{m=0}^{+\infty}(h^*(v,v))^m \frac{\dim H^0(L^m)}{\mathrm{Vol(M)}}. \tag{3.19}$$

So we are ready to prove the following (see [4])

**Theorem 3.16** (C. Arezzo, A. Loi, F. Zuddas)**.** *Let $(L,h)$ be a regular quantization and let $D_h = \{v \in L^* \mid \rho(v,v) = 1 - h^*(v,v) > 0\} \subset L^*$ be the disk bundle of $M$. Then the log–term of the Szegő kernel of $D_h$ is zero.*

*Dimostrazione.* First of all, observe that it is possible to extend $h^*(v,v)$ to $h^*(v,\bar{v}')$ for all $v,v' \in D_h \times \bar{D}_h$ except for a subset of $D_h \times \bar{D}_h$ of measure zero and $h^*(v,\bar{v}')$ is well defined by the Cauchy-Schwarz inequality. From the Riemann–Roch Theorem, for $m \gg 1$ $\dim H^0(L^m)$ is a monic polynomial $m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0$. Thus

$$\dim H^0(L^m) = \sum_{k=0}^{n} \binom{m+k}{m} d_k$$

with $d_n = n!$. Substituting in (3.19), the Szegő kernel of $D_h$ reads

$$\mathcal{S}(v,\bar{v}') = \sum_{m=0}^{+\infty} (h^*(v,\bar{v}'))^m \frac{\sum_{k=0}^{n} \binom{m+k}{m} d_k}{\mathrm{Vol(M)}} = \frac{1}{\mathrm{Vol(M)}} \sum_{k=0}^{n} d_k \sum_{m=0}^{+\infty} (h^*(v,\bar{v}'))^m \binom{m+k}{m}.$$

The last sum gives

$$\sum_{m=0}^{+\infty} (h^*(v,\bar{v}'))^m \binom{m+k}{m} = \frac{1}{(1-h^*(v,\bar{v}'))^{k+1}}, \tag{3.20}$$

where we are using that

$$\sum_{m=0}^{+\infty} x^m \binom{m+k}{m} = \frac{1}{(1-x)^{k+1}}.$$

Recall that the defining function of $D_h$ is $\rho(v,v) = (1 - h^*(v,v))$, with almost analytic extension $\rho(v,\bar{v}') = (1 - h^*(v,\bar{v}'))$ which substituted in (3.20) reads

$$\mathcal{S}(v,\bar{v}') = \frac{1}{\mathrm{Vol(M)}} \sum_{k=0}^{n} d_k \frac{1}{\rho(v,\bar{v}')^{k+1}}.$$

Now writing $\rho$ for $\rho(v,\bar{v}')$, a direct computation shows:

$$\mathcal{S}(v,\bar{v}') = \frac{1}{\mathrm{Vol(M)}} \left( \frac{d_0}{\rho} + \frac{d_1}{\rho^2} + \cdots + \frac{n!}{\rho^{n+1}} \right) = \frac{\frac{1}{\mathrm{Vol(M)}} \big( d_0 \rho^n + \cdots + d_{n-1}\rho + n! \big)}{\rho^{n+1}},$$

which compared with Fefferman's formula (2.6) yields

$$a(v,\bar{v}') = \frac{1}{\mathrm{Vol(M)}} \big( d_0 \rho(v,\bar{v}')^n + \cdots + d_{n-1}\rho(v,\bar{v}') + n! \big)$$

and clearly $b(v,\bar{v}') = 0$. $\square$

Observe that if $M = \mathbb{CP}^n$ and $L^m = \mathcal{O}(m)$, from Theorem 1.4 we have $\dim H^0(L^m) = \binom{m+n}{m}$. Recalling that $\mathrm{Vol}(\mathbb{CP}^\mathrm{n}) = \frac{4\pi^\mathrm{n}}{\mathrm{n}!}$ we find that the Szegő kernel of the disk bundle $D_{h_{FS}}$ of $\mathbb{CP}^n$ is

$$\mathcal{S}(v, \bar{w}) = \frac{1}{\mathrm{Vol(M)}} \sum_{m=0}^{+\infty} \binom{m+n}{m} (h^*(v, \bar{v}'))^m = \frac{n!}{4\pi^n} \frac{1}{\rho^{n+1}}. \tag{3.21}$$

## 3.2 The conjecture of Zhiqin Lu and Gang Tian

In [41] Z. Lu and G.Tian analyzed what happens to the log–term of the Szegő kernel of the disk bundle $D_h$ when one varies the metric $h$ by preserving the corresponding cohomology class.

In particular they conjectured the following

**Conjecture 3.17** (Z.Lu–G.Tian)**.** *Let $\omega \in [\omega_{FS}]$ be a Kähler metric on $\mathbb{CP}^n$ in the same cohomology class as the Fubini–Study metric $\omega_{FS}$. Let $(L, h)$ be the hyperplane bundle whose curvature is $\omega$, (i.e. $\mathrm{Ric}(h) = \omega$). If the log–term of the Szegő kernel of the unit disk bundle $D_h \subset L^*$ vanishes, then there is an automorphism $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $\varphi^*\omega = \omega_{FS}$.*

Moreover, in the same paper, they proved the local version of the conjecture, in fact the conjecture above holds true if the hermitian metric $h$ is close to $h_{FS}$ in the following sense

**Theorem 3.18** (Z. Lu–G. Tian)**.** *Let $L$ be the hyperplane bundle of $\mathbb{CP}^n$ and let $h$ be a hermitian metric on $L$ such that $\mathrm{Ric}(h) = \omega$. Assume that there exists $\epsilon > 0$ (depending only on $n$) for which*

$$\left\| \frac{h}{h_{FS}} - 1 \right\|_{C^{2n+4}} < \epsilon. \tag{3.22}$$

*If the log–term of the Szegő kernel of the unit disk bundle $D_h$ vanishes, then there exists an automorphism $\varphi$ of $\mathbb{CP}^n$ such that $\varphi^*(\omega) = \omega_{FS}$.*

The main result obtained by Z. Lu and G. Tian in [41] is the close relationship between the vanishing of the log–term of the Szegő kernel constructed on the disk bundle $D_h \subset L^*$ and the vanishing of the coefficients $a_k$ of the TYZ expansion of $(M, \omega)$ for $k > n$ when $M$ is compact.

More precisely, they proved the following theorem

**Theorem 3.19** (Z. Lu, G.Tian)**.** *Let $(L, h)$ be a positive line bundle over a complex compact manifold $(M, \omega)$ of dimension $n$ such that $\mathrm{Ric}(h) = \omega$. If the log–term of the Szegő kernel of $D_h \subset L^*$ vanishes then the coefficients $a_k$ of the TYZ expansion in* (3.7) *vanish for $k > n$.*

For completeness we report here the proof that can be found in [41].

*Dimostrazione.* Let $v, v' \in L^*$ be two points whose local coordinates are $v = (z, \alpha)$ and $v' = (w, \beta)$, respectively (in the same trivializing open set).
We consider $h(z, w)$ as the almost analytic expansion of $h(z)$ in $z$ and $w$ in the sense that $\bar{\partial}_z h(z, w)$ and $\partial_w h(z, w)$ vanish to infinite order at $z = w$ and $h(z) = h(z, z)$. Define a global function $\psi(v, v') = -i\rho(v, v')$ with

$$\psi(v, v') = \psi(z, \alpha, w, \beta) = -i \left(1 - h(z, w)^{-1} \alpha \bar{\beta}\right).$$

Moreover, if $v, v' \in X_h \subset L^*$, we can write

$$\alpha = \sqrt{h(z)} e^{i\theta}, \qquad \beta = \sqrt{h(w)} e^{i\theta'},$$

where $\theta, \theta'$ are real numbers. Thus on $X_h$, we have

$$\psi(v, v') = \psi(z, \alpha, w, \beta) = -i \left(1 - \sqrt{h(z)}\sqrt{h(w)} h(z, w)^{-1} e^{i(\theta - \theta')}\right), \qquad (3.23)$$

and by Fefferman's formula (2.6) (see also [7])

$$\mathcal{S}(v, v') = \frac{a(v, v')}{\rho(v, v')^{n+1}} + b(v, v') \log \rho(v, v'). \qquad (3.24)$$

In particular, from [59] and eq.(2.3) in [41], we know that the projection of $\mathcal{S}(v,v')$ onto $H^2_m(X_h)$ is related to $\mathcal{S}(v,v')$ by

$$\mathcal{S}_m(v,v') = \int_{\mathbb{S}^1} \mathcal{S}(v, r_\theta v') e^{im\theta} d\theta, \tag{3.25}$$

where $r_\theta : X_h \to X_h$ is given by $r_\theta v = r_\theta(z,\alpha) = (z, \alpha e^{i\theta}) = (z, \sqrt{h(z)} e^{2i\theta})$. If the log–term of the Szegő kernel vanishes, i.e. $b = 0$ in (3.24), and passing to points on the diagonal of $X_h \times X_h$ we have

$$\mathcal{S}_m(v,v) = \int_{\mathbb{S}^1} \frac{i^{n+1} a(v, r_\theta v)}{(1-e^{-i\theta})^{n+1}} e^{im\theta} d\theta,$$

where $\rho(v, r_\theta v) = \psi(v, r_\theta v) = -i\left(1 - \sqrt{h(z)}\sqrt{h(z)} h(z,w)^{-1} e^{i(\theta-2\theta)}\right)$. We need to prove that the above expression expands to a polynomial in the variable $m$. For that, take a real number $c > 1$ and consider

$$\mathcal{S}_m(v,v) = \lim_{c\to 1} \int_{\mathbb{S}^1} \frac{i^{n+1} a(v, r_\theta v)}{(c-e^{-i\theta})^{n+1}} e^{im\theta} d\theta.$$

Now, integrating by parts $n$ times, we get

$$\mathcal{S}_m(v,v) = \lim_{c\to 1} \int_{\mathbb{S}^1} \frac{\zeta(v,\theta,m)}{(c-e^{-i\theta})^{n+1}} e^{im\theta} d\theta,$$

where $\zeta(v,\theta,m)$ is a polynomial in the variable $m$ and the coefficients are smooth functions in $v$ and $\theta$. By the Riemann-Lebesgue Lemma, we know that the above expression has the same asymptotic expansion as

$$\mathcal{S}_m(v,v) = \zeta(v,0,m) \lim_{c\to 1} \int_{\mathbb{S}^1} \frac{1}{(c-e^{-i\theta})^{n+1}} e^{im\theta} d\theta.$$

In other words, there is a polynomial $P(x,m)$ of degree $\leq m$ such that

$$\mathcal{S}_m(v,v) \sim P(v,m),$$

in the sense that

$$|\mathcal{S}_m(v,v) - P(v,m)| < \frac{C}{m^k}$$

for any $k$. From Remark 3.15 we can compare this expansion of $\mathcal{S}_m$ with the TYZ expansion of $T_m$ in (3.7). In particular, $P(v,m)$ being of degree less than or equal to $m$ implies that the coefficients $a_j$ with $j > n$ of the TYZ expansion are all equal to zero. $\square$

Following this idea, in [54] the author showed the validity of the Lu–Tian's Conjecture for a family of Kähler forms in $\mathbb{CP}^2$ cohomologous to $2\omega_{FS}$ and which do not satisfy condition (3.22).

Consider for each $a > 0$, the one parameter family of Kähler forms on $\mathbb{CP}^2$ given by

$$\omega_a = \Phi^*\omega_{FS}, \tag{3.26}$$

where $a = |\alpha|^2, \alpha \in \mathbb{C}^*$ and $\Phi$ is the holomorphic Veronese–type embedding given by

$$\begin{aligned} \mathbb{CP}^2 &\xrightarrow{\Phi} \mathbb{CP}^5 \\ [Z_0, Z_1, Z_2] &\longmapsto [Z_0^2, Z_1^2, Z_2^2, \alpha Z_0 Z_1, \alpha Z_0 Z_2, \alpha Z_1 Z_2], \end{aligned}$$

where $Z_0, Z_1, Z_2$ are homogeneous coordinates on $\mathbb{CP}^2$. (Note that we are denoting by the same symbol the Fubini-Study form of $\mathbb{CP}^2$ and of $\mathbb{CP}^5$).

So the author has proved the following

**Theorem 3.20.** *Let $\omega_a$ be as above and let $h_a$ be the hermitian product on $\mathcal{O}(1) \to \mathbb{CP}^2$ such that $\mathrm{Ric}(h_a) = \omega_a$. If the log–term of the Szegő kernel of $D_{h_a}$ vanishes, then there is an automorphism $\varphi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\varphi^*\omega_a = \omega_{FS}$.*

*Dimostrazione.* Consider standard affine coordinates in $\mathbb{CP}^2$ in the chart $U_0 = \{Z_0 \neq 0\}$. Then the Kähler form $\omega_a$ in (3.29) is given in these coordinates by

$$\omega_a = \frac{i}{2}\partial\overline{\partial}\log(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)$$

with $a = |\alpha|^2$.

Suppose that the log–term of the Szegő kernel of

$$D_{h_a} = \{v \in L^* \mid \rho(v, v) := 1 - h_a^*(v, v) > 0\} \subset L^*,$$

with $L^* = \mathcal{O}(-1)$ vanishes. Then, by Theorem 3.19, the coefficients $a_k = 0$ for $k > 2$. In particular $a_3 = 0$, which combined with Theorem 3.10 gives the following equation

$$\begin{aligned} a_3 =& \frac{1}{8}\Delta\Delta\mathrm{Scal} + \frac{1}{24} divdiv(R, \mathrm{Ric}) - \frac{1}{6} divdiv(\mathrm{ScalRic}) + \frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\mathrm{Scal}^2) \\ &+ \frac{1}{48}\mathrm{Scal}(\mathrm{Scal}^2 - 4|\mathrm{Ric}|^2 + |R|^2) + \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) - R(\mathrm{Ric}, \mathrm{Ric})) = 0. \end{aligned}$$

A long but straightforward computation obtained also with the use of a computer program, gives that the function $a_3$ evaluated at the origin reads

$$\begin{aligned} a_3(0,0) =& \frac{1}{6}\frac{3a^6 - 30a^5 - 67a^4 + 278a^3 + 904a^2 - 704a - 2592)}{a^6} \\ =& \frac{1}{6}\frac{(3a^5 - 24a^4 - 115a^3 + 48a^2 + 1000a + 1296)(a-2)}{a^6} \end{aligned} \tag{3.27}$$

while evaluating $a_3$ at the point $(1,1)$ reads

$$\begin{aligned} a_3(1,1) =& -\frac{1}{3}\frac{28139a^8 - 526469a^7 - 57190a^6 + 6561820a^5 + 2946788a^4 +}{(1+a)} \\ & \frac{-22781096a^3 - 16867840a^2 + 19757632a + 16922624}{(a^2+8a+16)^4(a+4)} \\ =& -\frac{1}{3}\frac{(28139a^7 - 470191a^6 - 997572a^5 + 4566676a^4 + 12080140a^3}{(1+a)} \\ & \frac{+1379184a^2 - 14109472a - 8461312)(a-2)}{(a^2+8a+16)^4(a+4)}. \end{aligned} \tag{3.28}$$

With a bit of calculation and using Descartes' rule of signs and the intermediate value theorem, we found that the positive zeros of (3.27) are $x_1, x_2, x_3$ with $x_1 = 2$, $x_2 \in \left]\frac{31}{10}, \frac{32}{10}\right[$ and $x_3 \in ]11, 12[$ while the positive solutions of (3.28) are $y_1, y_2, y_3, y_4$ with $y_1 = 2$, $y_2 \in ]1, 2[$, $y_3 \in \left]\frac{34}{10}, \frac{35}{10}\right[$ and $y_4 \in ]18, 19[$. So we can conclude that the only value of $a$ for which the coefficient $a_3$ is zero for all points is $a = 2$, which is the only Fubini–Study metric of the family. □

Let us point out that the proof of Theorem 3.20 cannot be achieved by Lu–Tian's Theorem, since $h_a$ doesn't satisfy condition (3.22). Indeed, let $\sigma_{|U_0} : U_0 \to L \setminus \{0\}$ be the trivializing section given by

$$\sigma_{|U_0}([Z_0, Z_1, Z_2]) = ([1, z_1, z_2], (1, z_1, z_2)),$$

with $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$. Then the local expression of the hermitian metric $h_a$ and of the hermitian metric $h_{FS}^2$ such that $\mathrm{Ric}(h_{FS}^2) = 2\omega_{FS}$ are given by

$$h_a(\sigma_{|U_0}([Z_0, Z_1, Z_2]), \sigma_{|U_0}([Z_0, Z_1, Z_2])) = \frac{1}{(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)},$$

and

$$h_{FS}^2(\sigma_{|U_0}([Z_0, Z_1, Z_2]), \sigma_{|U_0}([Z_0, Z_1, Z_2])) = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2},$$

respectively. If condition (3.22) were satisfied then the quantity

$$\left\| \frac{(1 + |z_1|^4 + |z_2|^4 + 2|z_1|^2 + 2|z_2|^2 + 2|z_1|^2|z_2|^2)}{(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)} - 1 \right\|$$

would be bounded. By passing to polar coordinates $(z_1, z_2) = \rho(\cos\vartheta, \sin\vartheta)$ one gets

$$\lim_{\cos\vartheta\sin\vartheta \to -\frac{1}{a}} \lim_{\rho\to+\infty} \left\| \frac{(2-a)[\rho(\cos\vartheta + \sin\vartheta) + \rho^2\cos\vartheta\sin\vartheta]}{(1 + \rho^2 + a[\rho(\cos\vartheta + \sin\vartheta) + \rho^2\cos\vartheta\sin\vartheta]} \right\| = +\infty,$$

which yields the desired contradiction.

One could ask if a similar result holds for a more general family of forms on $\mathbb{CP}^n$. To answer this question, consider the three parameter family of Kähler forms on $\mathbb{CP}^2$ given by

$$\omega_{abc} = \Psi^*\omega_{FS} \tag{3.29}$$

where $a = |\alpha|^2, \alpha \in \mathbb{C}^*,\ b = |\beta|^2, \beta \in \mathbb{C}^*,\ c = |\gamma|^2, \gamma \in \mathbb{C}^*$ and $\Psi$ is the holomorphic Veronese–type embedding given by

$$\mathbb{CP}^2 \xrightarrow{\Psi} \mathbb{CP}^5$$

$$[Z_0, Z_1, Z_2] \longmapsto [Z_0^2, Z_1^2, Z_2^2, \alpha Z_0Z_1, \beta Z_0Z_2, \gamma Z_1Z_2],$$

where $Z_0, Z_1, Z_2$ are homogeneous coordinates on $\mathbb{CP}^2$. (Also in this case we are denoting by the same symbol the Fubini-Study form of $\mathbb{CP}^2$ and of $\mathbb{CP}^5$). Replacing the proof of Theorem 3.20 we find the following expressions for the coefficient $a_3$ evaluated in the points $(0,0)$ and $(0,1)$

$$\begin{aligned}
a_3(0,0) =&(a^4b^3 - 136a^3b^3 + a^3b^4)\mathbf{c^3}+\\
&+(-4a^5b^4 + 4a^5b^2 - 376a^2b^4 + 6a^3b^4 + 216a^4b^4 + 6a^4b^3 + 4a^2b^5 - 376a^4b^2 - 4a^4b^5)\mathbf{c^2}+\\
&+(8a^2b^5 + 408a^3b^5 - 80a^5b^5 + 8a^5b^2 - 240a^3b^3 - 680ab^5 + 3a^6b^5 - 10a^4b^5 - 10a^5b^4+\\
&+3a^5b^6 - 680a^5b + 408a^5b^3 - 6a^6b^3 - 6a^3b^6)\mathbf{c}+\\
&-2208a^6 + 192a^4b^4 - 160a^4b^6 - 384a^4b^2 + 1176a^6b^2 - 160a^6b^4 - 384a^2b^4 - 72a^6b^3+\\
&-72a^3b^6 + 12a^5b^6 + 96ab^6 + 96a^6b + 2a^7b^4 - 2208b^6 + 1176a^2b^6 + 12a^6b^5 + 2a^4b^7.
\end{aligned}$$

and

$$\begin{aligned}
a_3(0,1) =&4bc^7 + (56a + 48ab + 216b^3 - 96b^4 - 808b + 340b^2 - 4ab^2)c^6+\\
&+(576 + a^2b^3 - 40a^2b^2 - 10544ab + 384b + 92a^2b - 6512a - 600ab^4 + 184a^2 + 1180ab^2+\\
&+1428b^3a + 48b^2)c^5+\\
&+(-48b^4 + 304a^3 + 6496a^2b^3 - 8b^3a - 72a^3b^2 + 21116a^2b^2 - 49600b + 116a^3b - 64ab^2+\\
&-20960b^2 - 2976b^3 - 1670a^2b^4 + 8a^3b^3 + 800ab - 38400 + 1696a^2 + 5896a^2b + 1792a)c^4+\\
&+(296a^4 - 4704a^3b^2 + 992a^2b + 24a^2b^3 - 5120 + 40064a - 1536b^2 - 64b^3 - 160a^2b^2+\\
&+2816a^2 - 28968a^3b + 9a^4b^3 + 112a^4b + 30544b^3a - 13664a^3 + 98080ab^2 - 4228a^3b^4+\\
&+116672ab - 70a^4b^2 - 4554a^3b^3 + 2832ab^4 - 5376b)c^3+\\
&+(294912 + 2752a^3 + 1552a^4 - 94976a^2 + 168a^5 + 8384b^4 + 87168b^3 + 321792b^2 - 12800a+\\
&+509440b + 21120a^4b^2 + +80a^5b + 5792a^4b - 12288ab + 6506a^4b^3 - 1670a^4b^4 - 172480a^2b+\\
&+960a^3b + 32a^3b^3 + 128b^3a - 40a^5b^2 - 141632a^2b^2 - 144a^3b^2 - 2688ab^2 - 59856a^2b^3+\\
&-10144a^2b^4 + 2a^5b^3)c^2+\\
&+(12288 + 39936a^3 + 1664a^4 - 12800a^2 - 6592a^5 + 1536b^3 + 9216b^2 - 74752a + 18432b+\\
&-6a^6b^2 - 64a^4b^2 - 10608a^5b + 704a^4b - 314368ab - 12288a^2b + 116608a^3b + 30560a^3b^3+\\
&-161024b^3a + 1176a^5b^2 - 2688a^2b^2 + 98112a^3b^2 + 40a^6b - 359424ab^2 + 128a^2b^3 - 600a^5b^4+\\
&+1432a^5b^3 + 2832a^3b^4 - 25280ab^4 + 48a^6)c+\\
&-565248 - 5120a^3 - 38528a^4 + 294912a^2 + 512a^5 - 35328b^4 - 282624b^3 - 847872b^2+\\
&+12288a - 1130496b + 340a^6b^2 - 20992a^4b^2 + 320a^5b - 49728a^4b + 18432ab - 2976a^4b^3+\\
&-48a^4b^4 + 509440a^2b - 5376a^3b - 64a^3b^3 + 1536b^3a + 32a^5b^2 + 321792a^2b^2 - 96a^6b^4+\\
&-1536a^3b^2 + 4a^7b - 808a^6b + 216a^6b^3 + 9216ab^2 + 87168a^2b^3 + 8384a^2b^4.
\end{aligned}$$

and a similar (but more complicated) polynomial of degree 13 in $a$, $b$ and $c$ for $a_3(1,1)$.

# Capitolo 4

# Szegő kernel on Cartan–Hartogs domains

We have already seen that Z. Lu and G. Tian proved in [41] Theorem 3.19 which states that *if the Szegő kernel of the disk bundle of a compact complex manifold* $(M, \omega)$ *has no log–term, then the coefficients* $a_k$ *of the TYZ expansion vanish for* $k > n$. Observe that an analogous result, with a completely similar proof, can be stated also for the non–compact case:

**Theorem 4.1.** *Let* $X_h$ *be the unit circle bundle of* $L^*$ *over* $M$ *(not necessarily compact). If the function* $b$ *of the Szegő kernel of* $X_h$ *vanishes, then the coefficients* $a_k$ *of TYZ expansion vanish for* $k > n$.

It is natural to ask if the converse of Theorem 3.19 holds true. In fact, Z. Lu has conjectured (private communication) the following:

**Conjecture 4.2** (Lu)**.** *Let* $(L, h)$ *be a positive line bundle over a compact complex manifold* $(M, \omega)$ *of dimension* $n$*, such that* $\mathrm{Ric}(h) = \omega$*. If the coefficients* $a_k$ *of TYZ expansion in* (3.7) *vanish for all* $k > n$*, then the log–term of the Szegő kernel of the disk bundle over* $M$ *vanishes.*

In [38] we studied the analogue of this conjecture for the non–compact case, in particular for an important family of manifolds called Cartan–Hartogs domains.

## 4.1 The TYZ expansion on non-compact manifolds.

In this section we define the Kempf distortion function for a non–compact manifold $M$. Analogously to the compact case, we need the Kähler form $\omega$ to be integral, i.e. we require the existence of a linear holomorphic line bundle $(L, h)$ which polarizes $(M, \omega)$. For the sake of simplicity, we will assume that $M$ is contractible, a condition which is satisfied by the Cartan–Hartogs domains we are dealing with.

Consider the separable Hilbert space $\mathcal{H}_m$ consisting of $L^m$'s global holomorphic sections bounded with respect to the hermitian product $h_m = h^m$

$$\mathcal{H}_m = \left\{ s \in Hol(M) \mid \int_M h_m(s(x), s(x)) \frac{\omega^n}{n!} < \infty \right\}. \tag{4.1}$$

Observe that if $M$ is compact, the space $\mathcal{H}_m$ coincides with $H^0(L^m)$ and the Kempf distortion function is defined in Section 3.1. Consider the inner product

$$< s, t >_m = \int_M h_m(s(x), t(x)) \frac{\omega^n}{n!}(x)$$

for $s, t \in \mathcal{H}_m$. If $\mathcal{H}_m \neq \{0\}$, choose an orthonormal basis $s^m = (s_0^m, \dots, s_{N_m}^m)$ $(\dim \mathcal{H}_m = N_m + 1 \leq \infty)$ of $\mathcal{H}_m$ with respect to $h_m$ and define the Kempf distortion function as

$$T_m(x) := \sum_{j=0}^{N_m} h_m(s_j^m(x), s_j^m(x)) \tag{4.2}$$

where $T_m(x) \in C^\infty(M, \mathbb{R}^+)$. In this context, unlike in the compact case, we do not have a general theorem which ensure the existence of a TYZ expansion for $T_m$. A partial result in this direction was given by M. Engliš in [14], where he showed that if $M$ is a strictly pseudoconvex bounded domain in $\mathbb{C}^n$ with real analytic boundary and $M$ is a bounded symmetric domain equipped with its Bergman metric, then the Kempf distortion function $T_m(x)$ admits the asymptotic expansion

$$T_m(x) \sim \sum_{j=0}^{+\infty} a_j(x) m^{n-j} \tag{4.3}$$

where $a_j(x)$ are smooth coefficients and $a_0(x) = 1$. Equation (4.3) means that for every integer $l$, $r$ and every compact $H \subseteq M$

$$\left\| T_m(x) - \sum_{j=0}^{l} a_j(x) m^{n-j} \right\|_{C^r} \leq \frac{C_{l,r,H}}{m^{l+1}}$$

where $C_{l,r,H} > 0$ is a constant depending on $l$, $r$ and $H$ and on the Kähler form $\omega$. Moreover, $|| \cdot ||_{C^r}$ is the $C^r$ norm in local coordinates.

Later, in [16] Engliš also computed the first three coefficients of the TYZ expansion for these manifolds. A different approach to that problem was taken by X. Ma and G. Marinescu in [42, Th.6.1.1], where they proved the existence of a TYZ expansion of $T_m$ under some assumptions on the curvature of the bundles considered. More precisely, they proved the following:

**Theorem 4.3** (X. Ma–G. Marinescu)**.** *Let $(M, g, \omega = \mathrm{Ric}(h))$ be a complete Kähler manifold and for $m > 0$, $h_m = h^m$ the hermitian metric defined on $L^m$. Then the Kempf distortion function $T_m(x)$ admits an asymptotic expansion in $m$ with coefficients given by (4.3) if there exists $c > 0$ such that*

$$iR^{det} > -c\omega \tag{4.4}$$

*where $R^{det}$ denotes the curvature of the connection on $\det(T^{(1,0)}M)$ induced by $g$.*

**Remark 4.4.** Observe that Theorem 6.1.1 in [42] is stated in a more general setting. In particular, for the existence of the TYZ expansion of the Kempf distortion function of a manifold $(M, \omega)$, Ma and Marinescu required the existence of $\epsilon > 0$ and $C > 0$ such that

$$iR^L > \epsilon\omega, \quad i(R^{det} + R^E) > -C\omega Id_E, \quad |\partial\omega|_{g^TX} < C.$$

On the other hand, the last condition is trivially satisfied if $(M, \omega)$ is Kähler. Moreover, in our case, the bundle $E = M \times \mathbb{C}$ is the trivial line bundle endowed with the flat metric $h_E$, so $R^E = 0$. Finally, using that $iR^L = 2\omega$ (since the metric in $L$ is $h$, which induces the Kähler form $\omega$) the first condition is always satisfied if $0 < \epsilon < 2$ and there remains only (4.4).

## 4.2 Hartogs domains

Let $F : [0, x_0) \to (0, +\infty]$ be a non–increasing lower semicontinuous function from $[0, x_0) \subset \mathbb{R}$ ($x_0 \leq +\infty$) to the positive real numbers. The domain $D_F$ given by

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\}$$

is called the *Hartogs domain* corresponding to the function $F$. The lower semicontinuity of $F$ is needed to have that $D_F$ is an open set. If we assume that $F$ is $C^2$ in $[0, x_0)$, we can define a real 2–form $\omega_F$ by

$$\omega_F := \frac{i}{2}\partial\bar{\partial}\log\frac{1}{F(|z_1|^2) - |z_2|^2}.$$

In particular if (and only if) $\left(\frac{xF'}{F}\right)' < 0$ for all $x \in [0, x_0)$ then $\omega_F$ is Kähler, where the prime denotes the derivative with respect to $x$. For more details on this domains see for example [15]. Now we compute the Szegő kernel of the Hartogs domain $D_F$ using the volume form induced by the contact form $\alpha$ on $\partial D_F$.

**Example 4.5.** Let $D_F$ be the Hartogs domain defined by

$$D_F := \{(z_1, z_2) \in \mathbb{C}^2, F(|z_1|^2) - |z_2|^2 > 0\},$$

and consider the boundary $\partial D_F = \{(z_1, z_2) \in \mathbb{C}^2, F(|z_1|^2) - |z_2|^2 = 0\}$. By definition, the contact form $\alpha$ on $\partial D_F$ is given by $\alpha = -i\partial\rho_{|\partial D_F}$, where $\rho =$

$F(|z_1|^2) - |z_2|^2 > 0$ is the defining function of $D_F$. Thus, we get:

$$\alpha = -i(F'\bar{z}_1 dz_1 - \bar{z}_2 dz_2).$$

Furthermore, by $d\alpha = (\partial + \bar{\partial})\alpha = -i\bar{\partial}\partial\rho$, we get

$$d\alpha = -i[(F''|z_1|^2 + F')dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2].$$

The volume form $\alpha \wedge d\alpha$ reads

$$\alpha \wedge d\alpha = F'\bar{z}_1 dz_1 \wedge dz_2 \wedge d\bar{z}_2 + (F' + F''|z_1|^2)\bar{z}_2 dz_2 \wedge dz_1 \wedge d\bar{z}_1,$$

which in polar coordinates, restricted to $\partial D_F$, becomes

$$\alpha \wedge d\alpha = -\left(\frac{rF'}{F}\right)' F^2\, dr \wedge d\theta_1 \wedge d\theta_2.$$

For convenience in further computation, we set

$$c_k(F^\alpha) := \int_0^{x_0} t^k F(t)^\alpha \left(-\left(\frac{rF'}{F}\right)'\right) dt.$$

An orthogonal basis of the Hardy space of $D_F$, is given by the monomials $\{z_1^j z_2^k\}$ with $j, k \in \mathbb{N}$ (see [15, Sec. 3]) and norm

$$\begin{aligned} ||z_1^k z_2^j||^2 = & \int_{\partial D_F} |z_1^k z_2^j|^2 \alpha \wedge d\alpha = \int_{\partial D_F} r^k F^j F^2 \left(-\left(\frac{rF'}{F}\right)'\right) dr \wedge d\theta_1 \wedge d\theta_2 \\ = & 4\pi^2 \int_0^{x_0} r^k F^{j+2} \left(-\left(\frac{rF'}{F}\right)'\right) dr = 4\pi^2 c_k(F^{j+2}). \end{aligned} \tag{4.5}$$

From [15, eq. (3.30), p.445], there exists an infinite subset $E$ which contains all the integers greater or equal then 2 and a real number $\gamma$ such that for all $\alpha \in E$,

$$\sum_{k=0}^{\infty} \frac{t^k}{c_k(F^\alpha)} = (\alpha - 1 + \gamma)F(t)^{-\alpha} \qquad \forall t \in \partial D_F. \tag{4.6}$$

Then by the definition of the Szegő Kernel, using (4.5) and (4.6) and setting $t_1 = |z_1|^2$, $t_2 = |z_2|^2$ we compute

$$\begin{aligned}
\mathcal{S}(z_1, z_2) &= \sum_{k,j=0}^{\infty} \frac{|z_1|^{2k}|z_2|^{2j}}{||z_1^k z_2^j||^2} = \frac{1}{4\pi^2}\sum_{k,j=0}^{\infty} \frac{t_1^k t_2^j}{c_k(F^{j+2})} \\
&= \frac{1}{4\pi^2}\sum_{j=0}^{\infty} \frac{t_2^j(j+2-1+\gamma)}{F(t_1)^{j+2}} = \frac{1}{4\pi^2}\sum_{j=0}^{\infty}\left(\frac{t_2}{F(t_1)}\right)^j \frac{(j+1+\gamma)}{F(t_1)^2} \\
&= \frac{1}{4\pi^2 F(t_1)^2}\sum_{j=0}^{\infty}(j+1)\left(\frac{t_2}{F(t_1)}\right)^j + \gamma\left(\frac{t_2}{F(t_1)}\right)^j \qquad (4.7)\\
&= \frac{1}{4\pi^2 F(t_1)^2}\left[\frac{1}{\left(1-\frac{t_2}{F(t_1)}\right)^2} + \gamma\frac{1}{1-\frac{t_2}{F(t_1)}}\right] \\
&= \frac{1}{4\pi^2 F(t_1)^2}\frac{F(t_1)^2 + \gamma F(t_1)(F(t_1)-t_2))}{(F(t_1)-t_2)^2},
\end{aligned}$$

where we are using that $\sum_j^\infty (j+1)x^j = \frac{1}{(1-x)^2}$ and $\sum_j^\infty x^j = \frac{1}{(1-x)}$. Recall that the defining function of $D_F$ is $\rho(z_1, z_2) = F(|z_1|^2) - |z_2|^2$. We have

$$\mathcal{S}(z_1, z_2) = \frac{F(|z_1|^2) + \gamma\rho}{4\pi^2 F(|z_1|^2)\rho^2}.$$

In particular, the Szegő kernel of the Hartogs domain $D_F$ has vanishing log–term.

## 4.3 Cartan domains

Now we define an important family of domains called Cartan domains.

It is well known that every hermitian symmetric space of non–compact type of complex dimension $d$ is biholomorphically isometric to $(\Omega, cg_B)$, where $\Omega$ is a bounded symmetric domain of $\mathbb{C}^d$ endowed with its Bergman metric $g_B$ multiplied by a positive constant $c$. A globally defined potential for $g_B$ is given by $\Phi(z) = \log \mathrm{K}$, where K is the Bergman kernel of $\Omega$. The domain $\Omega$ can be chosen to be circular (i.e. $z \in \Omega,\ \theta \in \mathbb{R} \Rightarrow\ e^{i\theta}z \in \Omega$) and convex. Every bounded symmetric domain is the product of irreducible factors, called *Cartan domains*. From E. Cartan's classification, Cartan domains can be divided into two categories, classical and exceptional ones (see [32] for details). Classical domains can be described in terms of complex matrices as follows ($m$ and $n$ are non–negative integers, $n \geq m$):

$$\Omega_1[m,n] = \{Z \in M_{m,n}(\mathbb{C}),\ I_m - ZZ^* > 0\} \qquad (\dim(\Omega_1) = nm),$$

$$\Omega_2[n] = \{Z \in M_n(\mathbb{C}),\ Z = Z^T,\ I_n - ZZ^* > 0\} \qquad (\dim(\Omega_2) = \tfrac{n(n+1)}{2}),$$

$$\Omega_3[n] = \{Z \in M_n(\mathbb{C}),\ Z = -Z^T,\ I_n - ZZ^* > 0\} \qquad (\dim(\Omega_3) = \tfrac{n(n-1)}{2}),$$

$$\Omega_4[n] = \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ \sum_{j=1}^n |z_j|^2 < 1, 1 + |\sum_{j=1}^n z_j^2|^2 - 2\sum_{j=1}^n |z_j|^2 > 0\}$$

$$(\dim(\Omega_4) = n),\ n \neq 2,$$

where $I_m$ (resp. $I_n$) denotes the $m \times m$ (resp $n \times n$) identity matrix and $A > 0$ means that $A$ is positive definite. In the latter domain we are assuming $n \neq 2$ since $\Omega_4[2]$ is not irreducible (and hence is not a Cartan domain).

The reproducing kernels of some classical Cartan domains are given by

$$\mathrm{K}_{\Omega_1}(z,z) = \frac{1}{V(\Omega_1)}[\det(I_m - ZZ^*)]^{-(n+m)},$$

$$\mathrm{K}_{\Omega_2}(z,z) = \frac{1}{V(\Omega_2)}[\det(I_n - ZZ^*)]^{-(n+1)},$$

$$\mathrm{K}_{\Omega_3}(z,z) = \frac{1}{V(\Omega_3)}[\det(I_n - ZZ^*)]^{-(n-1)},$$

$$\mathrm{K}_{\Omega_4}(z,z) = \frac{1}{V(\Omega_4)}\left(1 + |\sum_{j=1}^n z_j^2|^2 - 2\sum_{j=1}^n |z_j|^2\right)^{-n}, \tag{4.8}$$

where $V(\Omega_j)$, $j = 1, \dots, 4$, is the total volume of $\Omega_j$ with respect to the Euclidean measure of the ambient complex Euclidean space (see [12] for details).

In general, every bounded symmetric domain $\Omega$ is uniquely determined by a triple of integers $(r, a, b)$. The genus $\gamma$ of $\Omega$ is $\gamma = (r-1)a + b + 2$ and the dimension $d$ is defined by $d = \frac{r(r-1)}{2}a + rb + r$. The table below summarizes the numerical invariants and the dimension of $\Omega$ according to its type

Tabella 4.1: Bounded symmetric domains, invariants and dimension.

| Type | $r$ | $a$ | $b$ | $\gamma$ | dimension |
|---|---|---|---|---|---|
| $\Omega_1[m,n]$ | $m$ | $2$ | $n-m$ | $n+m$ | $nm$ |
| $\Omega_2[n]$ | $n$ | $1$ | $0$ | $n+1$ | $n(n+1)/2$ |
| $\Omega_3[n]$ | $[n/2]$ | $4$ | $0$ ($n$ even)<br>$2$ ($n$ odd) | $n-1$ | $n(n-1)/2$ |
| $\Omega_4[n]$ | $2$ | $n-2$ | $0$ | $n$ | $n$ |

Denote by $N = N(z)$ the *generic norm* of $\Omega$, namely

$$N(z) = (V(\Omega)K(z,z))^{-\frac{1}{\gamma}},$$

where $V(\Omega)$ is the total volume of $\Omega$ with respect to the Euclidean measure of $\mathbb{C}^d$ and $K(z,z)$ is its Bergman kernel (see previous section or [1] for more details). In particular, every Cartan domain $\Omega$ can be endowed with its Bergman metric $g_B$ whose associated Kähler form is

$$\omega_B = -\frac{i}{2}\partial\bar{\partial}\log N^{\gamma}, \tag{4.9}$$

that is a Kähler form on $\Omega$. In the following, we consider the Cartan domain $\Omega$ endowed also with the form $\omega_\Omega(\mu) = -\frac{i}{2}\partial\bar{\partial}\log N^{\mu}$ for which the metric $g_\Omega(\mu)$ reads

$$g_\Omega(\mu) = \frac{\mu}{\gamma}g_B = \frac{\partial^2 \log N^{\mu}}{\partial z_j \partial \bar{z}_k}. \tag{4.10}$$

In particular, we have (see also [60])

$$g_\Omega(\mu) = -\frac{\partial^2 \log N^{\mu}}{\partial z_j \partial \bar{z}_k} = \frac{N^{\mu}_j N^{\mu}_{\bar{k}} - N^{\mu}_{j\bar{k}} N^{\mu}}{N^{2\mu}}, \tag{4.11}$$

for all $j, k = 0, \dots, d$ and where we denote by $N^{\mu}_j := \frac{\partial N^{\mu}}{\partial z_j}$, $N^{\mu}_{\bar{k}} := \frac{\partial N^{\mu}}{\partial \bar{z}_k}$ and $N^{\mu}_{j\bar{k}} := \frac{\partial^2 N^{\mu}}{\partial z_j \partial \bar{z}_k}$.

Finally, from the homogeneity of $\Omega$ it follows that (see [32, p.18-19]) $g_B$ is Kähler–

Einstein[1] and so

$$\det(g_B) = N^{-\gamma}. \tag{4.12}$$

## 4.4 Cartan–Hartogs domains

In 1998, Guy Roos and Weiping Yin [56] introduced the following Hartogs' type domains based on Cartan domains.

Given a bounded symmetric domain $\Omega \subset \mathbb{C}^d$ (i.e. the product of Cartan domains as defined in the previous section) of rank $r$ and positive invariant numbers $a$ and $b$, we can define a new family of domains in $\mathbb{C}^{d+d_0}$ in the following way:

**Definition 4.6.** *The Cartan-Hartogs domain $M_\Omega^{d_0}(\mu)$ based on $\Omega$ is the pseudoconvex domain of $\mathbb{C}^{d+d_0}$ defined by ($\mu > 0$ is a fixed constant):*

$$M_\Omega^{d_0}(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C}^{d_0},\ ||w||^2 < N^\mu(z)\right\}. \tag{4.13}$$

The Cartan–Hartogs domain $M_\Omega^{d_0}(\mu)$ can be equipped with the natural Kähler form

$$\omega_{d_0} = -\frac{i}{2}\partial\bar{\partial}\log(N^\mu(z) - ||w||^2).$$

Note that $M_\Omega^{d_0}(\mu)$ is a Hartogs domain (in the sense of the previous section) with $F = N^\mu$. The Cartan–Hartogs domain $(M_\Omega^{d_0}(\mu), \omega_{d_0})$ has been studied by several authors from different analytical and geometrical points of view (see for example [21] [20], [56], [57], [58], [60] and [62]). For all Cartan–Hartogs domains an important *inflation principle*, very useful for future computation, holds. From [56, Section 2.3], there exists a function $L(z, |w|^2)$ for which the reproducing kernel of $M_\Omega^1(\mu)$ can be written as

$$K_{M_\Omega^1}(z,w) = L(z, |w|^2),$$

because of the circular symmetry with respect to the variable $w$.

[1]A manifold $(M, g)$ is *Kähler–Einstein* if $M$ is Kähler (see Def.1.1) and if the Ricci tensor is such that $\mathrm{Ric} = \lambda g$ for same constant $\lambda$.

**Proposition 4.7** (Inflation principle)**.** *Let $M_\Omega^{d_0}(\mu)$ be the Cartan–Hartogs domain defined by*

$$M_\Omega^{d_0}(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C}^{d_0},\ ||w||^2 < N^\mu(z)\right\}.$$

*The reproducing kernel of $M_\Omega^{d_0}(\mu)$ is*

$$K_{M_\Omega^{d_0}}(z,w) = \frac{1}{d_0!}\frac{\partial^{d_0-1}}{\partial r^{d_0-1}} L(z,r)\Big|_{r=||w||^2},$$

*with $||w||^2 = |w_1|^2 + \cdots + |w_{d_0}|^2$.*

*Dimostrazione.* The proof can be obtained by straightforward adaptation of the proof given in Subsection 2.4 in [5] for the case of the Bergman kernel. □

This theorem tells us that we can compute the Szegő kernel of a Cartan–Hartogs domain $M_\Omega^{d_0}(\mu)$ of dimension $d + d_0$ in the variable $(z, w_1, \ldots w_{d_0})$ by simply replacing $|w|^2$ with $||w||^2 = |w_1|^2 + \cdots + |w_{d_0}|^2$ in the Szegő kernel of the Cartan–Hartogs of dimension $d+1$ in the variable $(z, w)$. Consider the line bundle $L = M_\Omega^{d_0}(\mu) \times \mathbb{C}$ on $M_\Omega^{d_0}(\mu)$ and observe that is a trivial bundle since $M_\Omega^{d_0}(\mu)$ is contractible and pseudoconvex, so any holomorphic line bundle over $M_\Omega^{d_0}(\mu)$ is holomorphically trivial. We can endowed $L$ with the following hermitian metric

$$h_{d_0}(z,w;\xi) = \left(N^\mu(z) - ||w||^2\right)|\xi|^2, \quad (z,w) \in M_\Omega^{d_0}(\mu),\ \xi \in \mathbb{C}, \tag{4.14}$$

which satisfies $\mathrm{Ric}(h_{d_0}) = \omega_{d_0}$. In the following lemma we show that the disk bundle $D_{h_{d_0}}$ of the Cartan–Hartogs domain $M_\Omega^{d_0}(\mu)$, is the Cartan–Hartogs domain $M_\Omega^{d_0+1}(\mu)$.

**Lemma 4.8.** *The disk bundle $D_{h_{d_0}} = \{v \in L^* | h_{d_0}^*(v,v) < 1\} \subset L^*$, with $L = M_\Omega^{d_0}(\mu) \times \mathbb{C}$ is a Cartan–Hartogs domain of dimension $d+d_0+1$, namely $M_\Omega^{d_0+1}(\mu)$.*

*Dimostrazione.* Without loss of generality, we prove this assertion for $d_0 = 1$. Let $M_\Omega^1(\mu)$ be the Cartan–Hartogs of dimension $d + 1$ defined as

$$M_\Omega^1(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C},\ |w|^2 < N^\mu(z)\right\}.$$

endowed with the Kähler form $\omega_1 = -\frac{i}{2}\partial\bar{\partial}\log(N^\mu(z) - |w|^2)$ such that $\mathrm{Ric}(h_1) = \omega_1$, with

$$h_1(z,w;\xi) = \left(N^\mu(z) - |w|^2\right)|\xi|^2, \quad (z,w) \in M^1_\Omega(\mu),\ \xi \in \mathbb{C}.$$

If a point $v = (z,w,\xi)$ belongs to the disk bundle $D_{h_1} \subset L^*$ then

$$1 - h_1^*(v,v) = 1 - |\xi|^2 h_1^{-1} = 1 - \frac{|\xi|^2}{(N^\mu(z) - |w|^2)} > 0, \tag{4.15}$$

where $h_1^* = h_1^{-1}$. Since $(z,w) \in M^1_\Omega(\mu)$, we have $(N^\mu(z) - |w|^2) > 0$, so the last part of (4.15) becomes

$$(N^\mu(z) - |w|^2) - |\xi|^2 > 0,$$

which implies that

$$N^\mu(z) > |w|^2 + |\xi|^2.$$

Comparing with (4.13) gives the assertion, where a point of $M^2_\Omega(\mu)$ is indicated by the triple $(z,w,\xi)$ with $z \in \Omega$ and $(w,\xi) \in \mathbb{C}^2$. $\square$

Now we are interested in the TYZ expansion of the Kempf distortion function of a Cartan–Hartogs domain. From Theorem 4.3, Remark 4.4 and the fact that $iR^{det} = \rho$, since the metric on $\det(T^{(1,0)}M)$ induced by $g$ is exactly $\omega$ (see [31, p.18]), the Kempf distortion function of the Cartan-Hartogs domain $M^{d_0}_\Omega(\mu)$ admits an asymptotic expansion if $\rho > -c\omega_{d_0}$. From [60], the Ricci form of the Cartan–Hartogs domain $(M^1_\Omega, \omega_1)$ of dimension $d+1$ reads

$$\begin{aligned}\rho =& \frac{\mu(d+1)-\gamma}{\mu}\frac{1}{N^{2\mu}}\begin{pmatrix} (N^\mu)_j(N^\mu)_{\bar{k}} - (N^\mu)_{j\bar{k}}N^\mu & 0 \\ 0\ldots 0 & 0\end{pmatrix} + \\ & -(d+2)\frac{1}{(N^\mu - |w|^2)^2}\begin{pmatrix} (N^\mu)_j(N^\mu)_{\bar{k}} - (N^\mu)_{j\bar{k}}(N^\mu - |w|^2) & -(N^\mu)_j w \\ -(N^\mu)_{\bar{k}}\bar{w} & N^\mu\end{pmatrix}\end{aligned} \tag{4.16}$$

where the metric $g_1(\mu)$ is

$$g_1(\mu) = \frac{1}{(N^\mu - |w|^2)}\begin{pmatrix} (N^\mu)_j(N^\mu)_{\bar{k}} - (N^\mu)_{j\bar{k}}(N^\mu - |w|^2) & -(N^\mu)_j w \\ -(N^\mu)_{\bar{k}}\bar{w} & N^\mu\end{pmatrix}.$$

Clearly if $\frac{\mu(d+1)-\gamma}{\mu} > 0$, then the previous condition holds for $c > d+2$. More generally, if $\frac{\mu(d+d_0)-\gamma}{\mu} > 0$ then the Kempf distortion function of $(M_\Omega^{d_0}, \omega_{d_0})$ admits an asymptotic expansion as in (4.3). The main result about the TYZ expansion for Cartan-Hartogs domains is expressed by the following recent result in [21], which shows that the expansion is indeed finite, namely it is a polynomial in $m$ of degree $d+d_0$ with computable (non-constant) coefficients.

**Theorem 4.9** (Z. Feng–Z. Tu)**.** *Let* $m > \max\left\{d+d_0, \frac{\gamma-1}{\mu}\right\}$, *then the Kempf distortion function associated to* $(M_\Omega^{d_0}(\mu), \omega_{d_0})$ *can be written as*

$$T_m(z,w) = \frac{1}{\mu^d}\sum_{k=0}^{d} \frac{D^k\tilde{X}(d)}{k!}\left(1-\frac{||w||^2}{N^\mu}\right)^{d-k}\frac{\Gamma(m-d+k)}{\Gamma(m-d-d_0)}, \tag{4.17}$$

*with*

$$D^k\tilde{X}(d) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\prod_{l=1}^{r}\frac{\Gamma(\mu(d-j)-\gamma+2-(l+1)\frac{a}{2}+b+ra)}{\Gamma(\mu(d-j)-\gamma+1+(l-1)\frac{a}{2})}.$$

In [21] Z. Feng and Z.Tu used Formula (4.17) to prove that if the coefficient $a_2$ of the TYZ expansion of $M_\Omega^{d_0}(\mu)$ is constant, then $M_\Omega^{d_0}(\mu)$ is the complex hyperbolic space. In [61], M. Zedda generalized this result by proving that if one of the coefficients $a_j$, $2 \le j \le d+d_0$, of the TYZ expansion associated to $M_\Omega^{d_0}(\mu)$ is constant, then the domain is biholomorphically equivalent to the complex hyperbolic space.

In our context, formula (4.17) implies, in particular, that $a_k = 0$ for $k > d+d_0$. Therefore it is natural to ask if Conjecture 4.2 holds true in this (non–compact) case. Observe that the boundary of $M_\Omega^{d_0+1}(\mu)$ is not smooth, being

$$\partial M_\Omega^{d_0}(\mu) = \partial\Omega \cup \{(z,w) \in \Omega \times \mathbb{C}^{d_0} \mid ||w||^2 = N^\mu\}.$$

More precisely, the only Cartan–Hartogs domain with smooth boundary is the Cartan–Hartogs domain of rank 1, i.e. when $\Omega$ is the complex hyperbolic space. Thus, it does not make sense to speak of the log–term of the Szegő kernel, since Fefferman's formula (2.5) applies only when the domain involved has smooth boundary. Nevertheless, in order to consider the case of Cartan-Hartogs domains, we give the following definition (which in the smooth boundary case coincides with the standard one).

**Definition 4.10.** Let $D \subset M$ be a strictly pseudoconvex domain in a complex $n$-dimensional manifold $M$. Let $X = \partial D$ be its boundary with defining function $\rho > 0$, i.e. $D = \{v \in M |\, \rho(v) > 0\}$. Assume that the points where $X$ fails to be smooth are of measure zero. We say that *the log–term of the Szegő kernel vanishes* if there exists a continuous function $a$ on $\bar{D}$ with $a \neq 0$ on $X$ such that $\mathcal{S}(v) = \frac{a(v)}{\rho(v)^n}$.

In the following section we prove that

**Theorem 4.11.** *The log–term of the Szegő kernel of a Cartan–Hartogs domain vanishes.*

## 4.5 The Szegő kernel of Cartan–Hartogs domains

In this section we obtain the proof of Theorem 4.11 by finding explicitly the Szegő kernel of the disk bundle of the Cartan–Hartogs domain $M^1_\Omega(\mu)$ of dimension $d+1$ and by Prop. 4.7 (inflation principle) we generalize this result to a Cartan–Hartogs domain of dimension $d + d_0$. First of all, we compute the volume form $\alpha \wedge (d\alpha)^d$ on the boundary $\partial M^1_\Omega(\mu)$ of the strictly pseudoconvex domain $M^1_\Omega(\mu)$.

**Lemma 4.12.** *The volume form $\alpha \wedge (d\alpha)^d$ on the boundary $\partial M^1_\Omega(\mu)$ is given in polar coordinates $(\rho, \theta)$ by*

$$\alpha \wedge (d\alpha)^d = \left(\frac{2\mu}{\gamma}\right)^d N^{\mu(d+1)-\gamma} d\theta_w \wedge \frac{\omega_0^d}{d!},$$

*where $\frac{\omega_0^d}{d!}$ is the standard volume form of $\mathbb{C}^d$ and $\theta_w = \theta_{d+1}$.*

*Dimostrazione.* By definition, the contact form $\alpha$ is given by $\alpha = -i\partial\rho_{|\partial M^1_\Omega(\mu)}$, where $\rho = N^\mu - |w|^2 > 0$ is the defining function of $M^1_\Omega(\mu)$. Thus, we get

$$\alpha = -i\left(\sum_{j=1}^{d} \partial_j N^\mu dz_j - \bar{w}dw\right).$$

Furthermore, by $d\alpha = (\partial + \bar{\partial})\,\alpha = -i\bar{\partial}\partial\rho$, we get

$$d\alpha = -i\left(\sum_{j,k=1}^{d} N^\mu_{j\bar{k}} dz_j \wedge d\bar{z}_k - dw \wedge d\bar{w}\right) = i\left(dw \wedge d\bar{w} - \sum_{j,k=1}^{d} N^\mu_{j\bar{k}} dz_j \wedge d\bar{z}_k\right),$$

$$(d\alpha)^d = i^d\left(\det(-N^\mu_{j\bar{k}})d\xi + \sum_{s,q=1}^{d} (-1)^{s+q} \det(-N^\mu_{j\bar{k}})_{s\bar{q}} d\zeta_{s\bar{q}}\right),$$

where we write $N^\mu_j = \partial N^\mu/\partial z_j$, $N^\mu_{\bar{k}} = \partial N^\mu/\partial \bar{z}_k$ and $N^\mu_{j\bar{k}} = \partial^2 N^\mu/\partial z_j \partial \bar{z}_k$ and denote by $d\xi = dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d$ and by $d\zeta_{\bar{q}}$, (resp. $d\zeta_{s\bar{q}}$) the form $d\xi$ where the term $d\bar{z}_q$ (resp. the terms $dz_s$, $d\bar{z}_q$) is replaced by $d\bar{w}$ (resp. $dz_s$ with $dw$ and $dz_{\bar{q}}$ with $d\bar{w}$). Further, we write $(-N^\mu_{j\bar{k}})_{s\bar{q}}$ for the matrix $(-N^\mu_{j\bar{k}})$ where the $s$-th row and the $q$-th column have been deleted. Thus, the volume form $\alpha \wedge (d\alpha)^d$ is given by

$$\alpha \wedge (d\alpha)^d = -i^{d+1}\left(\sum_{s,q=1}^{d} (-1)^{s+q} N^\mu_s \det(-N^\mu_{j\bar{k}})_{s\bar{q}} dz_s \wedge d\zeta_{s\bar{q}} + \right. \\ \left. -\bar{w}\det(-N^\mu_{jk})dw \wedge d\xi\right). \tag{4.18}$$

Observe first that

$$dz_s \wedge d\zeta_{s\bar{q}} = -dw \wedge d\zeta_{\bar{q}} = dw \wedge d\bar{w} \wedge d\xi_{\bar{q}},$$

where $d\xi_{\bar{q}}$ is the form $d\xi$ where the term $d\bar{z}_q$ is deleted. Further, evaluating at the boundary, turning to polar coordinates $(\rho, \theta)$ and denoting $\rho_{d+1}$ by $\rho_w$ and $\theta_{d+1}$ by $\theta_w$, from $\rho_w^2 = N^\mu$ we have $2\rho_w d\rho_w = \sum_{j=1}^{d} N^\mu_{\bar{j}} e^{-i\theta_j}\,(d\rho_j - i\rho_j d\theta_j)$,

$$\bar{w}dw \wedge d\xi = \rho_w(d\rho_w + i\rho_w d\theta_w) \wedge d\xi = iN^\mu d\theta_w \wedge d\xi, \tag{4.19}$$

and

$$dw \wedge d\bar{w} = -2i\rho_w d\rho_w \wedge d\theta_w = -i\sum_{j=1}^{d} N^\mu_{\bar{j}} d\bar{z}_j \wedge d\theta_w,$$

which yields

$$dz_s \wedge d\zeta_{s\bar{q}} = -iN^{\mu}_{\bar{q}} d\bar{z}_q \wedge d\theta_w \wedge d\xi_{\bar{q}} = -iN^{\mu}_{\bar{q}} d\theta_w \wedge d\xi. \tag{4.20}$$

Substituting (4.19) and (4.20) into (4.18) we get

$$\alpha \wedge (d\alpha)^d = i^d A\, d\theta_w \wedge d\xi = 2^d A\, d\theta_w \wedge \frac{\omega_0^d}{d!},$$

where we used that $\frac{\omega_0^d}{d!} = \left(\frac{i}{2}\right)^d d\xi$ and sets

$$A = N^{\mu} \det\left(\left[-N^{\mu}_{j\bar{k}}\right]\right) - \sum_{j,k=1}^{d} (-1)^{j+k} N^{\mu}_j N^{\mu}_{\bar{k}} \det\left(\left[-N^{\mu}_{p\bar{q}}\right]\right)_{j\bar{k}}.$$

It remains to show that

$$A = \left(\frac{\mu}{\gamma}\right)^d N^{\mu(d+1)-\gamma}. \tag{4.21}$$

In order to prove (4.21), consider the metric $g_{\Omega}$ on the domain $\Omega$ associated to $\omega_{\Omega}$ defined by equation (4.11).

A direct computation gives:

$$\begin{aligned}
\det(g_{\Omega}) &= \det\left(\left[\frac{N^{\mu}_j N^{\mu}_{\bar{k}} - N^{\mu}_{j\bar{k}} N^{\mu}}{N^{2\mu}}\right]\right) \\
&= \frac{1}{N^{2d\mu}} \det\left(\left[N^{\mu}_j N^{\mu}_{\bar{k}} - N^{\mu}_{j\bar{k}} N^{\mu}\right]\right) \\
&= \frac{N^{\mu}_1 \cdots N^{\mu}_d}{N^{2d\mu}} \det\left(\left[N^{\mu}_{\bar{k}} - \frac{N^{\mu}_{j\bar{k}} N^{\mu}}{N^{\mu}_j}\right]\right) \\
&= \frac{\prod_{h=1}^d N^{\mu}_h N^{\mu}_{\bar{h}}}{N^{2d\mu}} \det\left([1] + \left[-\frac{N^{\mu}_{j\bar{k}} N^{\mu}}{N^{\mu}_j N^{\mu}_{\bar{k}}}\right]\right) \\
&= \frac{1}{N^{d\mu}} \det\left(\left[-N^{\mu}_{j\bar{k}}\right]\right) - \frac{1}{N^{\mu(d+1)}} \sum_{j,k=1}^{d} (-1)^{j+k} N^{\mu}_j N^{\mu}_{\bar{k}} \det\left(\left[-N^{\mu}_{p\bar{q}}\right]\right)_{j\bar{k}} \\
&= \frac{A}{N^{\mu(d+1)}}.
\end{aligned}$$

The conclusion follows with the help of

$$\det(g_{\Omega}) = \left(\frac{\mu}{\gamma}\right)^d \det(g_B) = \left(\frac{\mu}{\gamma}\right)^d N^{-\gamma},$$

where $g_B$ is the Bergman metric on $\Omega$ defined by (4.10) and we use (4.12). $\square$

Now we prove the Theoren 4.11

*Proof of Theorem 4.11.* Observe first that by Prop.4.7 (inflation principle) (see also Section 2.3 in [56]) we can assume without loss of generality that $d_0 = 1$. In this case the defining function is $\rho(z, w) = N^\mu(z) - |w|^2$ and

$$\partial M^1_\Omega(\mu) = \partial\Omega \cup \{(z, w) \in \Omega \times \mathbb{C} \mid |w|^2 = N^\mu\}.$$

Although $\partial M^1_\Omega(\mu)$ is not smooth, the points where it fails to be smooth make up a set of measure zero, so we can use Definition 4.10. From Lemma 4.12 the volume form $d\nu = \alpha \wedge (d\alpha)^d$ reads

$$d\nu = \alpha \wedge (d\alpha)^d = \left(\frac{2\mu}{\gamma}\right)^d N^{\mu(d+1)-\gamma} d\theta_w \wedge \frac{\omega_0^d}{d!}, \tag{4.22}$$

where $\frac{\omega_0^d}{d!}$ is the standard Lebesgue measure on $\mathbb{C}^d$ ($\omega_0$ is the flat Kähler form on $\mathbb{C}^d$). In order to compute the Szegő kernel $\mathcal{S}_{M^1_\Omega(\mu)}$ of $M^1_\Omega(\mu)$ one needs to find an orthonormal basis of the separable Hilbert space $\mathcal{H}^2(\partial M^1_\Omega(\mu))$ (Hardy space) consisting of all holomorphic functions $\hat{s}$ on $M^1_\Omega(\mu)$, continuous on $\partial M^1_\Omega(\mu)$ and such that

$$\int_{\partial M^1_\Omega(\mu)} |\hat{s}|^2 d\nu < \infty.$$

Consider the Hilbert space

$$\mathrm{H}^2_m(\Omega) = \left\{ s \in \mathrm{Hol}(\Omega) \;\middle|\; \int_\Omega N^{\mu m} |s(z)|^2 \frac{\omega_\Omega^d}{d!} < \infty \right\},$$

(where $\omega_\Omega = \frac{\gamma}{\mu}\omega_B$ is the Kähler form in $\Omega$ given by $\omega_\Omega = -\frac{i}{2}\partial\bar{\partial}\log N^\mu$ with $\omega_B$ given by (4.9)) and the map

$$\wedge : \mathrm{H}^2_m(\Omega) \to \mathcal{H}^2(\partial M^1_\Omega(\mu)) : \quad s \mapsto \hat{s} \tag{4.23}$$

defined by

$$\hat{s}(v) = 2^{-\frac{d}{2}} N(z, z)^{-\frac{\mu(d+1)}{2}} w^m s(z), \;\; v = (z, w) \in \partial M^1_\Omega(\mu).$$

Notice that the Hardy space $\mathcal{H}^2(\partial M^1_\Omega(\mu))$ admits a Fourier decomposition into irreducible factors with respect to the natural $\mathbb{S}^1$–action, i.e.

$$\mathcal{H}^2(\partial M^1_\Omega(\mu)) = \bigoplus_{m=0}^{+\infty} \mathcal{H}^2_m(\partial M^1_\Omega(\mu)),$$

where $\mathcal{H}^2_m(\partial M^1_\Omega(\mu)) := \{\hat{s} \in \mathcal{H}^2(\partial M^1_\Omega(\mu)) \mid \hat{s}(\lambda v) = \lambda^m \hat{s}(v)\}$ and $\lambda v := (z, \lambda w)$, for $v = (z, w)$. Since

$$\frac{\omega_\Omega^d}{d!} = \left(\frac{\mu}{\gamma}\right)^d N^{-\gamma} \frac{\omega_0^d}{d!},$$

it is not hard to see that the map $\wedge$ defines an isometry between $\mathrm{H}^2_m(\Omega)$ and $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$. Thus, if we consider the orthogonal projection of the Szegő kernel on each $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$, we get

$$\mathcal{S}_{M^1_\Omega(\mu)}(v) = \sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty} \hat{s}^m_j(v)\overline{\hat{s}^m_j(v)} = 2^{-d} N^{-\mu(d+1)} \sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty} |w|^{2m} |s^m_j(z)|^2, \quad (4.24)$$

where $s^m_j$, $j = 0, 1, \dots$ is an orthonormal basis of $\mathrm{H}^2_m(\Omega)$ and $\hat{s}^m_j = \wedge(s^m_j)$ is the corresponding orthonormal basis for $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$.

It is well-known (for a proof, see e.g. [18, p.77] or [19, Ch. XIII.1]) that $\sum_{j=0}^{\infty} N^{\mu m} |s^m_j(z)|^2$ is a polynomial in $m$ of degree $d = \dim \Omega$. Hence it can be written as

$$\sum_{j=0}^{\infty} N^{\mu m} |s^m_j(z)|^2 = \sum_{l=0}^{d} b_l \binom{m+l}{l},$$

where $b_l$ depends on the metric $g_\Omega$ associated to $\omega_\Omega$. Thus, this formula together with (4.24) yields

$$\begin{aligned}
\mathcal{S}_{M^1_\Omega(\mu)}(v) =& 2^{-d} N^{-\mu(d+1)} \sum_{m=0}^{\infty}\sum_{l=0}^{d} |w|^{2m} N^{-\mu m} b_l \binom{m+l}{l} \\
=& 2^{-d} N^{-\mu(d+1)} \sum_{l=0}^{d} b_l \sum_{m=0}^{\infty} \binom{m+l}{l} (|w|^2 N^{-\mu})^m \\
=& 2^{-d} N^{-\mu(d+1)} \sum_{l=0}^{d} b_l \frac{1}{(1 - |w|^2 N^{-\mu})^{l+1}}.
\end{aligned}$$

That is

$$\begin{aligned}\mathcal{S}_{M^1_\Omega(\mu)}(v) =&2^{-d}N^{-\mu(d+1)}\left[\frac{b_0 N^{\mu}}{(N^\mu-|w|^2)}+\cdots+\frac{b_d N^{\mu(d+1)}}{(N^\mu-|w|^2)^{d+1}}\right]\\ =&2^{-d}\frac{b_0N^{-\mu d}\left(N^\mu-|w|^2\right)^d+\cdots+b_{d-1}N^{-\mu}\left(N^\mu-|w|^2\right)^2+b_d}{\left(N^\mu-|w|^2\right)^{d+1}}.\end{aligned}$$

Observe that in the above expression, all terms except $b_d = d!\,m^d$ vanish once evaluated at the boundary $\partial M^1_\Omega(\mu)$. The vanishing of the log–term of $\mathcal{S}_{M^1_\Omega(\mu)}$ (as in Definition 4.10) then follows by setting

$$a(v) = 2^{-d}\left(b_0N^{-\mu d}\left(N^\mu-|w|^2\right)^d+\cdots+b_{d-1}N^{-\mu}\left(N^\mu-|w|^2\right)^2+b_d\right).$$

□

This result together with Lemma 4.8 implies the following

**Corollary 4.13.** *The log–term of the Szegő kernel of the disk bundle over a Cartan–Hartogs domain vanishes.*

Thus the Cartan–Hartogs domains are a family of non–compact manifolds for which Conjecture 4.2 holds true.

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