Rendiconti Seminario Facoltà Scienze Università Cagliari • Vol. 74, Fasc. 1-2 (2004)

Three-dimensional potentials producing families of straight lines (FSL)

GEORGE BOZIS (*), THOMAS A. KOTOULAS (*)

Abstract. *We identify a given two-parametric family of regular orbits given in the form* $f(x, y, z) = c_1$, $g(x, y, z) = c_2$ *in the 3-D Cartesian space by two functions* $\alpha(x, y, z)$ *and* $\beta(x, y, z)$ *including first order derivatives of f and g in x, y, z and defined by the system:* $dy/dx = \alpha$, $dz/dx = \beta$ *whose general solution is the family of orbits. Then, from the inverse-problem viewpoint, we find three necessary and sufficient differential conditions which the functions* $\alpha(x, y, z)$ *and* $\beta(x, y, z)$ *must satisfy when the given family* $\{\alpha, \beta\}$ *is indeed a two-parametric family of straight lines (FSL) and is actually created by a potential* $V(x, y, z)$*. From the direct-problem viewpoint, we establish two differential conditions which are satisfied by all genuine 3-D potentials producing two-parametric (FSL). Some pertinent theorems are shown and certain examples are worked out.*

Riassunto. *In un sistema di riferimento inerziale, individuato da una terna cartesiana ortogonale Oxyz, sia assegnata una famiglia a due parametri di orbite regolari di equazioni* $f(x, y, z) = c_1$, $g(x, y, z) = c_2$*. Si dimostra che tale famiglia può essere identificata con due funzioni* $\alpha(x, y, z)$ *e* $\beta(x, y, z)$*, dipendenti dalle derivate parziali prime di f e g rispetto ad x, y, z e definite dal sistema di equazioni differenziali:* $dy/dx = \alpha$, $dz/dx = \beta$ *la cui soluzione generale è la data famiglia di orbite. Ponendosi poi dal punto di vista del problema inverso della Dinamica, si stabiliscono tre condizioni differenziali necessarie e sufficienti a cui devono soddisfare le funzioni* $\alpha(x, y, z)$ *e* $\beta(x, y, z)$ *affinché la data famiglia di curve sia una famiglia di orbite rettilinee (FSL) creata da un potenziale* $V(x, y, z)$*. Si stabiliscono infine due condizioni differenziali, utili dal punto di vista del problema diretto della Dinamica, che sono soddisfatte da tutti i genuini potenziali 3-dimensionali capaci di produrre come orbite le rette della data famiglia 2-parametrica (FSL). Si dimostrano inoltre alcuni teoremi pertinenti e si danno alcuni esempi.*

Key words*: Potentials, trajectories, Inverse Problem of Dynamics, Mechanics.*

(*) University of Thessaloniki, Department of Physics, Section of Astrophysics, Astronomy and Mechanics GR-541 24 Thessaloniki, Greece. e-mail: tkoto@skiathos.physics.auth.gr

Presentato il 13/12/2004.

1. INTRODUCTION

Isolated straight lines produced by two-dimensional potentials $V = V(x, y)$ in inertial frames *Oxy* have been found e.g. by Antonov and Timoshkova (1993), with the aid of Szebehely's PDE (1974), by Van de Merwe (1991) with the aid of Rajaraman's integro-differential equation (1979), by Contopoulos and Zikides (1980), by Caranicolas and Innanen (1992) and, for three-dimensional potentials, by Caranicolas (1994). Such isolated straight lines were used also in the reasoning of Yoshida's theorem (1987) concerning nonintegrability of homogeneous 2-D potentials.

In the framework of the inverse problem of Dynamics, planar *monoparametric families* of orbits

$$f(x, y) = c \tag{1}$$

are identified as the general solution of an ordinary first order D.E.

$$\frac{dy}{dx} = -\frac{1}{\gamma(x, y)},$$

where «the slope function» $\gamma(x, y)$ is defined by

$$\gamma(x, y) = \frac{f_y}{f_x} \tag{2}$$

In particular for families g(*x*, *y*) of straight lines (*FSL*) it is

$$\gamma\gamma_x - \gamma_y = 0 \tag{3}$$

Such families have been studied by Bozis and Anisiu (2001) as an exceptional case of the general planar inverse problem. The main findings of this study are:

- (i) Only potentials $V = V(x, y)$ satisfying the differential condition

$$V_x V_y (V_{xx} - V_{yy}) = V_{xy}(V_x^2 - V_y^2) \tag{4}$$

can create *FSL* among the other orbits.

- (ii) To any genuinely 2-D potential, solution of the PDE (4), there corresponds the unique *FSL* defined by the slope function

$$\gamma(x, y) = -\frac{V_x}{V_y} \tag{5}$$

- (iii) If $V = u(x, y)$ is one particular solution of (4), then all functions $V(x, y) = F(u)$, with F arbitrary C^2-function, are also solutions of (4) and they create the same *FSL* which $u(x, y)$ creates.
- (iv) Given a *FSL* $\gamma(x, y)$ which satisfies (3), there exist, according to (5), infinitely many potentials which can create $\gamma(x, y)$.

Certain classes of solutions of the equation (4) (e.g. separable in Cartesian or in polar coordinates, homogeneous etc.) were studied and several *FSL* were found.

In the present paper we address the question of finding compatible pairs of *FSL* and potentials $V = V(x, y, z)$ in a three-dimensional inertial Cartesian *Oxyz* frame.

In Section 2 we give a *new* account of the basic facts regarding the three-dimensional inverse problem. We introduce two functions $\alpha = \alpha(x, y, z)$ and $\beta = \beta(x, y, z)$ which can represent uniquely the two-parametric family of spatial curves (6) in a manner analogous to that of the function $\gamma(x, y)$ representing the monoparametric family (1).

In Section 3 we derive two differential conditions which the pair (α, β) has to satisfy in order to represent a spatial *FSL* and an additional condition so that this *FSL* can be created by a potential $V(x, y, z)$.

In Section 4 we find certain easily detected pairs of *FSL* and potentials and we comment on the meaning of complex potentials.

In Section 5 we look at the problem from the direct viewpoint and we derive two conditions which all genuine 3-D potentials $V(x, y, z)$ producing *FSL* have to satisfy. We then compare them with the two-dimensional findings.

In Section 6 we present an example of a compatible *FSL* and a potential and a counterexample of a *FSL* which is not produced by a potential.

Finally, Section 7 is devoted to certain comments regarding this work as a whole.

2. THE INVERSE PROBLEM IN THREE DIMENSIONS

After Newton’s era, Darboux, Dainelli, Suslov, Joukovski were some of the authors who studied various versions of the inverse problem in Dynamics. An account of the pertinent references may be found in Shorokov (1988) and in Ramirez and Sadovskaia (1996).

Following Szebehely (1974), the general three-dimensional version of the inverse problem for a monoparametric family of spatial orbits was first studied by Érdi (1982), then, for a two-parametric family, by Bozis (1983), by Váradi and Érdi (1983) and later, from a different viewpoint, by Bozis and Nakhla (1986), by Shorokov (1988) and by Puel (1992). At almost the same period the group at Cagliari (Melis and Piras 1982, 1985, Melis and Borghero 1986, Borghero 1987 and Borghero and Melis 1990) generalized Szebehely’s equation to account for holonomic systems with n degrees of freedom.

The version of the inverse problem studied in the present paper is the following: In the inertial frame *Oxyz*, let $V = V(x ,y, z)$ be a potential which gives rise to the preassigned two-parametric family of regular orbits given in the form

$$f(x, y, z) = c_1,\ g(x, y, z) = c_2 \tag{6}$$

and traced by a material point $P(x, y, z)$ of unit mass with total energy

$$E = E(f, g) \tag{7}$$

which is constant on each member (c_1, c_2) of the family (6).

Remark 1: Two-parametric families given e.g. in the form $\phi_1(x, y, z, c_1) = c_2$, $\phi_2(x, y, z) = 0$ are essentially *two-dimensional* entities (a set of curves on one specific surface) and are not considered in the present study. A version of this problem was studied by Mertens (1981) and by Bozis and Mertens (1985).

For $i = 1, 2, 3$, let δ_i and Δ_i be respectively the components of the vectors

$$\vec{\delta} = \nabla f \times \nabla g \text{ and } \vec{\Delta} = \vec{\delta} \times \vec{a} \tag{8}$$

where $\vec{a}$ is the vector with components

$$a_i = \vec{\delta} \cdot \nabla \delta_i. \tag{9}$$

Before proceeding more, we note here that

Remark 2: The transformation $x \to y \to z \to x$ brings $\delta_1 \to \delta_2 \to \delta_3 \to \delta_1$.

Remark 3: The transformation: $f \to g \to f$ brings the triplet $(\delta_1, \delta_2, \delta_3)$ to $(-\delta_1, -\delta_2, -\delta_3)$ and leaves unaltered the ratios (11) below, as it should.

Two basic facts for this problem are (e.g. Bozis and Nakhla, 1986):

- (i) The potential satisfies the following linear in $V(x, y, z)$ PDE:

$$\delta_2 V_x - \delta_1 V_y = \frac{2\Delta_3}{|\vec{\delta}|^2}(E - V) \quad \text{and} \quad \delta_1 V_z - \delta_3 V_x = \frac{2\Delta_2}{|\vec{\delta}|^2}(E - V) \tag{10}$$

- (ii) The necessary and sufficient requirements for the above system (10) to be compatible lead to a linear system of PDEs in $E = E(f, g)$ with coefficients depending on the orbital data. As these equations in the unique unknown function E must be compatible themselves, we conclude that: in general, a preassigned family (6) does not result from an autonomous potential $V(x, y, z)$ unless certain conditions are fulfilled by the «given» functions $f(x, y, z)$, $g(x, y, z)$.

We mentioned in the *Introduction* that, for the planar family (1), the ratio $f_y/f_x = \gamma(x, y)$ specifies the monoparametric family (1). Accordingly, we show here that the two-parametric families (6) in the *Oxyz* space may be identified by the two ratios

$$\alpha(x,y,z) = \frac{\delta_2}{\delta_1} \quad \text{and} \quad \beta(x,y,z) = \frac{\delta_3}{\delta_1} \tag{11}$$

This is meant in the following sense: For all one dimensional curves

$$\vec{r} = x\vec{i} + y(x)\vec{j} + z(x)\vec{k} \tag{12}$$

($\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along the perpendicular axes *Ox*, *Oy*, *Oz*) parametrized by the ordinate x and defined by (6), we have

$$f_x + f_y y' + f_z z' = 0 \text{ and } g_x + g_y y' + g_z z' = 0 \tag{13}$$

where primes denote differentation with respect to x. According to (8) and (11), we obtain from (13)

$$y' = \alpha(x, y, z),\ z' = \beta(x, y, z) \tag{14}$$

The general solution of the system (14) of the two ordinary D.E. in the two unknown functions $y = y(x)$ and $z = z(x)$ includes two arbitrary constants c_1 and c_2 and, by its very structure, is given by the two equations (6).

Remark 4: Instead of the equations (6), let us represent the two-parametric family by (what apparently is equivalent) the equations $F(f, g) = c_1$, $G(f, g) = c_2$ with

$$\frac{\partial(F,G)}{\partial(f,g)} \neq 0.$$

If we calculate now the triplet $(\delta_1, \delta_2, \delta_3)$ from (8) and insert into (11), we see that the functions $\alpha(x, y, z)$ and $\beta(x, y, z)$ remain unaltered, as they should.

Remark 5: At any point $P(x, y, z)$, the tangent of the orbit $\vec{r} = \vec{r}(x)$ passing through P has the direction of the vector $\{1, y', z'\}_P = \{1, \alpha, \beta\}_P$.

Remark 6: The regularity (already assumed) for the curves (6) implies the nonzeroing of at least one of the functions δ_1, δ_2, δ_3. Without loss of generality (and in view of the above *Remark 2*) we can take $\delta_1 \neq 0$, so that the ratios (11) may be defined. In addition to that and because in this study we are searching for genuinely 3-D potentials $V(x, y, z)$, *we assume* that $\delta_1 \delta_2 \delta_3 \neq 0$, i.e. we assume that both α and β are not zero (see also *Remark 7* in Section 3).

3. SZEBEHELY'S EQUATIONS FOR STRAIGHT LINES IN 3-D SPACE AND CONDITIONS FOR THEM

We prove first that, if the equations (6) represent straight lines, the r.h.s. of both equations (10) vanish. Indeed, for any selection of the pair (c_1, c_2), a curve (6) may be thought of as given by (12). Let us then consider that (12) represents a straight line, in which case, its curvature

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

must vanish. This leads to $(y'z'' - y''z')\,\vec{i} - z''\,\vec{j} + y''\,\vec{k} = 0$ which implies that $y'' = 0$, $z'' = 0$ and, in view of (14),

$$\alpha_x + \alpha\alpha_y + \beta\alpha_z = 0 \text{ and } \beta_x + \alpha\beta_y + \beta\beta_z = 0. \tag{15}$$

The equations (15) are the necessary and sufficient conditions which the «slope functions» $\alpha(x, y, z)$, $\beta(x, y, z)$ must satisfy so that the equations (6) represent a set of straight lines. In view of the notations introduced in the previous Section 2 and after some straightforward algebra it can be shown that each of the two equations (15) implies the vanishing of the corresponding numerator of the fraction appearing in each of the two equations (10).

Indeed, from the second of equations (8), we have $\Delta_3 = \delta_1 a_2 - \delta_2 a_1$, which, in view of (9), is written as

$$\Delta_3 = \delta_1^2 \left\{ \delta_1 \left(\frac{\delta_2}{\delta_1} \right)_x + \delta_2 \left(\frac{\delta_2}{\delta_1} \right)_y + \delta_3 \left(\frac{\delta_2}{\delta_1} \right)_z \right\} \tag{16}$$

and, in view of (11), as

$$\Delta_3 = \delta^3{}_1(\alpha_x + \alpha\alpha_y + \beta\alpha_z). \tag{17}$$

Thus, because of the first of (15),

(18) $$\Delta_3 = 0.$$

In a similar manner one can show that

(19) $$\Delta_2 = 0.$$

Therefore, for families of straight lines (*FSL*), Szebehely's equations (10), according to (15), reduce to

(20) $$\alpha V_x - V_y = 0, \text{ and } \beta V_x - V_z = 0.$$

or, in view of (11), to the system

(21) $$V_x = \delta_1,\ V_y = \delta_2,\ V_z = \delta_3,$$

which is compatible if

(22) $$rot\vec{\delta} = \vec{0}$$

The total energy E no longer appears in (20) and, in this sense, we are in front of a special case of the inverse problem: In order to find the potential $V(x, y, z)$, we need only give in advance the *FSL* and not the energy dependence (7).

Remark 7: As stated in *Remark 6* of Section 2, we keep assuming that $\alpha\beta \neq 0$. For a *FSL* with e.g. $\beta = 0$, equation (20b) implies that $V_z = 0$, i.e. the potential $V = V(x, y)$ is two-dimensional. Equation (15b) is satisfied identically, whereas (15a) gives $\alpha_x + \alpha\alpha_y = 0$. Combining this last result with (20a) we reobtain (4). As expected, potentials $V(x, y)$ generating *FSL* in the *Oxy* plane also allow for such families lying on planes parallel to *Oxy*.

Remark 8: For families given by the equations (6) the compatibility condition (22) is tested directly, with the aid of (8a). For families given by the pair (α, β) (i.e. a pair satisfying (15)), it is not sure that there exists indeed a potential $V(x, y, z)$ which creates this *FSL*; because, now, *two* equations (i.e. equations (20)) have to be satisfied by the unique unknown function $V(x, y, z)$. All these statements are, of course, in agreement with the basic fact (ii) of Section 2.

Having in mind the previous *Remark 8* let us ask: «*Given an appropriate pair* (α, β) *which satisfies the equations (15), is there a potential satisfying the two equations (20)?*».

For an affirmative answer to this question, a *third* linear PDE in $V(x, y, z)$ must by necessity be true (Favard 1963, Smirnov 1964 or the equation (6) of the article by Bozis and Nakhla, 1986). For the case at hand, this third PDE reduces to: $V_x(\alpha\beta_x - \beta\alpha_x - \beta_y + \alpha_z)$

= 0 and consequently, since $V_x \neq 0$, to

$$\alpha\beta_x - \beta\alpha_x = \beta_y - \alpha_z \tag{23}$$

This is a new restriction for the «given» pair (α, β), *to be added* to the two conditions (15a, b). In conclusion, if the family is given in the form (6) and satisfies (22), the potential can be found also from (21). If, more generally (as we assume in this study), the family is given by the pair {α, β} and satisfies the conditions (15a, b) and (23), the potential is found from the two (compatible) equations (20).

4. CERTAIN COMPATIBLE PAIRS OF FSL AND POTENTIALS

Two easily detected cases of pairs (α, β) satisfying the three PDEs (15a, b) and (23) are the following:

- *Case* (a): $\alpha = \alpha_0$, $\beta = \beta_0$ (α_0, β_0 not zero constants): As can be seen from the system (14), these constants lead to a two-parametric set of parallel straight lines as the intersections of the two sets of planes

$$f(x, y, z) = \alpha_0 x - y = c_1, \quad g(x, y, z) = \beta_0 x - z = c_2. \tag{24}$$

In view of the *Remark 4* of Section 2, the very same *FSL* (24) may be represented as the intersection of the two sets of cylindrical surfaces

$$F(\alpha_0 x - y, \beta_0 x - z) = c_1, \;\; G(\alpha_0 x - y, \beta_0 x - z) = c_2 \tag{25}$$

with generatrices parallel to the same direction $\{1, \alpha_0, \beta_0\}$.

The potentials which create the above *FSL* are found either from (20) or (because from (8a) we have $\vec{\delta} = \vec{i} + \alpha_0 \vec{j} + \beta_0 \vec{k}$) from (21). So or otherwise there results

$$V(x, y, z) = A(x + \alpha_0 y + \beta_0 z), \tag{26}$$

where A = arbitrary function of its argument.

- *Case* (b): $\alpha = y/x$, $\beta = z/x$: It corresponds to a *FSL* passing through the origin *O* and is represented by the intersection of the planes

$$\frac{y}{x} = c_1, \quad \frac{z}{x} = c_2 \tag{27}$$

or as the intersection of the conic surfaces

(28) $$F\left(\frac{y}{x},\frac{z}{x}\right)=c_1,\quad G\left(\frac{y}{x},\frac{z}{x}\right)=c_2.$$

According to (20), the corresponding (central) potential is:

(29) $$V(x, y, z) = A(x^2 + y^2 + z^2),$$

where A = arbitrary function of its argument.

Remark 9: Generalizing slightly the *Case* (a), let us examine if *linear expressions* in x, y, z for the functions $\alpha(x, y, z)$ and $\beta(x, y, z)$ may be good choices to represent a *FSL* produced by a potential. In treating the three equations (15a,b) and (23) with the aid of *MATHEMATICA*, we come to understand that, except for α = const., β = const., no other *real* solution exists. There exist, however *complex* slope functions α, β associated with 3D *complex potentials*. As an example consider the pair

(30) $$\alpha = 1 + z + iy,\ \ \beta = y - i(1 + z)$$

corresponding to the potential

(31) $$V = 2(x + y + yz) + i(y^2 - 2z - z^2).$$

Two-dimensional complex potentials are conceived as formal mathematical entities (Contopoulos and Bozis, 2000) and they are met as such in the literature referring to the problem of integrability (Hietarinta, 1984, Ramani *et al*., 1982) and to the inverse problem (Bozis and Grigoriadou, 1993).

Remark 10: In generalizing the *Case* (b), we easily show that the only functions of the form $\alpha = \alpha(y/x)$, $\beta = \beta(z/x)$ which satisfy (15a,b) and (23) are those given in *Cases* (a) and (b), i.e. either $\alpha = \alpha_0$, $\beta = \beta_0$ or $\alpha = y/x$, $\beta = z/x$.

5. POTENTIALS ADMITTING TWO-PARAMETRIC FSL

We now approach the problem from the direct point of view: We give an «appropriate» genuine 3-D potential $V(x, y, z)$ and we ask for the *FSL* which this potential generates. In terms of α and β, the answer is given directly from formulae (20). To find the family in the form (6), we need solve the system (14) which is feasible because the potential is «appropriate». There remains the question: Which potentials are *appropriate*?

Solving equations (20) for $\alpha = V_y/V_x$ and $\beta = V_z/V_x$ and inserting into (15a, b), we obtain correspondingly the two equations:

$$(32)\qquad \begin{aligned} &V_{xy}(V^2_x - V^2_y) - V_x V_y (V_{xx} - V_{yy}) + V_z(V_x V_{yz} - V_y V_{xz}) = 0 \\ &V_{xz}(V^2_z - V^2_x) + V_x V_z (V_{xx} - \mathrm{V}_{zz}) + V_y(V_z V_{xy} - V_x V_{yz}) = 0. \end{aligned}$$

We observe also that (23) becomes an identity. Therefore, we state the following.

Theorem 1: The two conditions (32a, b) are necessary and sufficient so that a potential $V = V(x, y, z)$ can produce a two-parametric *FSL*, given by the pair (α, β) in eqs. (20).

The following theorem is analogous to that reported in Section 1 (iii) and is valid for the spatial case too:

Theorem 2: If $V = u(x, y, z)$ is one particular solution of the system of PDEs (32a, b), then all functions $V = F(u(x, y, z))$, with F arbitrary C^2-function are also solutions of (32a, b).

This theorem is shown easily by direct computations. In view of this theorem and of the two Cases treated in Section 4, we can verify that both equations (32a, b), as expected, are valid for the («linear») potentials (26) and for the (central) potentials (29). The corresponding *FSL* are (25) and (28) respectively.

Theorem 3: For a_i, b_i = constants (i = 1, 2, 3) and for potentials of the form

$$(33)\qquad V(x, y, z) = \Phi(a, b),$$

where Φ is an arbitrary function of the two arguments

$$(34)\qquad a = a_1 x + a_2 y + a_3 z \quad \text{and} \quad b = b_1 x + b_2 y + b_3 z,$$

the two equations (32a, b) reduce to the *unique* equation

$$(35)\qquad (K\Phi_a + L\Phi_b)\Phi_b \Phi_{aa} + (M\Phi^2_b - K\Phi^2_a)\Phi_{ab} - (L\Phi_a + M\Phi_b)\Phi_a \Phi_{bb} = 0$$

where

$$(36)\qquad K = a^2_1 + a^2_2 + a^2_3, \quad L = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad M = b^2_1 + b^2_2 + b^2_3.$$

This is shown by direct computations (aided by *MATHEMATICA*): We prepare, in view of (33) and (34), the derivatives $V_x, V_y, ..., V_{zz}$ (e.g. $V_x = a_1 V_a + b_1 V_b$) and insert them both into (32a) and (32b). As a result, we obtain the unique second order nonlinear PDE (35) with (constant) coefficients given by (36). We note here that the equipotential surfaces of (33) are cylinders with generatrices parallel to the straight lines $a = b = 0$.

To pursue the solution of (35), let us introduce the variable $z = z\,(a, b)$ by

(37) $$\Phi_b = z\,\Phi_a$$

and express Φ_{ab}, Φ_{bb} in terms of Φ_a, Φ_{aa}. Inserting into (35) we obtain the PDE

(38) $$(K + Lz)z_a + (L + Mz)z_b = 0$$

whose general solution is

(39) $$(L + Mz)a - (K + Lz)b = A(z)$$

where $A(z)$ is arbitrary.

For any particular solution $z = z\,(a, b)$, resulting from (39) after selecting $A(z)$, the PDE (37) is solved to completion, provided that the ODE

$$\frac{db}{da} = -\frac{1}{z(a,b)}$$

can be solved by quadratures. Thus, e.g., for $A = 0$, from (39) we obtain

$$z = \frac{Kb - La}{Ma - Lb}$$

and, from (35),

(40) $$\Phi(a, b) = \mathfrak{B}(Kb^2 - 2Lab + Ma^2)$$

where $\mathfrak{B}$ is arbitrary function of its argument.

Remark 11: For

(41) $$K = L = M\ (\neq 0)$$

the general solution of (35) can be given with two arbitrary functions. Indeed, the general solution of (38) is $z = A(a - b)$, A = arbitrary and of (37) is:

$$\Phi(a, b) = G(b + H(a - b))$$

where G, H are arbitrary functions of their arguments. In view of (36) it can be shown that only complex sets of values of the constants a_1, a_2, a_3, b_1, b_2, b_3 exist to account for (41).

An indicative set of such values is: $a_1 = i$, $a_2 = -1$, $a_3 = 1$, $b_1 = -1$, $b_2 = -1$, $b_3 = i$.

6. EXAMPLES

The examples of this section serve to verify the formulae established in the paper and also to indicate how one can try to establish analytically compatible pairs of *FSL* (α, β) and potentials $V(x, y, z)$.

Example 1: For $\alpha = \alpha_0 = const \neq 0$, equation (15a) is satisfied identically, whereas equation (23) implies that

$$\beta = b(\phi, z) \tag{42}$$

where b is an arbitrary function of $\phi = x + \alpha_0 y$ and of z. We now try to specify b so that the condition (15b) is also satisfied. To this end, we must have

$$(1 + \alpha_0^2)b_\phi + bb_z = 0 \tag{43}$$

with solutions $b = b(\phi, z)$ given implicitly by the relation

$$b = B(b\phi - (1 + \alpha_0^2)z), \tag{44}$$

where B is an arbitrary function of the unique argument $w = b\phi - (1 + \alpha_0^2)z$. Selecting $B = w$ in (44), we get the *FSL*

$$\alpha = \alpha_0, \quad \beta = \frac{(1+\alpha_0^2)z}{x+\alpha_0 y-1}, \tag{45}$$

which, with the aid of (20), leads to the potential

$$V(x, y, z) = (x + \alpha_0 y - 1)^2 + (1 + \alpha_0^2)z^2 \tag{46}$$

and, of course (according to the *Theorem 2* of Section 5), to any function of (46).

Let us now select $B = \sqrt{2w}$ in (44). We obtain the two pairs of *FSL*

$$\alpha = \alpha_0, \quad \beta = [(x + \alpha_0 y) \pm \sqrt{(x + \alpha_0 y)^2 - 2(1 + \alpha_0^2)z}], \tag{47}$$

for which we know that potentials $V(x, y, z)$ creating these families do exist but we cannot find them analytically.

Example 2: As a counterexample, let us consider the positive parameters c_1 and c_2 and the (monoparametric) family of hyperboloids of one sheet

$$4x^2 + y^2 - z^2 = c_1. \tag{48}$$

It is known that, *for any definite* $c_1 > 0$, each surface (48) contains the (*monoparametric*) set of straight lines

$$2x + z = c_2(\sqrt{c_1} + y), \quad 2x - z = \frac{1}{c_2}(\sqrt{c_1} - y) \tag{49}$$

parametrized by $c_2 > 0$. If we free c_1 as a parameter, the *two-parametric FSL* (49) may be written in the form (6) as follows:

$$f(x,y,z) = 4x^2 + y^2 - z^2 = c_1, \quad g(x,y,z) = \frac{\sqrt{4x^2 + y^2 - z^2} - y}{2x - z} = c_2 \tag{50}$$

It is

$$\alpha = \frac{4xy + 2z\sqrt{4x^2 + y^2 - z^2}}{z^2 - y^2}, \quad \beta = \frac{4xz + 2y\sqrt{4x^2 + y^2 - z^2}}{z^2 - y^2}$$

and the conditions (15a, b) are fulfilled, meaning that, indeed, the equations (50) represent a *FSL*. However, the condition (23) *is not* satisfied and this is interpreted to mean that *no potential* $V(x, y, z)$ exists which can create the *FSL* (50).

Reamark 12: In view of (8a) we can check that the (vectorial) condition (22), as expected, is not satisfied for the family (50). However, this negative result gives us no information as to whether (50) is not a *FSL* or if perhaps (50) is a *FSL* but not traced by a potential.

7. CONCLUDING COMMENTS

• **(a)** The literature in the version of the three-dimensional inverse problem considered here (already reported in Section 2) uses the two families (6) to represent the orbital data. In the present study we replaced *f* and *g* by the functions α and β, given in (11), the merits of which are:

(i) Lower order partial derivatives enter into the calculations and into the pertinent formulae as e.g.: formulae (15a,b) and (23). As a result we have an appreciable simplification in direct problem considerations, i.e. when the *FSL* is demanded.

(ii) As the equations (14) imply, the functions α, β give the curvature of the orbits. For the present study of the *FSL*, this curvature

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

is zero; for any other families it is

$$\kappa = \frac{\sqrt{(\alpha\vec{\varepsilon}\cdot\nabla\beta - \beta\vec{\varepsilon}\cdot\nabla\alpha)^2 + (\vec{\varepsilon}\cdot\nabla\alpha)^2 + (\vec{\varepsilon}\cdot\nabla\beta)^2}}{(1+\alpha^2+\beta^2)^{3/2}} \tag{51}$$

where $\vec{\varepsilon} = \vec{i} + \alpha\,\vec{j} + \beta\,\vec{k}$ and the dot ($\cdot$) denotes the scalar product of two vectors.

(iii) Using α and β, we can write down the system of PDEs (20) in order to find the potential $V(x, y, z)$, even if we do not possess the family (6) itself. It suffices to have at our disposal the system of ODEs (14) which *may or may not be solvable* to give us its solution in the form (6).

(iv) The three conditions (15a, b) and (23) are more informative compared to the three (analytic) conditions implied by (22). This was explained in *Remark 12* of Section 6.

- **(b)** This study, as a whole, answers the following two questions:

(1) *Direct problem*: The three-dimensional potential $V = V(x, y, z)$, allowing for the creation of ∞^5 orbits is given. Is there, among these orbits, a two-parametric set of straight lines? (The answer is given by the conditions (32)). In case that the answer is affirmative, we obtain the functions $\alpha(x, y, z)$ and $\beta(x, y, z)$ from (20). If possible, we solve the system (14) to find the *FSL* in the form (6).

(2) *Inverse problem*: Either the family (6) or the first order system of ODEs (14) which is solvable or not in the two unknown functions $y = y(x)$ and $z = z(x)$ is given. Then, a two-parametric set of spatial orbits is at our disposal or hidden behind the given system. Do these orbits represent a *FSL*? (The answer is given by the conditions (15a, b)). If so, is this *FSL* generated by a potential? (The answer is given by the condition (23)). If so, then the pertinent potential is found from the system (20). We note here that the conditions (15a, b) are written as: $\vec{\varepsilon}\cdot\nabla\alpha = 0$, $\vec{\varepsilon}\cdot\nabla\beta = 0$ and the condition (21) as: $\vec{\varepsilon}\cdot(\nabla\times\vec{\varepsilon}) = 0$. So, if the vector ε satisfies these relations, the pair $\{\alpha, \beta\}$ represents a *FSL* created by a potential.

- **(c)** In order to *establish* compatible *FSL* (α, β) and potentials $V(x, y, z)$ we act as follows: Either (i) we find adequate pairs (α, β) satisfying the system (15a,b) and (23), in which case the equations (20) are also compatible and give the corresponding potential (inverse problem) or (ii) we find an appropriate potential satisfying the system (32a, b) in which case the same equations (20) offer the corresponding *FSL* (α, β) (direct problem).

So or otherwise, as we have more partial differential equations to be satisfied by less unknown functions, the problem generally is not expected to admit of a solution. Under these circumstances, it is usual to try to determine simpler functions to fit into a foreseen *form of solution*. This is e.g. what we did in the *Example 1* of Section 6. We may also use (successfully or not) the method of the determination of constants, as we do below.

• **(d)** For an indicative *numerical* check of the results of the present study, we worked with the potential (46) for

$$\alpha = \alpha_0 = 1, \quad \beta = \frac{2z}{x+y-1}.$$

For a particle P of unit mass and with initial conditions

(52) $$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1, \quad \dot{x}_0 = 1, \quad \dot{y}_0 = 1, \quad \dot{z}_0 = 1,$$

we integrated numerically the (linear) system

(53) $$\ddot{x} = -2(x+y-1), \quad \ddot{y} = -2(x+y-1), \quad \ddot{z} = -4z$$

and we found, indeed, a bounded rectilinear motion of total energy $E = (T + V)_P = 6$. The initial conditions for the velocity $\vec{u}_0(1, 1, 2)$ in (52) were taken so that $\vec{u}_0$ is parallel to the vector $\{1, \alpha, \beta\}$ calculated at the point $P(1\ ,1, 1)$. It is understood that any multiple of $\vec{u}_0$ could be used. The material point P would then trace the same straight line $y = x, z = 2x - 1$ with other values of the total energy

• **(e)** Linear as it is, the system (53) can be solved also analytically. For the initial conditions (52) it is

(54) $$x(t) = \frac{1}{2}(1+\cos(2t)+\sin(2t)), \quad y(t) = \frac{1}{2}(1+\cos(2t)+\sin(2t)),$$
$$z(t) = \cos(2t)+\sin(2t).$$

The linear oscillation (54) lies in the box

(55) $$\frac{1}{2}(1-\sqrt{2}) \le x, y \le \frac{1}{2}(1+\sqrt{2}), \quad -\sqrt{2} \le z \le \sqrt{2}.$$

Judging from the general solution of the system (53) we can tell that all rectilinear motions are oscillatory. We have also checked that there exist unbounded motions but not rectilinear.

• **(f)** If, instead of (46), and in view of the *Theorem 2* of Section 5, we had used the potential $\sqrt{V(x,y,z)}$, the (non-linear) system integrated numerically for the same initial conditions (52), would give the same rectilinear motion, now traced with total energy $E = 3 + \sqrt{3}$.

Acknowledgements

We thank Prof. Francesco Borghero at the University of Cagliari for critical reading of the manuscript and for writing the Abstract in Italian.

The work of Thomas A. Kotoulas was financially supported by the scientific program «EPEAEK II, PYTHAGORAS», No. 21878, of the Greek Ministry of Education and E.U.

REFERENCES

[1] Antonov V.A., Timoshkova E.I.: 1993, *Simple trajectories in a rotationally symmetric gravitational field*. Astron. Rep. **37**, (2), 138-144.

[2] Borghero F.: 1987, *Variational determination of the generalized Szebehely's equations*. Rend. Sem. Mat., Vol. **45**, 3, 125-135, Univers. Politecn. Torino.

[3] Borghero F., Melis A.: 1990, *On Szebehely's problem for holonomic systems involving generalized potential functions*. Celest. Mech. and Dyn. Astr. **49**, 273-284.

[4] Bozis G., Mertens R.: 1985, *On Szebehely's inverse problem for a particle describing orbits on a given surface*. ZAMM, **65**, 8, 383-384.

[5] Bozis G., Nakhla A.: 1986, *Solution of the three-dimensional inverse problem*. Celest. Mech., **38**, 357-375.

[6] Bozis G., Grigoriadou S.: 1993, *Families of planar orbits generated by homogeneous potentials*. Celest. Mech. and Dyn. Astr., **57**, 461-472.

[7] Bozis G., Anisiu M.C.: 2001, *Families of straight lines in planar potentials*. Rom. Astron. J., Vol. 11, No. 1, 27-43.

[8] Caranicolas N.D., Innanen K.A.: 1992, *Periodic motion in perturbed elliptic oscillators*. Astron. J., **103**, **4**, 1328-1312.

[9] Caranicolas N.D.: 1994, *1:1:1 resonant periodic orbits in 3-dimensional galactic-type Hamiltonians*. Astron. Astrophys., **282**, 34-36.

[10] Contopoulos G., Zikides M.: 1980, *Periodic orbits and ergodic components of a resonant dynamical system*. Astron. Astrophys., **90**, 198.

[11] Contopoulos G., Bozis G.: 2000, *Complex force fields and complex orbits*. J. Inv. Ill-Posed Problems, Vol. **8**, No. 2, 1-14.

[12] Erdi B.: 1982, *A generalization of Szebehely's equation for three dimensions*. Celest. Mech. **28**, 209-218.

[13] Favard J.: 1963, *Cours d'Analyse de l'École Polytechnique*, Tome 3^2, p. 39, Gauthier-Villars, Paris.

[14] Hietarinta J.: 1984, *New Integrable Hamiltonians with Transcedental Invariants*. Phys. Rev. Letters, **52**, No. 13, 1057.

[15] Melis A., Piras B.: 1982, *On a generalization of Szebehely's problem.* Rendiconti del Seminario della Facoltà di Scienze dell'Università di Cagliari, Vol. LII, (1) 73-78.

[16] Melis A., Piras B.: 1985, *An observation on the inverse extended problem of Szebehely*. Celest. Mech., **36**, 83-87.

[17] Melis A., Borghero F.: 1986, *On Szebehely's problem extended to holonomic systems with a given integral of motion*. Meccanica, **21**, 71-74.

[18] Mertens R.: 1981, *On Szebehely's equation for the potential energy of a particle describing orbits on a given surface*. ZAMM, **61**, pp. T252-T253.

[19] Puel F.:1992, *Explicit solutions of the three-dimensional inverse problem of Dynamics, using the Frenet reference frame*. Celest. Mech. and Dyn. Astron., **53**, 207-218.

[20] Rajaraman R.: 1979, *Solitons of Coupled Scalar Field Theories in Two Dimensions*. Physical Review Letters, Vol. **42**, No. **4**, 200-204.

[21] Ramani A., Dorizzi B., Grammaticos B.: 1982, *Painlevé Conjecture Revisited*. Phys. Rev. Letters, **49**, 1539-1541.

[22] Ramirez R., Sadovskaia N.: 1996, *On the construction of dynamic systems from given integrals*. Cel. Mech. and Dyn. Astr., **63**, 149-170.

[23] Shorokhov S.G.: 1988, *Solution of an inverse problem of the Dynamics of a particle*. Celest. Mech., **44**, 193-206.

[24] Smirnov V.I.: 1964, *A course of Higher Mathematics*. Vol. IV, p. 359, Pergamon Press, Oxford.

[25] Szebehely V.:1974, *On the determination of the potential by satellite observations*. In G. Proverbio (ed.), Proc. of the Int. Meeting on Earth's Rotation by Satellite Observation, The Univ. of Cagliari, Bologna, Italy 31-35.

[26] Váradi F., Érdi B.: 1983, *Existence of the solution of Szebehely's equation in three dimensions using a two-parametric family of orbits*. Celest. Mech., **32**, 395-405.

[27] van de Merwe P. du T.: 1991, *Solvable forms of a generalized Hénon-Heiles system*. Physics Letters A, Vol. **156**, No. **5**, 216-220.

[28] Yoshida H.: 1987, *A criterion for the non-existence of an additional integral in Hamiltonian systems with a homogeneous potential*. Physica D, **29**, 128.