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k-symplectic affine Lie algebras(*)

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Abstract. *The notion of k-symplectic structures was introduced by A. Awane in his dissertation in 1984 ([2], [14]). Here we are interested by the classification of Lie algebras provided with such a structure. We introduce also the notion of affine structure associated to a k-symplectic structure on a Lie algebra.*

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1. INTRODUCTION AND DEFINITIONS

Let G be a Lie group of dimension $n(k+1)$. A left invariant k-symplectic structure $(\theta^1, \dots, \theta^k; E)$, is given by an integrable left-invariant n-codimensional subbundle E of TG and $\theta^1, \dots, \theta^k$, left-invariant closed 2-forms on G vanishing on the cross sections of E with transversal characteristic spaces [6]. The left-invariance conditions show that we can define a such structure on the corresponding Lie algebras.

Definition 1.1. Let G be a $n(k+1)$-dimensional Lie algebra over K ($K = \mathbb{R}$ or $\mathbb{C}$), θ^1,

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$\dots, \theta^k$ closed 2-forms of $\Lambda^2(G)$ and H a Lie subalgebra of G of codimension n. We recall that $(\theta^1, \dots, \theta^k; H)$ is a k-symplectic structure on G if the following conditions are satisfied:

1. The system $(\theta^1, \dots, \theta^k)$ is non degenerated, that is

$$A(\theta^1) \cap \dots \cap A(\theta^k) = (0)$$

where $A(\theta^p)$ is the associated space to θ^p:

$$A(\theta^p) = \{X \in G \text{ such that } i(X)\,\theta^p = 0\}$$

2. H is a totally isotropic subspace of G relative to the system $\{\theta^1, \dots, \theta^k\}$, that is, $\theta^p(x, y) = 0$ for all $x, y \in H$ and $p = 1, \dots, k$.

We recall also, that G is an exact k-symplectic Lie algebra if in addition, the following properties are satisfied:

(i) H is an ideal of G,

(ii) the 2-forms defining the k-symplectic structure are exact: $\theta^1 = d\omega^1, \dots, \theta^k = d\omega^k$, where $\omega^1, \dots, \omega^k$ are independent linear forms on the Lie algebra G,

(iii) $G = H + (\ker \omega^1 \cap \dots \cap \ker \omega^k)$.

In this paper we complete the study presented in [2] on the k-symplectic Lie algebras and one gives some properties, existence theorems and classifications of the k-symplectic Lie algebras in 1-codimensional case.

In the second part of this work, we introduce and studie the left symmetric k-symplectic Lie algebras that is k-symplectic Lie algebras provided with an associated affine structure.

Let us remind here that the introduction of k-symplectic structures was led by the local study of Pfaffian systems and Nambu's statistical mechanics [2] and [6]. By the Heisenberg group of rank k in the sense of Goze-Haraguchi ([4] and [9]), we see that the k-symplectic geometry is related to the k-contact systems in analogy with the well known relationship between symplectic and contact structures. Let us note also that Kostant-Souriau's geometric prequantization is obtained in the context of these structures [14]. Also, we recall that the left symmetric algebras appeared for the first time, in the literature, in the works of E. Cartan, they were used in the bounded homogeneous domains by J.L. Koszul and in the convex homogeneous domains by E.B. Vinberg. The left symmetric algebras were the object of the thesis of D. Burde ([7], and M. Goze and E. Remm [11], [15]) studied these algebras with the operads point of view.

2. $(k + 1)$-DIMENSIONAL k-SYMPLECTIC LIE ALGEBRAS

We will call a Lie algebra G provided with a k-symplectic structure a k-symplectic Lie algebra. The general classification of k-symplectic Lie algebras in a given dimension is still hard (the simplest case corresponding to $k = 1$ is not completely solved [8] or [12]).

We propose in this section to classify real or complex k-symplectic Lie algebras when the dimension is equal to $k + 1$. Thus the codimension of the subalgebra H is equal to 1.

2.1. Case H abelian

Let G be a $n(k + 1)$-dimensional Lie algebra over $\mathbb{K}(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$, $\theta^1, \ldots, \theta^k$ closed 2-forms of $\Lambda^2(G)$ and H a Lie subalgebra of G of codimension n.

Theorem 2.1. Let G be a $(k+1)$-dimensional Lie algebra provided with a k-symplectic structure $(\theta^1, \ldots, \theta^k; H)$ with $\dim H = k$. If H is abelian then G is an extension by derivation of a k-abelian Lie algebra.

Proof. Let $(X_i)_{1 \leq i \leq k+1}$ be a basis of G and $(\omega^i)_{1 \leq i \leq k+1}$ its dual basis. We suppose that the subalgebra H is defined by the equation $\omega^{k+1} = 0$. In these conditions the Maurer-Cartan equations of G are written:

$$d\omega^p = \left(\sum_{q=1}^{k} a^p_q\, \omega^q\right) \wedge \omega^{k+1} \ (1 \leq p \leq k) \text{ and } d\omega^{k+1} = \left(\sum_{q=1}^{k} b_q\, \omega^q\right) \wedge \omega^{k+1}.$$

As the forms $\theta^1, \ldots, \theta^k$ vanish on H, then

$$\theta^p = \left(\sum_{q=1}^{k} A^p_q \omega^q\right) \wedge \omega^{k+1},\ p = 1, ..., k.$$

The exterior system $\{\theta^1, \ldots, \theta^k\}$ is nondegenerated which implies that the determinant $\det (A^p_q)_{1 \leq p,\, q \leq k}$ is non zero. As the 2-forms of the system are closed, we have

$$\left(\sum_{q=1}^{k} A^p_q \omega^q\right) \wedge d\omega^{k+1} = 0, \text{ for } p = 1, \ldots, k.$$

But $\det (A^p_q)_{1 \leq p,\, q \leq k} \neq 0$ implies that $d\omega^{k+1} = 0$. We have proved that the Maurer-Cartan equations of G are

$$(*)\ d\omega^p = \left(\sum_{q=1}^{k} a^p_q\, \omega^q\right) \wedge \omega^{k+1}, 1 \leq p \leq k,\ \ d\omega^{k+1} = 0$$

and the k-symplectic structure is given by

$$\theta^p = \left(\sum_{q=1}^{k} A^p{}_q \omega^q\right) \wedge \omega^{k+1}, \quad 1 \leq p \leq k, \ \text{ with } \ \det\,(A^p{}_q)_{1 \leq p,\, q \leq k} \neq 0$$

and H $= \ker \omega^{k+1}$. This proves that adX_{k+1} acts as an external derivation on H. □

Corollary 2.1. With the previous hypothesis, the subalgebra H is an abelian ideal of G.

Let f be a derivation of H of matrix $A = (a_{ij})$. Then the structural constants of the Lie algebra G defined as a 1-dimensional extension of H by the derivation f are given in (*).

On the algebra $M_k(\mathbb{K})$ of matrices of order k we define the equivalence relation:

$$A \,\Re\, B \Leftrightarrow \exists \alpha \in \mathbb{K}^*, \exists\, P \in GL_k(\mathbb{K}) \mid B = \alpha\, PAP^{-1}, \forall A, B \in M_k(\mathbb{K}).$$

For $A \in M_k(\mathbb{K})$ we note by G_A the Lie algebra defined by the structural equations:

$$\begin{pmatrix} d\omega^1 \\ \vdots \\ d\omega^k \end{pmatrix} = A \begin{pmatrix} \omega^1 \wedge \omega^{k+1} \\ \vdots \\ \omega^k \wedge \omega^{k+1} \end{pmatrix} = \begin{pmatrix} \sum_{q=1}^{k} a^1{}_q \omega^q \\ \cdots \\ \sum_{q=1}^{k} a^k{}_q \omega^q \end{pmatrix} \wedge \omega^{k+1} \text{ and } d\omega^{k+1} = 0.$$

Proposition 2.1. If $A \,\Re\, B$ then the Lie algebras G_A and G_B are isomorphic.

Proof. It is clear that if $B = \alpha A$ with $\alpha \in \mathbb{K}^*$, then G_A and G_B are isomorphic. It is sufficient to consider the change of basis: $X_i \to X_i$ $(1 \leq i \leq k)$ and $X_{k+1} \mapsto \alpha X_{k+1}$. If $B = PAP^{-1}$, with $P = (p_{ij})_{1 \leq i,j \leq k} \in GL_k(\mathbb{K})$, then the isomorphism whose matrix is

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of Lie algebras between G_A and G_B. □

The following proposition, which is the converse of theorem 2.1, ends the description of this family of k-symplectic Lie algebras.

Proposition 2.2. Every one-dimensional extension by derivation of a k-dimensional abelian Lie algebra H

$$0 \to \mathrm{H} \to \mathrm{G} \to \mathbb{K} \to 0$$

is provided with a k-symplectic structure.

Proof. Let f be in $Der(\mathrm{H})$ and G the corresponding extension of H . The Maurer-Cartan equations are given by (*). For every regular matix $M = (m_i^{\ j})$, the exterior 2-forms given by

$$\theta^p = (\sum_{q=1}^{k} m^p_{\ q}\omega^q) \wedge \omega^{k+1}, \quad p = 1,...,k,$$

and H = ker ω^{k+1} define a k-symplectic structure on G associated to H . In particular, if A, the matrix of f is nondegenerated we take $M = A$ and in this case the k-symplectic structure constructed on G is exact. □

Remarks

1. The Lie algebra G is nilpotent if and only if the matrix A is nilpotent. In fact, the matrix A is the matrix of the derivation associated to the one dimensional extension of the abelian lie algebra. The extension is nilpotent if and only if the derivation is nilpotent.

2. The classification of the $(k+1)$-dimensional k-symplectic abelian Lie algebras is reduced to the classification of the endomorphisms of $\mathbb{K}^n$.

2.2. Case H non-abelian

Let us begin by an example. Let G be the 4-dimensional Lie algebra whose Maurer-Cartan equations are:

$$\begin{cases} d\omega^1 = 0, \\ d\omega^p = \omega^1 \wedge \omega^p, \ p = 2,3 \\ d\omega^4 = -\omega^1 \wedge \omega^4 \end{cases}$$

From the previous calculus, this Lie algebra admits a 3-symplectic structure, the corresponding subalgebra H being abelian and generated by X_2, X_3, X_4. A such Lie algebra admits also another 3-symplectic structure given by $\theta_1 = = \omega^1 \wedge \omega^4$, $\theta_2 = = \omega^2 \wedge \omega^4$, $\theta_3 =$ $= \omega^3 \wedge \omega^4$ and H $= X_1, X_2, X_3$. In this case the subalgebra H is nonabelian. We will see that this example is generic.

Lemma 2.1. There exists a basis $\omega^1,\ldots,\omega^{k+1}$ of G* such that:

1. The k-symplectic structure of G is given by:

$$\theta^p = (\sum_{q=1}^{k} A^p_q\, \omega^q) \wedge \omega^{k+1}, \quad 1 \leq p \leq k,$$

with $\det (A^p_q)_{1 \leq p,\, q \leq k} \neq 0$ and H $= \mathit{ker}\, \omega^{k+1}$.

2. The Maurer-Cartan equations of G are:

$$\begin{cases} d\omega^p = (-\sum_{i=1}^{k} C^{k+1}_{ik+1}\omega^i) \wedge \omega^p + (\sum_{i=1}^{k} C^p_{ik+1}\omega^i) \wedge \omega^{k+1}, \text{ for } 1 \leq p \leq k \\ d\omega^{k+1} = (\sum_{i=1}^{k} C^{k+1}_{ik+1}\omega^i) \wedge \omega^{k+1} \end{cases}$$

with

(a) $\sum^{k}_{i=1,\, i \neq p} C^{k+1}_{ik+1}\, C^i_{jk+1} = 0$ if $j \neq p$,
(b) $C^{k+1}_{ik+1}\, C^p_{jk+1} - C^{k+1}_{ik+1}\, C^p_{ik+1} = 0$ if $i \neq p$ and $\partial \neq p$.

Proof. Let $(X_i)_{1 \leq i \leq k+1}$ be a basis of G and let $(\omega^i)_{1 \leq i \leq k+1}$ be the dual basis.
We suppose that the subalgebra H is defined by the equation $\omega^{k+1} = 0$. Then we have

$$d\omega^p = -\sum_{1 \leq i < j \leq k}^{k} C^p_{ij}\, \omega^i \wedge \omega^j - (\sum_{i=1}^{k} C^p_{ik+1}\, \omega^i) \wedge \omega^{k+1}$$

for $1 \leq p \leq k$ and

$$d\omega^{k+1} = -(\sum_{i=1}^{k} C^{k+1}_{ik+1}\, \omega^i) \wedge \omega^{k+1}.$$

As H is in the kernel of the 2-forms θ^p and as the exterior system $\theta^1,...,\theta^k$ is non-degenerated, we have:

$$\theta^p = (\sum_{q=1}^{k} A^p_q\, \omega^q) \wedge \omega^{k+1}, \; 1 \leq p \leq k, \text{ and } \det(A^p_q)_{1 \leq p,\, q \leq k} \neq 0.$$

At last let us prove that $C^l_{ij} = 0$ if $l \neq i$ and $l \neq j$ and $C^i_{ij} = C^{k+1}_{jk+1}$.

The 2-forms $\theta^p = (\sum_{i=1}^{k} A^p_i\, \omega^i) \wedge \omega^{k+1}$ are closed, then for all p we have:

$$d\theta^p = -\sum_{1\leq i<j\leq k}((\sum_{l=1}^{k} A^p_l C^l_{ij}) + (A^p_i C^{k+1}_{jk+1} - A^p_j C^{k+1}_{ik+1}))\omega^i \wedge \omega^j \wedge \omega^{k+1} = 0$$

which implies:

$$A^p_1\, C^1_{ij} + \ldots + A^p_i(C^i_{ij} - C^{k+1}_{jk+1}) + \ldots + A^p_j(C^j_{ij} + C^{k+1}_{ik+1}) + \ldots + A^p_k\, C^k_{ij} = 0$$

for all $1 \leq i < j \leq k$. But $\det(A^p_q) \neq 0$. Thus $C^l_{ij} = 0$ if $l \neq i$, $l \neq j$ and $C^i_{ij} = C^{k+1}_{jk+1}$.

These relations give:

$$\begin{cases} d\omega^p = -(\sum_{i=1}^{k} C^{k+1}_{ik+1}\omega^i) \wedge \omega^p - (\sum_{i=1}^{k} C^p_{ik+1}\omega^i) \wedge \omega^{k+1} \\ d\omega^{k+1} = -(\sum_{i=1}^{k} C^{k+1}_{ik+1}\omega^i) \wedge \omega^{k+1}. \end{cases}$$

The Jacobi relations imply:

(a) $(\sum_{i=1}^{k}{}_{i\neq p} C^{k+1}_{ik+1}C^i_{jk+1}) + C^{k+1}_{jk+1}C^p_{pk+1} = 0$ if $j \neq p$,

(b) $C^{k+1}_{ik+1}C^p_{jk+1} - C^{k+1}_{jk+1}C^p_{ik+1} = 0$ if $i \neq p$ and $j \neq p$.

As $C^p_{pk+1} = C^{k+1}_{k+1,k+1} = 0$, we have proved the lemma. □

Consequences

1. The subalgebra H is solvable (D^2(H) = {0}). If it is non-abelian then it is non nilpotent.

2. H is an ideal if and only if it is abelian.

In fact, from the previous lemma, we have:

$$[X_i, X_j] = C^{k+1}_{jk+1}\ X_i - C^{k+1}_{ik+1}\ X_j,\ \text{ for all } 1 \leq i < j \leq k$$

and H is solvable. Moreover the equations

$$[X_r, [X_i, X_j]] = C^{k+1}_{jk+1}[X_r, X_i] - C^{k+1}_{ik+1}[X_r, X_j] = -C^{k+1}_{rk+1}[X_i, X_j]$$

show that H is not nilpotent. □

Theorem 2.2. Let G be a real or complex $(k+1)$-dimensional Lie algebra provided with a k-symplectic structure $(\theta^1, \dots, \theta^k; H)$ where H is a non abelian subalgebra. Then G is isomorphic to $sl(2)$ or is a one dimensional extension by derivation of an abelian k-dimensional Lie algebra. In this last case, it admits another nonisomorphic k-symplectic structure whose associated subalgebra H_1 is abelian.

Proof. From the lemma there exists a basis $\{X_1,\dots, X_{k+1}\}$ of G satisfying:

$$\begin{cases} [X_i, X_j] = C^{k+1}_{jk+1}X_i - C^{k+1}_{ik+1}X_j, \quad 1 \le i < j \le k \\ [X_i, X_{k+1}] = \sum_{j=1}^{k} C^{j}_{ik+1}Xj + C^{k+1}_{ik+1}X_{k+1}, \quad 1 \le i \le k. \end{cases}$$

Let us put $a_i = C^{k+1}_{ik+1}$ and $b^j_i = C^j_{ik+1}$. As H is non-abelian, we can suppose that $a_1 \neq 0$. If we note

$$Y_1 = -\frac{X_1}{a_1} \text{and } Y_i = X_i + a_iY_1, \quad 2 \le i \le k,$$

then, in the basis $Y_1,\dots, Y_k, X_{k+1}\}$, the brackets of G are given by

$$\begin{cases} [Y_1, Y_i] = Y_i, \quad 2 \le i \le k \\ [Y_1, X_{k+1}] = \alpha^1_1 Y_1 + \dots + \alpha^k_1 Y_k - X_{k+1} \\ [Y_i, X_{k+1}] = \alpha^1_i Y_1 + \dots + \alpha^k_i Y_k + \alpha^{k+1}_i X_{k+1}, \quad 2 \le i \le k. \end{cases}$$

The matrix of ad_{Y_1} is written:

$$\begin{pmatrix} 0 & 0 & 0 & \alpha^1_{k+1} \\ & 1 & & \\ \vdots & \ddots & \vdots & \vdots \\ & & 1 & \alpha^k_{k+1} \\ 0 & & 0 & -1 \end{pmatrix}$$

Let Y_{k+1} be a nonzero eigenvector of ad_{Y_1} associated to the eigenvalue – 1. We have:

$$\begin{cases} [Y_1, Y_i] = Y_i, & 2 \le i \le k \\ [Y_1, Y_{k+1}] = -Y_{k+1} \end{cases}$$

and the Jacobi identities give $[Y_i, Y_{k+1}] = \alpha_i Y_1$, $2 \le i \le k$. If the dimension of G is equal to 3, then we have the two possibilities $a_2 \ne 0$ or $a_2 = 0$. The first case corresponds to *sl*(2). If the dimension is greater than 3, then the Jacobi conditions imply that the constants α_i are null, then G contains a 1-codimension abelian ideal and H is an abelian subalgebra.

This case concerns the first part of this study. But the associated *k*-symplectic structure is not isomorphic to the given one. □

REMARK: a 2-symplectic structure on *sl*(2)

We have proved that *sl*(2) admits a 2-symplectic structure. We describe briefly this structure. Let $\{X_1, X_2, X_3\}$ the classical basis of *sl*(2), that is satisfying

$$[X_1, X_2] = -2X_2, \ [X_1, X_3] = 2X_3, \ [X_2, X_3] = X_1.$$

Let $\omega_1, \omega_2, \omega_3$ the dual basis. The two-forms

$$\theta_1 = \omega_2 \wedge \omega_1, \ \theta_2 = \omega_2 \wedge \omega_3$$

are closed and define a 2-symplectic structure with H generated by X_1, X_3 which is not abelian.

2.3. Classification in dimension 3

Let G be a real or complex 3-dimensional 2-symplectic Lie algebra and let $(\theta^1, \theta^2; H)$ be its 2-symplectic structure. If H is an abelian subalgebra then, from the previous results, it is an abelian ideal and G is solvable. But every 3-dimensional solvable Lie algebra admits a such ideal. Then every 3-dimensional solvable Lie algebra admits a 2-symplectic structure. If H is not abelian, from the previous theorem G is isomorphic to *sl*(2).

PROPOSITION 2.3. Every 3-dimensional complex Lie algebra admits a 2-symplectic structure. Every 3-dimensional real Lie algebra non isomorphic to *so*(3) is provided with a 2-symplectic structure.

We can note that it does not exist classical symplectic structure on semi-simple Lie algebra. On other hand $sl(2, \mathbb{R})$ is provided with a 2-symplectic structure but it is the only one which is *n*-dimensional and endowed to a $(n-1)$-symplectic structure.

2.4. Classification in dimension 4

If a 4-dimensional Lie algebra is provided with a 3-symplectic structure then G is solvable. From the classification of Levy-Nahas ([18]), such an algebra is isomorphic to one of the following one:

1. $(G_1)^4$ the abelian algebra
2. $(G_1)^2 \times G_2$ defined by: $[X_3, X_4] = X_4$
3. $G_1 \times G_{3,1}$ defined by: $[X_2, X_3] = X_4$
4. $G_1 \times G_{3,2}(\alpha)$ defined by: $[X_2, X_3] = X_3$ and $[X_2, X_4] = \alpha X_4$, with $|\alpha| \geq 1$
5. $G_1 \times G_{3,3}$ defined by: $[X_2, X_3] = X_3 + X_4$ and $[X_2, X_4] = X_4$
6. $G_1 \times G_{3,4}(\alpha)$ defined by: $[X_2, X_3] = \alpha X_3 - X_4$ and $[X_2, X_4] = X_3 + \alpha X_4$, with $\alpha \geq 0$
7. $G_{4,3}$ defined by: $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$,
8. $G_{4,4}$ defined by: $[X_1, X_2] = X_3$ and $[X_1, X_4] = X_4$,
9. $G_{4,5}(\alpha, \beta)$ defined by: $[X_1, X_2] = X_2$, $[X_1, X_3] = \alpha X_3$ and $[X_1, X_4] = \beta X_4$, with $-1 < \alpha \leq \beta < 0$ or $(-1 \leq \alpha < 0$ and $0 < \beta \leq 1)$ or $(0 < \alpha \leq \beta \leq 1)$
10. $G_{4,6}(\alpha)$ defined by: $[X_1, X_2] = \alpha X_2$, $[X_1, X_3] = X_3 + X_4$ and $[X_1, X_4] = X_4$ with $\alpha \neq 0$
11. $G_{4,7}$ defined by: $[X_1, X_2] = X_2 + X_3$, $[X_1, X_3] = X_3 + X_4$ and $[X_1, X_4] = X_4$.
12. $G_{4,8}(\alpha, \beta)$ defined by: $[X_1, X_2] = \alpha X_2$, $[X_1, X_3] = \beta X_3 - X_4$ and $[X_1, X_4] = X_3 + \beta X_4$, with $\alpha > 0$.

We have seen that every $(k+1)$-dimensional Lie algebra which has a 1-codimension abelian ideal admits a k-symplectic structure. From the above classification we can deduce:

Proposition 2.4. A 4-dimensional real Lie algebra G which has a k-symplectic structure is isomorphic to one of the following: $(G_1)^4$, $(G_1)^4 \times G_2$, $G_1 \times G_{3,1}$, $G_1 \times G_{3,2}(\alpha)$, $G_1 \times G_{3,3}$, $G_1 \times G_{3,4}(\alpha)$, $G_{4,3}$, $G_{4,4}$, $G_{4,5}(\alpha, \beta)$, $G_{4,6}(\alpha)$, $G_{4,7}$ and $G_{4,8}(\alpha, \beta)$.

3. LEFT-SYMMETRIC *k*-SYMPLECTIC LIE ALGEBRAS

3.1. Affine structures on Lie algebras

Let A be a vector space on the field $\mathbb{K}(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$. A bilinear mapping $\nabla: (X, Y) \mapsto \nabla_X Y = X.Y$, of A x A with values into A is called *left-symmetric product* if it satisfies:

$$(X \cdot Y) \cdot Z - X \cdot (Y \cdot Z) = (Y \cdot X) \cdot Z - Y \cdot (X \cdot Z),$$

for all $X, Y, Z \in$ A. In this case (A, ∇) is called a left-symmetric algebra or a Vinberg algebra.

If ∇ is a left-symmetric product then the bracket $[X, Y] = X \cdot Y - Y \cdot X$, satisfies the identities of Jacobi, that is (A,[,]) is a Lie algebra subordinated to the left-symmetric algebra (A, ∇).

Let us consider a Lie algebra G. We say that G is endowed with an affine structure if there exists on the underlying vector space of G a left-symmetric product ∇ such that (G,

[,]) is the Lie algebra subordinated to the Vinberg algebra (G, ∇).

3.2. Affine structures associated to a k-symplectic structure

Let us recall that if G is a symplectic Lie algebra, then there always exists an affine structure associated to the symplectic form:

$$\theta(Y,[X, Z]) = -\theta(\nabla_X Y, Z).$$

Here we study the analogous existence problem for the k-symplectic case.

Let G be a $n(k+1)$-dimensional Lie algebra on $\mathbb{K}$, provided with a k-symplectic structure $(\theta^1, \ldots, \theta^k; H)$.

Definition 3.1. We say that an affine structure ∇ on the k-symplectic Lie algebra G is compatible with the k-symplectic structure $\theta^1, \ldots, \theta^k; H)$ if it satisfies the following property:

$$\theta^p(\nabla_X Y, Z) = -\theta^p(Y, [X, Z])$$

for all $p = 1, \ldots, k$ and $X, Y, Z \in$ G.

We denote by j the mapping of G with values in $Hom(G, \mathbb{K}^k)$ given by:

$$j(Z) = (i(Z)\,\theta^1, \ldots, i(Z)\theta^k).$$

Let be $f_{X, Y} \in Hom\ (G, \mathbb{K}^k)$ defined by:

$$f_{X, Y}(Z) = -(\theta^1(Y, [X, Z]), \ldots, \theta^k(Y, [X, Z]))\ \forall X, Y \in G \text{ and } Z \in G.$$

As the k-symplectic system is a nondegenerated exterior system, the map j is injective. Thus the following properties are equivalent:

1. ∇ is compatible with the k-symplectic structure $(\theta^1, \ldots, \theta^k; H)$,
2. $j(\nabla_X Y) = f_{X,Y}$ for all $X, Y \in$ G.

Theorem 3.1. Let G be a $n(k+1)$-dimensional k-symplectic Lie algebra. We suppose that there exists on G an affine structure compatible with the k-symplectic structure. Then if $k \geq 2$, we have the following properties:

1. H is an abelian ideal for the Lie algebra structure of G.
2. H is an ideal of the Vinberg algebra (G, ∇).
3. $\nabla_H H = 0$.

Proof. Let us suppose that ∇ is an affine structure compatible with the k-symplectic structure $(\theta^1, \ldots, \theta^k; H)$. From the classification theorem [6] there exists a basis $(e_{pi}, e_i)_{1 \leq p \leq k, 1 \leq i \leq n}$ of G, called the k-symplectic basis, such that

$$\theta^p = \sum_{i=1}^{n} \omega^{pi} \wedge \omega^i, \quad (p = 1, \ldots, k)$$

and

$$H = \ker \omega^1 \cap \ldots \cap \ker \omega^n,$$

where $(\omega^{pi}, \omega^i)_{1 \leq p \leq k, 1 \leq i \leq n}$ is the dual basis of $(e_{pi}, e_i)_{1 \leq p \leq k, 1 \leq i \leq n}$. □

Lemma 3.1. If $k \geq 2$ then the subalgebra H is abelian.

Proof of the lemma. For all $p = 1, \ldots, k$, let H_p be the subspace of G generated by the vectors $(e_{pi})_{1 \leq i \leq n}$. As $H = \oplus_p H_p$ we can show that $[H_p, H_q] = \{0\}$ for all p, q. As the affine structure is adapted to the k-symplectic structure we have for all p and for all $r \neq p$:

$$\theta^p(\nabla_{e_{li}} e_j, e_{rs}) = 0 = -\theta^p(e_j, [e_{li}, e_{rs}]).$$

Then $\omega([e_{li}, e_{rs}] = 0$ for all li and rs with $r \neq p$. This is equivalent to

$$[H_p, H_q] = \{0\}$$

with $p \neq q$ and

$$[H_p, H_p] \subset H_p.$$

Now consider $e_{pi} \in H_p$ and $e_{qj} \in H_q$. We have

$$\theta(\nabla_{e_{pi}} e_k, e_{qj}) = 0 = -\theta^p(\nabla_{e_{qj}} e_k, e_{pi}).$$

Thus $\nabla_{e_{pi}} e_k \in H$ and

$$\theta^q(\nabla_{e_{qj}} e_k, e_{qs}) = 0 = -\theta^q(e_k, [e_{qj}, e_{qs}]).$$

As $[e_{qj}, e_{qs}] \in H_q$, the previous identity shows that $[e_{qj}, e_{qs}] = 0$ and

$$[H_q, H_q] = \{0\}.$$

Lemma 3.2. The following properties are equivalent:

1. H is an ideal for the structure of Lie algebra of G.
2. 2H p $_p$ is a left ideal for the structure of Vinberg algebra (G, ∇).
3. ∇ is trivial on H.

Proof. Let us suppose that H is an ideal for the Lie algebra structure and let $h, h' \in$ H, $x \in$ G. We have $\theta^p(x.h, h') = -\theta^p(h, [x, h']) = 0$. This shows that $x.h \in$ H, that is H is a left ideal for the Vinberg structure.

Conversely if H is a left ideal for the Vinberg structure given by ∇, then $\theta^p(h.h', x) = -\theta^p(h', [h, x]) = \theta^p(h', [x, h]) = \theta^p(x.h', h) = 0$, for all $h, h' \in$ H, and $x \in$ G. Thus ∇_H H = 0.

Now if we suppose that ∇ is null on H, thus $\theta^p(h.h', x) = -\theta^p(h', [h, x]) = 0$, for all $h, h' \in$ H and $x \in$ G, that is $[h, x] \in$ H.

We prove by similar arguments:

Lemma 3.3. The following properties are equivalent:
1. H is a Lie abelian subalgebra.
2. H is a right ideal for the Vinberg structure.

Let us take again the proof of the theorem. Using the previous lemmas, it is enough to prove that H is an abelian ideal for the Lie structure. As H is an abelian Lie subalgebra, let us prove that $C^l_{pi,j} = 0$. But for $p \neq q$, we have $0 = \theta^q(e_j.e_{ql}, e_{pi}) = -\theta^q(e_{ql}, [e_j, e_{pi}]) = C^l_{pi,j}$. □

Remark. The case $k = 1$.

In the theorem we consider $k \geq 2$. This hypothesis is fundamental and the theorem is false for $k = 1$. For example let us consider the $2n$-dimensional solvable Lie algebra G given by the Maurer-Cartan equations:

$$\begin{cases} d\omega^1 = \omega^1 \wedge \omega^2, \\ d\omega^{n+1} = \omega^2 \wedge \omega^{n+1}, \\ d\omega^i = 0 \quad \text{for} \quad i \neq 1 \quad \text{and} \quad n+1, \end{cases}$$

and H the Lie subalgbera of G defined by the equations $\omega^{n+1} = \ldots = \omega^{2n} = 0$, $\{\omega^1, \ldots, \omega^{2n}\}$ being a basis of G*. We verify that the pair (θ ,H), where $\theta = \omega^1 \wedge \omega^{n+1} + \ldots + \omega^n \wedge \omega^{2n}$ is a 1-symplectic structure. Then G admits an affine structure compatible with the symplectic structure and H is not abelian.

Example

1. Let G be the solvable 6-dimensional Lie algebra given by the Maurer-Cartan equations:

$$\begin{cases} d\omega^1 = \omega^1 \wedge \omega^5 + \omega^3 \wedge \omega^6 \\ d\omega^2 = \omega^2 \wedge \omega^5 + \omega^4 \wedge \omega^6 \\ d\omega^3 = \omega^3 \wedge \omega^5 \\ d\omega^4 = \omega^4 \wedge \omega^5 \\ d\omega^5 = 0 \\ d\omega^6 = 0, \end{cases}$$

$\{\omega^1,\ldots,\omega^6\}$ being a basis of G*. Let us consider

$$\begin{cases} \theta^1 = d\omega^1 = \omega^1 \wedge \omega^5 + \omega^3 \wedge \omega^6 \\ \theta^2 = d\omega^2 = \omega^2 \wedge \omega^5 + \omega^4 \wedge \omega^6 \end{cases}$$

and H = ker $\omega^5 \cap$ ker ω^6. Then (θ^1, θ^2, H) is a 2-symplectic structure on G. It is also provided with a compatible affine structure defined by:

$$\begin{cases} X_1.X_5 = X_1 \\ X_2.X_5 = X_2 \\ X_3.X_5 = X_3 \\ X_3.X_6 = X_1 \\ X_4.X_5 = X_4 \\ X_4.X_6 = X_2 \\ X_5.X_5 = X_5 \\ X_5.X_6 = X_6 \\ X_6.X_5 = X_6 \end{cases}$$

where $\{X_1,\ldots,X_6\}$ is the basis of G whose dual basis is $\{\omega^1,\ldots,\omega^6\}$. We can note that G is not a symplectic Lie algebra.

3.3. Affine structures on $(k+1)$-dimensional k-symplectic Lie algebras

Suppose that G is a $(k+1)$-dimensional Lie algebra provided with a k-symplectic structure $(\theta^1,\ldots,\theta^k; H)$. If there is an affine structure adapted to the k-symplectic structure then H is an abelian ideal. From the description of these Lie algebras proposed in the previous section, G is a one dimensional extension by derivation of H. Let $(e_1,\ldots,e_{k+1})$ a basis of G such that H is generated by $(e_1,\ldots,e_k)$. Let $(\omega_1,\ldots,\omega_{k+1})$ the dual basis. Then, if we put $\theta^i = \omega_1 \wedge \omega_{k+1})$, the system

$$(\theta^1, \ldots, \theta^k, \mathcal{G})$$

is a *k*-symplectic system on G. Let us put

$$\begin{cases} e_i \bullet e_j = 0, \quad 1 \leq i, j \leq k \\ e_i \bullet e_{k+1} = [e_i, e_{k+1}], \quad 1 \leq i \leq k \\ e_{k+1} \bullet e_i = 0, \quad 1 \leq i \leq k \\ e_{k+1} \bullet e_{k+1} = \alpha e_{k+1}. \end{cases}$$

This product satisfies

$$e_i \bullet e_j - e_j \bullet e_i = [e_i, e_j].$$

If we denote by (x, y, z) the associator of the product • concerning the vectors x, y, z, we have

$$(e_i, e_j, e_{k+1}) = (e_i, e_{k+1}, e_j) = (e_{k+1}, e_j, e_i) = 0$$

for all $i, j \leq k$. Moreover

$$(e_{k+1}, e_{k+1}, e_i) = (e_{k+1}, e_i, e_{k+1}) = 0$$

for all $i \leq k$, and

$$(e_i, e_{k+1}, e_{k+1}) = \alpha[e_i, e_{k+1}] - [[e_i, e_{k+1}], e_{k+1}].$$

If the product • is left symmetric, we can have $\alpha = 0$ and $ad(e_{k+1})^2 = 0$ or $ad(e_{k+1})^2 = \alpha\, ad(e_{k+1})$.

In the last case, if $\alpha \neq 0$, then $ad(e_{k+1})$ is diagonalizable and $\alpha = 1$ or -1.

Then a such Lie algebra G is provided with an affine structure. Moreover

$$\theta^p(e_i \bullet e_{k+1}, e_{k+1}) = \theta^p([e_i, e_{k+1}], e_{k+1}) = -\theta^p(e_{k+1}, [e_i, e_{k+1}])$$

and

$$\theta^p(e_{k+1} \bullet e_i, e_{k+1}) = 0 = -\theta^p(e_i, [e_{k+1}, e_{k+1}]).$$

At last

$$\theta^p(e_{k+1} \bullet e_{k+1}, e_p) = \theta^p(e_p, \alpha e_{k+1}) = \alpha = -\theta^p(e_{k+1}, [e_p, e_{k+1}]).$$

Then the affine structure is associated to the k-symplectic structure.

THEOREM 3.2. Every (k +1)-dimensional k-symplectic Lie algebra which admits an affine structure associated to the k-symplectic structure if it is isomorphic to a one dimensional extension by derivation of a k-dimensional abelian Lie algebra such that the derivation is nilpotent and it is not nilpotent and satisfies $f = Id$.

Proof. Let $(\theta^1, \ldots, \theta^k, H)$ the k-symplectic structure on G. If H is abelian, we have construct above the corresponding affine structure. If H is not abelian, if G is not isomorphic to $sl(2)$, then it is a one dimensional extension of an abelian ideal and G admits another k-symplectic structure $(\vartheta^1,\ldots, \vartheta^k, H')$ with H′ abelian. In this case *e* find again the first case. □

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