CLASSIFICATION RESULTS FOR BIHARMONIC SUBMANIFOLDS IN SPHERES

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Dedicated to Professor Vasile Oproiu on his 65th birthday

ABSTRACT. We study biharmonic submanifolds of the Euclidean sphere that satisfy certain geometric properties. We classify: (i) the biharmonic hypersurfaces with at most two distinct principal curvatures; (ii) the conformally flat biharmonic hypersurfaces. We obtain some rigidity results for pseudo-umbilical biharmonic submanifolds of codimension 2 and for biharmonic surfaces with parallel mean curvature vector field. We also study the type, in the sense of B-Y. Chen, of compact proper biharmonic submanifolds with constant mean curvature in spheres.

1. Introduction

The study of biharmonic maps between Riemannian manifolds, as a generalization of harmonic maps, was suggested by J. Eells and J.H. Sampson in [9]. They define the *energy* of a smooth map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds, by $E(\phi) = \frac{1}{2}\int_M |d\phi|^2\, v_g$, and say that ϕ is *harmonic* if it is a critical point of the energy. The Euler-Lagrange equation associated to E is given by the vanishing of the *tension field* $\tau(\phi) = \operatorname{trace} \nabla d\phi$.

By integrating the square of the norm of the tension field one can consider the *bienergy* of a smooth map ϕ, $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2\, v_g$, and call its critical points *biharmonic* maps (see [17]). The first variation formula for the bienergy, derived in [12], shows that the Euler-Lagrange equation corresponding to E_2 is given by the vanishing of the *bitension field* $\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = -\Delta\tau(\phi) - \operatorname{trace}\ R^N(d\phi, \tau(\phi))d\phi$, where J^{ϕ} is formally the Jacobi operator of ϕ. The operator J^{ϕ} is obviously linear, thus any harmonic map is biharmonic. We call *proper biharmonic* the non-harmonic biharmonic maps.

During the last decade important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps. In differential geometry, a special attention has been payed to the study of biharmonic submanifolds, i.e. submanifolds such that the inclusion map is a biharmonic map.

Moreover, the non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

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encouraged the study of proper biharmonic submanifolds in spheres and other curved spaces [2, 3, 10, 11, 13, 15].

Although important results and examples were obtained, the classification of proper biharmonic submanifolds in spheres is still an open problem.

This paper is fully devoted to the classification of proper biharmonic submanifolds in spheres that satisfy certain geometric properties.

It is organized as follows. In the preliminary section we gather some fundamental known results on proper biharmonic submanifolds of space forms and, in particular, of the Euclidean sphere. This section also contains some basic information on finite type Euclidean submanifolds. Although defined in a different manner, finite type submanifolds are, in a natural way, solutions of a variational problem. They are critical points of the volume functional for a certain class of directional deformations (see [6]).

In the third section we study the type of compact proper biharmonic submanifolds of constant mean curvature in $\mathbb{S}^n$ and prove that, depending on the value of the mean curvature, they are of $1-$type or of $2-$type as submanifolds of $\mathbb{R}^{n+1}$.

The fourth section is devoted to the complete classification of the proper biharmonic hypersurfaces with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. We prove that they are open parts either of the hypersphere $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of the Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$ (Theorem 4.3). A similar classification is obtained for conformally flat biharmonic hypersurfaces in spheres. On the contrary, for the hyperbolic space $\mathbb{H}^{m+1}$ we prove the non-existence of such hypersurfaces.

In the last section we prove that the pseudo-umbilical biharmonic submanifolds in spheres have constant mean curvature and we give an estimate for their scalar curvature. Then we classify the proper biharmonic pseudo-umbilical submanifolds of codimension 2 (Theorem 5.3). We also prove that the only biharmonic surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ are the minimal surfaces of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ (Theorem 5.6).

2. Preliminaries

2.1. Biharmonic submanifolds.

Let $\phi : M \to \mathbb{E}^n(c)$ be the canonical inclusion of a submanifold M in a constant sectional curvature c manifold, $\mathbb{E}^n(c)$. The expressions assumed by the tension and bitension fields are

$$\tau(\phi) = mH, \qquad \tau_2(\phi) = -m(\Delta H - mcH),$$

where H denotes the mean curvature vector field of M in $\mathbb{E}^n(c)$.

The attempt of classifying the biharmonic submanifolds in space forms was initiated in [7] and [2] with the following characterization results, obtained by splitting the bitension field in its normal and tangent components.

Theorem 2.1. [2, 5]. *The canonical inclusion $\phi : M^m \to \mathbb{E}^n(c)$ of a submanifold M in an n-dimensional space form $\mathbb{E}^n(c)$ is biharmonic if and only if*

$$\begin{cases} -\Delta^{\perp} H - \operatorname{trace} B(\cdot, A_H \cdot) + mcH = 0, \\ \\ 2 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + \frac{m}{2} \operatorname{grad}(|H|^2) = 0, \end{cases} \tag{2.1}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $\nabla^{\perp}$ and $\Delta^{\perp}$ the connection and the Laplacian in the normal bundle of M in $\mathbb{E}^n(c)$.

For hypersurfaces, this result becomes

Proposition 2.2. *Let M be a hypersurface of $\mathbb{E}^{m+1}(c)$. Then M is proper biharmonic if and only if*

$$\begin{cases} \Delta^{\perp}H - (mc - |A|^2)H = 0, \\ 2A\big(\operatorname{grad}(|H|)\big) + m|H|\operatorname{grad}(|H|) = 0. \end{cases} \tag{2.2}$$

In the case of the hyperbolic space some non-existence results were given. We recall here

Theorem 2.3. [3]. *Any biharmonic pseudo-umbilical submanifold M^m, $m \neq 4$, of the hyperbolic space $\mathbb{H}^n$ is minimal.*

For the sphere, using the canonical inclusion in the Euclidean space, the next caracterization result was obtained

Theorem 2.4. [3]. *If $\phi : (M, g) \to \mathbb{S}^n$ is a Riemannian immersion and $\varphi = \mathbf{i} \circ \phi$, where $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical inclusion, then*

$$\tau_2(\phi) = \tau_2(\varphi) + 2m\tau(\varphi) + \{2m^2 - |\tau(\varphi)|^2\}\varphi.$$

The first achievement towards the classification problem is represented by the complete classification of proper biharmonic submanifolds of the 3-dimensional unit Euclidean sphere, obtained in [2].

Theorem 2.5. [2].

a) *An arc length parameterized curve $\gamma : I \to \mathbb{S}^3$ is proper biharmonic if and only if it is either the circle of radius $\frac{1}{\sqrt{2}}$, or a geodesic of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ with slope different from ± 1.*
b) *A surface M is proper biharmonic in $\mathbb{S}^3$ if and only if it is locally a piece of $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$. Furthermore, if M is compact and orientable, then it is proper biharmonic if and only if $M = \mathbb{S}^2(\frac{1}{\sqrt{2}})$.*

Then, inspired by the $3-$dimensional case, two methods for constructing proper biharmonic submanifolds in $\mathbb{S}^n$ were given.

Theorem 2.6. [3]. *Let M be a minimal submanifold of $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if $a = \frac{1}{\sqrt{2}}$.*

Remark 2.7.

a) This result proved to be quite useful for the construction of proper biharmonic submanifolds in spheres. For instance, it implies the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$ (see [3]).
b) All minimal submanifolds of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^n$ are pseudo-umbilical, have parallel mean curvature vector in $\mathbb{S}^n$ and $|H| = 1$.

Non pseudo-umbilical examples were also produced by proving the following

Theorem 2.8. [3]. *Let $M_1^{m_1}$ and $M_2^{m_2}$ be two minimal submanifolds of $\mathbb{S}^{n_1}(r_1)$ and $\mathbb{S}^{n_2}(r_2)$, respectively, where $n_1 + n_2 = n - 1$, $r_1^2 + r_2^2 = 1$. Then $M_1 \times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if $r_1 = r_2 = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$.*

Remark 2.9.

a) The proper biharmonic submanifolds of $\mathbb{S}^n$ constructed as above are not pseudo-umbilical, but they have parallel mean curvature vector field, thus constant mean curvature, i.e. constant norm of the mean curvature vector field, and $|H| \in (0,1)$.
b) The generalized Clifford torus, $\mathbb{S}^{n_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{n_2}(\frac{1}{\sqrt{2}})$, $n_1 + n_2 = n-1$, $n_1 \neq n_2$, was the first example of proper biharmonic submanifold in $\mathbb{S}^n$ (see [12]).

We end this section with a partial classification result for constant mean curvature biharmonic submanifolds in spheres. The result was obtained in [14] and due to its importance for our paper we shall present it with its proof.

Theorem 2.10. [14]. *Let M be a proper biharmonic submanifold with constant mean curvature $|H|$ in $\mathbb{S}^n$. Then $|H| \in (0,1]$. Moreover, if $|H| = 1$, then M is a minimal submanifold of a hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^n$.*

Proof. Let M be a constant mean curvature biharmonic submanifold of $\mathbb{S}^n$. The first equation of (2.1) implies that

$$\langle \Delta^\perp H, H\rangle = m|H|^2 - |A_H|^2,$$

and by using the Weitzenböck formula,

$$\frac{1}{2}\Delta |H|^2 = \langle \Delta^\perp H, H\rangle - |\nabla^\perp H|^2,$$

it follows

$$m|H|^2 = |A_H|^2 + |\nabla^\perp H|^2. \tag{2.3}$$

Let now $\{X_i\}$ be a local orthonormal basis such that $A_H(X_i) = \lambda_i X_i$. From

$$\lambda_i = \langle A_H(X_i), X_i\rangle = \langle B(X_i, X_i), H\rangle$$

and

$$\sum \lambda_i = m|H|^2, \qquad \sum (\lambda_i)^2 = |A_H|^2,$$

using (2.3) we obtain

$$\sum \lambda_i = \sum (\lambda_i)^2 + |\nabla^\perp H|^2 \geq \frac{(\sum \lambda_i)^2}{m} + |\nabla^\perp H|^2. \tag{2.4}$$

Thus

$$m|H|^2 \geq m|H|^4 + |\nabla^\perp H|^2.$$

Consequently, if $|H| > 1$, the last inequality leads to a contradiction.

If $|H| = 1$, then the last inequality implies $\nabla^\perp H = 0$ and $\sum(\lambda_i)^2 = \frac{(\sum \lambda_i)^2}{m} = m$, thus we get $\lambda_1 = \ldots = \lambda_m$. Therefore M is a minimal submanifold of the hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. □

2.2. **Pseudo-umbilical submanifolds in spheres.**

Definition 2.11. *A submanifold M of a Riemannian manifold N is said to be pseudo-umbilical if there exists a function $\lambda \in C^\infty(M)$, such that $A_H = \lambda\,\mathrm{Id}$, where A_H is the Weingarten operator associated to the mean curvature vector field H of M in N.*

Remark 2.12. If M is a pseudo-umbilical submanifold of N, one can immediately prove that $\lambda = |H|^2$.

We also recall here two important geometric properties of pseudo-umbilical submanifolds in spheres.

Theorem 2.13. [4, p.173]. *Let M be an $m-$dimensional pseudo-umbilical submanifold of an $n-$dimensional unit Euclidean sphere $\mathbb{S}^n$. Then the scalar curvature s of M satisfies*

$$s \leq m(m-1)(1+|H|^2).$$

The equality holds if and only if M is contained in an $m-$sphere $\mathbb{S}^m\big(\frac{1}{\sqrt{1+|H|^2}}\big)$ of $\mathbb{S}^n$.

Theorem 2.14. [4, p.180]. *Let M^m be a pseudo-umbilical submanifold in $\mathbb{S}^{m+2}$. If M has constant mean curvature, then M is either a minimal submanifold of $\mathbb{S}^{m+2}$ or a minimal hypersurface of a hypersphere of $\mathbb{S}^{m+2}$.*

2.3. Finite type submanifolds in Euclidean spaces.

Definition 2.15. *An isometric immersion $\varphi : M \to \mathbb{R}^n$ is called of finite type if φ can be expressed as a finite sum of $\mathbb{R}^n-$valued eigenfunctions of the Laplacian Δ of M. When M is compact it is called of $k-$type if the spectral decomposition of φ contains exactly k non-zero terms, excepting the center of mass.*

The following result constitutes a useful tool in determining whether a compact submanifold of $\mathbb{R}^n$ is of finite type.

Theorem 2.16. *(Minimal Polynomial Criterion).*[5, 6]. *Let $\varphi : M^m \to \mathbb{R}^n$ be an isometric immersion of a compact Riemannian manifold M into $\mathbb{R}^n$ and denote by H^0 the mean curvature vector field of M in $\mathbb{R}^n$. Then*

a) *M is of finite type if and only if there exists a non-trivial polynomial $Q(t)$ such that $Q(\Delta)H^0 = 0$.*
b) *M is of finite type k if and only if there exists a unique monic (i.e. with leading coefficient equal to 1) polynomial $P(t)$ with exactly k distinct positive roots, such that $P(\Delta)H^0 = 0$.*

3. The type of compact proper biharmonic submanifolds in spheres

In this section, by applying the preliminary results to the biharmonic case, we intend to analyze the type of proper biharmonic submanifolds of $\mathbb{S}^n$, as submanifolds in $\mathbb{R}^{n+1}$.

We prove the following

Theorem 3.1. *Let M^m be a compact constant mean curvature, $|H|^2 = k$, submanifold in $\mathbb{S}^n$. Then M is proper biharmonic if and only if*

either

a) *$|H|^2 = 1$ and M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$,*

or

b) *$|H|^2 = k \in (0, 1)$ and M is a 2-type submanifold of $\mathbb{R}^{n+1}$ with the eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.*

Proof. We apply directly Theorem 2.4. Denote by $\phi : M \to \mathbb{S}^n$ the inclusion of M in $\mathbb{S}^n$ and by $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ the canonical inclusion. Let $\varphi : M \to \mathbb{R}^{n+1}$, $\varphi = \mathbf{i} \circ \phi$, be the inclusion of M in $\mathbb{R}^{n+1}$. Denote by H the mean curvature vector field of M in $\mathbb{S}^n$ and by H^0 the mean curvature vector field of M in $\mathbb{R}^{n+1}$.

The tension fields of the immersions ϕ and φ are related by

$$\tau(\varphi) = \tau(\phi) - m\varphi$$

and from here it follows that $H^0 = H - \varphi$.

Also, from Theorem 2.4, we get that $\tau_2(\phi) = 0$ if and only if

$$\Delta H^0 - 2mH^0 + m(|H|^2 - 1)\varphi = 0. \tag{3.1}$$

There are two situations to be analyzed.

If $|H|^2 = 1$, then $\Delta H^0 - 2mH^0 = 0$, and Theorem 2.16 implies that M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$.

If $|H|^2 = k \in (0,1)$, then equation (3.1) implies

$$\begin{aligned} 0 &= \Delta\Delta H^0 - 2m\Delta H^0 + m(k-1)\Delta\varphi \\ &= \Delta\Delta H^0 - 2m\Delta H^0 - m^2(k-1)H^0. \end{aligned}$$

The monic polynomial with positive distinct roots described in Theorem 2.16, which provides the type of the submanifold M, is

$$P(\Delta) = \Delta^2 - 2m\Delta^1 - m^2(k-1)\Delta^0,$$

so M is a 2-type submanifold with eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.

For the converse, let first M be a constant mean curvature $|H| = 1$ submanifold of $\mathbb{S}^n$. Suppose it is of 1−type with eigenvalue $\lambda = 2m$ in $\mathbb{R}^{n+1}$. This means that $\Delta\varphi = 2m\varphi$, and, by applying Δ, it implies $\Delta H^0 - 2mH^0 = 0$. From here we see that M satisfies equation (3.1), i.e. it is biharmonic in $\mathbb{S}^n$.

When $|H|^2 = k \in (0,1)$ and M is a 2−type submanifold in $\mathbb{R}^{n+1}$ with eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$ we have

$$\varphi = x_1 + x_2,$$

where $\Delta x_i = \lambda_i x_i$, $i = 1, 2$. Applying the Laplacian we obtain

$$H^0 = -\{x_1 + x_2 + \sqrt{k}(x_1 - x_2)\} = -\varphi - \sqrt{k}(x_1 - x_2)$$

and

$$\Delta H^0 = -m\{(k+1)\varphi + 2(-\varphi - H^0)\} = -m\{(k-1)\varphi - 2H^0\}.$$

Finally, using (3.1), M is biharmonic in $\mathbb{S}^n$. □

Remark 3.2. Note that, using Theorem 2.10, we can conclude that all proper biharmonic submanifolds of $\mathbb{S}^n$ with $|H| = 1$ are 1−type submanifolds in $\mathbb{R}^{n+1}$, independently on whether they are compact or not.

4. The classification of biharmonic hypersurfaces with at most two distinct principal curvatures in spheres

We recall that if M is a proper biharmonic umbilical hypersurface in $\mathbb{S}^{m+1}$, then it is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ and that there exist no proper biharmonic umbilical hypersurfaces in $\mathbb{R}^{m+1}$ or in the hyperbolic space $\mathbb{H}^{m+1}$.

Similarly to the case of the Euclidean space (see [8]), the study of proper biharmonic hypersurfaces with at most two distinct principal curvatures constitutes the next natural step for the classification of proper biharmonic hypersurfaces in space forms.

We underline the fact that there exist examples of hypersurfaces with at most two distinct principal curvatures and non-constant mean curvature in any space form. In the following we show that, by adding the hypothesis of biharmonicity, the mean curvature proves to be constant.

Theorem 4.1. *Let M be a hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$. If M is proper biharmonic in $\mathbb{E}^{m+1}(c)$, then it has constant mean curvature.*

Proof. If M is umbilical we immediately get to the conclusion.

For M non-umbilical, suppose that $|H|$ is not constant. This, together with the hypothesis for M to be proper biharmonic with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$, implies the existence of an open subset U of M, with

$$\begin{cases} \operatorname{grad}_p f \neq 0, \\ f(p) > 0, & \forall p \in U \\ k_1(p) \neq k_2(p), \\ m_1, m_2 \quad \text{constant}, \end{cases} \tag{4.1}$$

where, denoting by η the unit section in the normal bundle, f is the mean curvature function of U in $\mathbb{E}^{m+1}(c)$, i.e. $H = \frac{1}{m}(\operatorname{trace} A)\eta = f\eta$, and k_1, k_2 are the principal curvature functions w.r.t. η, with multiplicities m_1, m_2.

Under these hypotheses we shall prove that f is constant on U, contradicting the condition $\operatorname{grad}_p f \neq 0, \forall p \in U$.

Since M is proper biharmonic in $\mathbb{E}^{m+1}(c)$, from (2.2) we have

$$\begin{cases} \Delta f = (mc - |A|^2) f, \\ A(\operatorname{grad} f) = -\frac{m}{2} f \operatorname{grad} f. \end{cases} \tag{4.2}$$

Consider now $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ on U. Then X_1 is a principal direction with principal curvature $k_1 = -\frac{m}{2} f$. Suppose that there are m_1 principal directions of principal curvature k_1 and m_2 principal directions of principal curvature $k_2 \neq k_1$ and recall that $mf = m_1 k_1 + m_2 k_2$.

We shall use the moving frames method and denote by $X_1, \{X_i\}_{i=2}^{m_1}, \{X_\alpha\}_{\alpha=m_1+1}^{m}$ the orthonormal frame field of principal directions and by $\{\omega^a\}_{a=1}^{m}$ the dual frame field of $\{X_a\}_{a=1}^{m}$ on U.

Obviously,

$$X_i(f) = \langle X_i, \operatorname{grad} f\rangle = |\operatorname{grad} f| \langle X_i, X_1\rangle = 0, \qquad i = 2, \ldots, m_1$$

and analogously $X_\alpha(f) = 0$, $\alpha = m_1 + 1, \ldots, m$, thus

$$\operatorname{grad} f = X_1(f) X_1.$$

We write

$$\nabla X_a = \omega_a^b X_b, \qquad \omega_a^b \in C(T^*U).$$

From the Codazzi equations for M we get, for distinct $a, b, d = 1, \ldots, m$,

$$X_a(k_b) = (k_a - k_b)\omega_a^b(X_b) \tag{4.3}$$

and

$$(k_b - k_d)\omega_b^d(X_a) = (k_a - k_d)\omega_a^d(X_b). \tag{4.4}$$

We shall show, in the first place, that $m_1 = 1$.
Consider in equation (4.3), $a = 1$ and $b = i$. This leads to $X_1(k_1) = 0$, thus $|\operatorname{grad} f| = 0$ on U and we have a contradiction. From here it results that $m_1 = 1$, thus

$$k_2 = \frac{3m}{2(m-1)} f.$$

Consider now in (4.3), $a = 1$ and $b = \alpha$. We obtain

$$3X_1(f) = -(m+2)f\omega_1^\alpha(X_\alpha). \tag{4.5}$$

For $a = \alpha$ and $b = 1$, as $0 = X_\alpha(k_1)$, equation (4.3) leads to $\omega_1^\alpha(X_1) = 0$ and we can write

$$\omega_1^a(X_1) = 0, \quad \forall a = 1, \dots, m. \tag{4.6}$$

From (4.4), for $a = 1$, $b = \alpha$ and $d = \beta$, with $\alpha \neq \beta$, we get

$$\omega_1^\beta(X_\alpha) = 0, \qquad \forall \alpha \neq \beta. \tag{4.7}$$

We now compute

$$\begin{aligned} \Delta f &= -\mathrm{div}(\mathrm{grad}\, f) = -\langle \nabla_{X_1} \mathrm{grad}\, f, X_1\rangle - \sum_{\alpha=2}^{m} \langle \nabla_{X_\alpha} \mathrm{grad}\, f, X_\alpha\rangle \\ &= -X_1\big(X_1(f)\big) - X_1(f) \sum_{\alpha=2}^{m} \omega_1^\alpha(X_\alpha). \end{aligned} \tag{4.8}$$

By using (4.5) we get that

$$f\Delta f = -fX_1\big(X_1(f)\big) + \frac{3(m-1)}{m+2}\big(X_1(f)\big)^2. \tag{4.9}$$

As $|A|^2 = k_1^2 + (m-1)k_2^2 = \frac{m^2(m+8)}{4(m-1)} f^2$ and M is biharmonic,

$$\Delta f = (mc - |A|^2)f = \Big(mc - \frac{m^2(m+8)}{4(m-1)} f^2\Big) f,$$

and equation (4.9) becomes

$$fX_1\big(X_1(f)\big) - \frac{3(m-1)}{m+2}\big(X_1(f)\big)^2 - \frac{m^2(m+8)}{4(m-1)} f^4 + mcf^2 = 0. \tag{4.10}$$

We shall now use the Gauss and the Cartan structural equations in order to obtain other information on f. We have

$$d\omega_1^\alpha = -\sum_{a=1}^{m} \omega_1^a \wedge \omega_\alpha^a - (k_1k_2 + c)\omega^1 \wedge \omega^\alpha,$$

thus, using equations (4.6) and (4.7), we get

$$d\omega_1^\alpha(X_1, X_\alpha) = -k_1k_2 - c = \frac{3m^2}{4(m-1)} f^2 - c. \tag{4.11}$$

On the other hand from (4.6) and (4.7) we obtain $\omega_1^\alpha = \omega_1^\alpha(X_\alpha)\omega^\alpha$, thus (4.5) implies

$$3X_1(f)\omega^\alpha = -(m+2)f\omega_1^\alpha. \tag{4.12}$$

By differentiating (4.12) we obtain

$$3d\big(X_1(f)\big) \wedge \omega^\alpha + 3X_1(f)d\omega^\alpha = -(m+2)(df \wedge \omega_1^\alpha + fd\omega_1^\alpha). \tag{4.13}$$

Now, substituting

$$\begin{gathered} \big(dX_1(f) \wedge \omega^\alpha\big)(X_1, X_\alpha) = X_1\big(X_1(f)\big), \\ d\omega^\alpha(X_1, X_\alpha) = \omega_1^\alpha(X_\alpha), \\ (df \wedge \omega_1^\alpha)(X_1, X_\alpha) = X_1(f)\omega_1^\alpha(X_\alpha) \end{gathered}$$

in (4.13) and taking into account (4.11), we obtain

$$fX_1\big(X_1(f)\big) - \frac{m+5}{m+2}\big(X_1(f)\big)^2 + \frac{m^2(m+2)}{4(m-1)}f^4 - \frac{m+2}{3}cf^2 = 0. \tag{4.14}$$

Consider now an arbitrary integral curve γ of X_1 and denote by f' and f'' the first and the second derivatives of f along this curve. Equations (4.10) and (4.14) become, respectively,

$$ff'' - \frac{3(m-1)}{m+2}(f')^2 - \frac{m^2(m+8)}{4(m-1)}f^4 + mcf^2 = 0 \tag{4.15}$$

and

$$ff'' - \frac{m+5}{m+2}(f')^2 + \frac{m^2(m+2)}{4(m-1)}f^4 - \frac{m+2}{3}cf^2 = 0, \tag{4.16}$$

along γ.

Multiplying by $(m+5)$ equation (4.15) and by $-3(m-1)$ equation (4.16) and summing up, we get

$$(4-m)ff'' = \frac{m^2(m^2+4m+9)}{2(m-1)}f^4 - (m^2+3m-1)cf^2. \tag{4.17}$$

For $m=4$, equation (4.17) implies immediately that f is constant.
For $m \neq 4$, multiply equation (4.17) by f'/f, integrate the result and obtain

$$(f')^2 = \frac{m^2(m^2+4m+9)}{8(4-m)(m-1)}f^4 - \frac{(m^2+3m-1)}{2(4-m)}cf^2 + C. \tag{4.18}$$

On the other hand, multiplying by -1 equation (4.15) and adding it to equation (4.16) leads to

$$(f')^2 = \frac{m^2(m+5)(m+2)}{4(4-m)(m-1)}f^4 - \frac{(2m+1)(m+2)}{3(4-m)}cf^2 \tag{4.19}$$

From (4.18) and (4.19) we conclude that f is the solution of a polynomial equation, thus f is constant along γ. Since γ is an arbitrary integral curve for X_1 we have $X_1(f) = 0$ on U, thus f is constant. □

To strengthen the Generalized Chen's Conjecture, as an immediate consequence of Theorem 4.1, we have the following non-existence result.

Theorem 4.2. *There exist no proper biharmonic hypersurface with at most two distinct principal curvatures in* $\mathbb{H}^{m+1}$.

Proof. Suppose that M is a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{H}^{m+1}$. From Theorem 4.1, the mean curvature of M is constant, and applying Proposition 2.2 we obtain $|A|^2 = -m$ and we conclude. □

The case of the sphere is essentially different. Theorem 4.1 proves to be the main ingredient for the following complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures.

Theorem 4.3. *Let* M^m *be a proper biharmonic hypersurface with at most two distinct principal curvatures in* $\mathbb{S}^{m+1}$. *Then* M *is an open part of* $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ *or of* $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Proof. By Theorem 4.1, the mean curvature of M in $\mathbb{S}^{m+1}$ is constant and, by using Proposition 2.2, we obtain $|A|^2 = m$. These imply that M has constant principal curvatures.

For $|H|^2 = 1$ we conclude that M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.

For $|H|^2 \in (0, 1)$ we deduce that M has two distinct constant principal curvatures. Proposition 2.5 in [16] implies that M is an open part of the product of two spheres $\mathbb{S}^{m_1}(a) \times \mathbb{S}^{m_2}(b)$, such that $a^2 + b^2 = 1$, $m_1 + m_2 = m$. Since M is biharmonic in $\mathbb{S}^n$, from Theorem 2.8, it follows that $a = b = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$. □

Remark 4.4. Note that, for $m = 2$, we recover the result in Theorem 2.5 *b*).

We recall that a Riemannian manifold is called conformally flat if for every point it admits an open neighborhood conformally diffeomorphic to an open set of an Euclidean space. Also, a hypersurface $M^m \subset N^{m+1}$ which admits a principal curvature of multiplicity at least $m - 1$ is called quasi-umbilical.

Theorem 4.5. *Let M^m, $m \geq 3$, be a proper biharmonic hypersurface in $\mathbb{S}^{m+1}$. The following statements are equivalent*

a) *M is quasi-umbilical,*
b) *M is conformally flat,*
c) *M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m-1}(\frac{1}{\sqrt{2}})$.*

Proof. By Theorem 4.3 we get that *a*) is equivalent to *c*). Also, note that *c*) obviously implies *b*).

In order to prove that *b*) implies *a*), remind that, for $m \geq 4$, by a well-known result (see, for example, [4]), any conformally flat hypersurface of a space form is quasi-umbilical and we conclude.

For $m = 3$, as M is conformally flat, it results that the $(0, 2)-$tensor field $L = -\operatorname{Ricci} + \frac{s}{4}\langle\ ,\ \rangle$, where s is the scalar curvature of M, is a Codazzi tensor field, i.e.

$$(\nabla_X L)(Y, Z) = (\nabla_Y L)(X, Z), \quad \forall X, Y, Z \in C(TM). \tag{4.20}$$

Using the notations from the proof of Theorem 4.1, the Gauss equation implies

$$\operatorname{Ricci}(X, Y) = 2\langle X, Y\rangle + 3f\langle A(X), Y\rangle - \langle A(X), A(Y)\rangle$$

and

$$s = 6 + 9f^2 - |A|^2. \tag{4.21}$$

We use the same techniques as in the proof of Theorem 4.1. Suppose the existence of an open subset U of M with 3 distinct principal curvatures.

If f is constant on U, using the above expressions, we conclude that U is flat and that the product of any of its two principal curvatures is -1, thus we get to a contradiction.

Assume that f is not constant on U. We can suppose that $\operatorname{grad}_p f \neq 0$, $\forall p \in U$. Consider $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$. As M is proper biharmonic, X_1 gives a principal direction with principal curvature $k_1 = -\dfrac{3}{2}f$. From $k_1+k_2+k_3 = 3f$, we can write $k_2 = \dfrac{9}{4}f + \varepsilon$ and $k_3 = \dfrac{9}{4}f - \varepsilon$, $\varepsilon \in C^\infty(U)$. Using the Codazzi and Gauss equations and equations (4.20) and (4.21) we show that $f = a\varepsilon^5$, $a \in \mathbb{R}$, and combining all these relations we obtain that ε is a solution of a polynomial equation. Thus ε and f are constant.

Finally, it results that M has at most two distinct principal curvatures and we conclude. □

For what concerns proper biharmonic hypersurfaces with constant mean curvature in spheres we also have the following geometric property

Proposition 4.6. *Let M be a proper biharmonic hypersurface with constant mean curvature $|H|^2 = k$ in $\mathbb{S}^{m+1}$. Then M has constant scalar curvature,*

$$s = m^2(1+k) - 2m.$$

Proof. Since M is proper biharmonic of constant mean curvature, the squared norm of its second fundamental form is $|A|^2 = m$. By applying the Gauss equation, we conclude. □

In view of the above results we propose the following

Conjecture. *The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ and of generalized Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

5. Codimension 2 biharmonic pseudo-umbilical submanifolds in spheres

We shall first prove a general result concerning the mean curvature of biharmonic pseudo-umbilical submanifolds in spheres

Theorem 5.1. *Let M be a pseudo-umbilical submanifold of $\mathbb{S}^n$, $m \neq 4$. If M is biharmonic, then it has constant mean curvature.*

Proof. Consider $x \in M$ and let $\{X_i\}_{i=\overline{1,m}}$ be a local orthonormal frame field geodesic in x. As M is biharmonic, from (2.1), we get

$$\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) = -\frac{m}{4}\operatorname{grad}(|H|^2). \tag{5.1}$$

On the other hand, in x, by standard computations, we get

$$\begin{aligned}
\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) &= \sum_{i,j}\big\{X_i\langle\nabla^{\mathbb{S}^n}_{X_j}X_i, H\rangle - \langle\nabla^{\mathbb{S}^n}_{X_i}\nabla^{\mathbb{S}^n}_{X_j}X_i, H\rangle\big\}X_j,\\
\sum_{i,j}X_i\langle\nabla^{\mathbb{S}^n}_{X_j}X_i, H\rangle X_j &= \sum_i \nabla_{X_i}A_H(X_i),\\
\sum_{i,j}\langle\nabla^{\mathbb{S}^n}_{X_i}\nabla^{\mathbb{S}^n}_{X_j}X_i, H\rangle X_j &= \frac{m}{2}\operatorname{grad}(|H|^2).
\end{aligned}$$

Now, as M is pseudo-umbilical, $\sum_i \nabla_{X_i}A_H(X_i) = \operatorname{grad}(|H|^2)$, thus

$$\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) = \frac{2-m}{2}\operatorname{grad}(|H|^2). \tag{5.2}$$

By putting together expressions (5.1) and (5.2), we conclude. □

The first consequence of this result is an estimate for the scalar curvature of biharmonic pseudo-umbilical submanifolds in spheres.

Proposition 5.2. *Let M^m be a biharmonic pseudo-umbilical submanifold of $\mathbb{S}^n$, $m \neq 4$. Then its scalar curvature s satisfies*

$$s \leq 2m(m-1).$$

The equality holds if and only if M is open in $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.

Proof. From Theorem 2.13, the scalar curvature s of a pseudo-umbilical submanifold M of $\mathbb{S}^n$ satisfies $s \leq m(m-1)(1+|H|^2)$, and equality holds if and only if M is contained in an m-sphere of $\mathbb{S}^n$.

By using Theorem 5.1 and Theorem 2.10, as M is biharmonic, it follows that its constant mean curvature satisfies $|H| \in (0,1]$, and this completes the proof. □

For what concerns biharmonic pseudo-umbilical submanifolds of codimension two we obtain the following rigidity result.

Theorem 5.3. *Let M^m be a pseudo-umbilical submanifold of $\mathbb{S}^{m+2}$, $m \neq 4$. Then M is proper biharmonic if and only if it is minimal in $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$.*

Proof. From the hypotheses, using Theorem 5.1, we deduce that M has constant mean curvature. Now, by using Theorem 2.14, it follows that any such submanifold is either minimal in $\mathbb{S}^{m+2}$ or minimal in a hypersphere of $\mathbb{S}^{m+2}$. But M is proper biharmonic in $\mathbb{S}^{m+2}$ and, from Theorem 2.6, we conclude. □

Replace now the condition on M to be pseudo-umbilical with that of being a hypersurface of a hypersphere in $\mathbb{S}^{m+2}$.

Theorem 5.4. *Let M^m be a hypersurface of $\mathbb{S}^{m+1}(a) \subset \mathbb{S}^{m+2}$, $a \in (0,1)$. Assume that M is not minimal in $\mathbb{S}^{m+1}(a)$. Then it is biharmonic in $\mathbb{S}^{m+2}$ if and only if $a > \frac{1}{\sqrt{2}}$ and M is open in $\mathbb{S}^m(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+1}(a)$.*

Proof. Note that the converse follows immediately from Theorem 2.6.

In order to prove the other implication, denote by $\mathbf{j}$ and $\mathbf{i}$ the inclusion maps of M in $\mathbb{S}^{m+1}(a)$ and of $\mathbb{S}^{m+1}(a)$ in $\mathbb{S}^{m+2}$, respectively.

We consider

$$\mathbb{S}^{m+1}(a) = \Big\{(x^1, \ldots, x^{m+2}, \sqrt{1-a^2}) \in \mathbb{R}^{m+3} : \sum_{i=1}^{m+2} (x^i)^2 = a^2\Big\} \subset \mathbb{S}^{m+2}.$$

Then

$$C\big(T\mathbb{S}^{m+1}(a)\big) = \Big\{(X^1, \ldots, X^{m+2}, 0) \in C(T\mathbb{R}^{m+3}) : \sum_{i=1}^{m+2} x^i X^i = 0\Big\},$$

while

$$\eta = \frac{1}{c}\Big(x^1, \ldots, x^{m+2}, -\frac{a^2}{\sqrt{1-a^2}}\Big)$$

is a unit section in the normal bundle of $\mathbb{S}^{m+1}(a)$ in $\mathbb{S}^{m+2}$, where $c^2 = \frac{a^2}{1-a^2}$, $c > 0$.

By computing the tension and bitension fields of $\phi = \mathbf{i} \circ \mathbf{j}$, one gets

$$\tau(\phi) = \tau(\mathbf{j}) - \frac{m}{c}\eta,$$

and

$$\tau_2(\phi) = \tau_2(\mathbf{j}) - \frac{2m}{c^2}\tau(\mathbf{j}) + \frac{1}{c}\big\{|\tau(\mathbf{j})|^2 - \frac{m^2}{c^2}(c^2-1)\big\}\eta.$$

By the hypotheses M is biharmonic in $\mathbb{S}^{m+2}$, thus

$$|\tau(\mathbf{j})|^2 = \frac{m^2}{c^2}(c^2-1) = \frac{m^2}{a^2}(2a^2-1)$$

and, since $\tau(\mathbf{j}) \neq 0$, this implies $a > \frac{1}{\sqrt{2}}$.

Also,

$$|\tau(\phi)|^2 = |\tau(\mathbf{j})|^2 + \frac{m^2}{c^2} = m^2.$$

This implies that the mean curvature of M in $\mathbb{S}^{m+2}$ is 1, thus, using Theorem 2.10, M has to be a minimal submanifold of the hypersphere $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+2}$, i.e. it is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^{m+2}$.

Since $M \subset \mathbb{S}^{m+1}(a)$ is pseudo-umbilical in $\mathbb{S}^{m+2}$ it results pseudo-umbilical, and thus totally umbilical, in $\mathbb{S}^{m+1}(a)$. From here follows that M is an open subset of a hypersphere $\mathbb{S}^m(r)$ in $\mathbb{S}^{m+1}(a)$. But it is proper biharmonic in $\mathbb{S}^{m+2}$, thus $r = \frac{1}{\sqrt{2}}$ and we conclude. □

Corollary 5.5. *Let M be a proper biharmonic hypersurface of a hypersphere $\mathbb{S}^{m+1}(a)$ in $\mathbb{S}^{m+2}$, $a \in (0,1)$. Then $a \geq \frac{1}{\sqrt{2}}$. Moreover,*

a) *if $a = \frac{1}{\sqrt{2}}$, then M is minimal in $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$*

b) *if $a > \frac{1}{\sqrt{2}}$, then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.*

We also use Theorem 5.4 in order to prove

Theorem 5.6. *Let M^2 be a proper biharmonic surface with parallel mean curvature vector field in S^n. Then M is minimal in $S^{n-1}(\frac{1}{\sqrt{2}})$.*

Proof. B-Y. Chen and S-T. Yau proved (see [4, p.106]) that the only non-minimal surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ are either minimal surfaces of small hyperspheres $\mathbb{S}^{n-1}(a)$ of $\mathbb{S}^n$ or surfaces with constant mean curvature in 3−spheres of $\mathbb{S}^n$.

If M is a minimal surface of a small hypersphere $\mathbb{S}^{n-1}(a)$, then it is biharmonic in $\mathbb{S}^n$ if and only if $a = \frac{1}{\sqrt{2}}$ (see Theorem 2.6).

If M is a surface in a 3−sphere $\mathbb{S}^3(a)$, $a \in (0,1]$, of $\mathbb{S}^n$ then we can consider the composition

$$M \longrightarrow \mathbb{S}^3(a) \longrightarrow \mathbb{S}^4 \longrightarrow \mathbb{S}^n.$$

Note that M is biharmonic in $\mathbb{S}^n$ if and only if it is biharmonic in $\mathbb{S}^4$. From Theorem 5.4, for $a \in (0,1)$, we conclude that either $a = \frac{1}{\sqrt{2}}$ and M is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$, or $a > \frac{1}{\sqrt{2}}$ and M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$. For $a = 1$, from Theorem 2.5, also follows that M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$.

In all cases M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. □

Remark 5.7. All the results we have proved so far could suggest that the codimension 2 biharmonic submanifolds of $\mathbb{S}^n$ arise from minimal submanifolds of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. This is not the case as shown by the following

Theorem 5.8. [1]. *Let $\phi : M^3 \to \mathbb{S}^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $x_0 = \mathbf{i} \circ \phi = x_0(u,v,w)$ of M in $\mathbb{R}^6$ is given by*

$$x_0(u,v,w) = \tfrac{1}{\sqrt{2}} e^{iw}(e^{iu}, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v),$$

where $\mathbf{i} : \mathbb{S}^5 \to \mathbb{R}^6$ is the canonical inclusion.

Remind that if we consider a Sasakian manifold (N, Φ, ξ, η, g) and a submanifold M tangent to ξ, M is called *anti-invariant* if Φ maps any tangent vector to M which is normal to ξ to a vector which is normal to M. Also, a map $\phi : M \to \mathbb{S}^n$ is said to be *full* if the image $\phi(M)$ is contained in no hypersphere of $\mathbb{S}^n$.

Note that ϕ is a full proper biharmonic anti-invariant immersion from a $3-$dimensional torus into $\mathbb{S}^5$. The immersion ϕ has constant mean curvature, is not pseudo-umbilical and its mean curvature vector is not parallel in $\mathbb{S}^5$. In addition to these properties, since $|H| = 1/3$, from Theorem 3.1 we conclude that x_0 is a $2-$type submanifold of $\mathbb{R}^6$ with eigenvalues 2 and 4.

We also note that the product $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times M^m$, where M is a minimal non-totally geodesic hypersurface of $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$, is a full proper biharmonic submanifold of $\mathbb{S}^{m+3}$ of codimension 2.

Since all the known examples of proper biharmonic submanifolds in spheres have constant mean curvature we propose the following

Conjecture. *Any biharmonic submanifold in* $\mathbb{S}^n$ *has constant mean curvature.*

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