Motion on a given surface: Potentials producing geodesic lines as trajectories

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Abstract. *In the light of inverse problem of dynamics, we consider the motion of a material point on an arbitrary two-dimensional surface, submersed in* $\mathbb{E}^3$*. We study* two-*dimensional potentials which produce a mono-parametric family of geodesic lines as trajectories. We establish a* new, non-linear *partial differential condition for the potential function* $V = V(u, v)$*. With the aid of this condition, we examine if a given potential produces a family of geodesic lines on a certain surface or not. On the other hand, we can check if a given family of regular orbits is indeed a family of geodesic lines on a certain surface and then find the potential function* $V = V(u, v)$ *which gives rise to this family of orbits. Special cases are also studied and pertinent examples are worked out.*

Keywords. *Motion on a surface, Mono-parametric families of orbits, Potentials, Lagrangian Mechanics.*

1. INTRODUCTION

The inverse problem of dynamics, as introduced by Szebehely (1974), seeks for all the potentials $V = V(x, y)$ which can generate a mono-parametric family of planar orbits $f(x, y) = c$, traced in the xy Cartesian plane by a material point of unit mass, with a pre-assigned dependence $\mathcal{E} = \mathcal{E}(f(x, y))$ of its total energy on the given family. There results a first-order partial differential equation, linear in the unknown function $V = V(x, y)$ and the coefficients depend on the family of orbits. Szebehely's equation was studied by many authors later (e.g. Bozis (1983), Puel (1984)). Bozis (1984) presented a second order linear partial differential equation giving all the potential functions $V = V(x, y)$ which give

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rise to a pre-assigned family of planar curves $f(x, y) = c$. Bozis' equation does not include the energy $\mathcal{E}$ and consequently no assumption about the energy dependence $\mathcal{E} = \mathcal{E}(f)$ needs to be made.

Mertens (1981) studied a family of curves $f(u, v) = c$ on a surface S in 3-D space using Szebehely's method and obtained a linear partial differential equation in the potential function $V(u, v)$. Furthermore, Bozis and Mertens (1985) derived a second order partial differential equation of hyperbolic type for the potential V in which all the coefficients are known functions of the coordinates (u, v) and gave some examples.

Borghero (1986) determined the expressions for the covariant components Q_1, Q_2 of forces acting on a test particle which describes orbits on a given surface, using the procedure of Dainelli (Whittaker, 1994).

Bozis and Borghero (1995) introduced the notion of the family boundary curves (FBC) for that version of the inverse problem of dynamics which combines the potential $V(u, v)$ with a mono-parametric family of regular orbits $f(u, v) = c$ on the configuration manifold (M_2, g) of a conservative holonomic system with $n = 2$ degrees of freedom.

Several examples were given there. Puel (2002) gave a geometrical interpretation for the deflection at the origin of rectilinear orbits in a central field. This interpretation was based on the correspondence between the plane orbits of a conservative force field and the geodesics of a certain surface. Recently, Kotoulas (2005a) studied the case of a generalized force field which gives rise to a *two*-parametric family of curves on a given surface. Among other curves, helical lines were also studied there.

A solvable version of the inverse problem of dynamics was studied by the same author (Kotoulas, 2005b). A review on basic facts of inverse problem in dynamics was made by Bozis (1995) and recently by Anisiu (2003).

In the present work we shall deal with the PDE given by Mertens (1981). Especially, we shall study geodesic lines on a given surface and we shall find potentials $V = V(u, v)$ producing them as trajectories. Simple as it may sound, the study of this problem leads to the understanding of how *Differential Geometry is linked to Lagrangian Dynamics*. In Section 2 we give a full description of the problem and produce a new formulation of Mertens' equation (1981). This equation takes a simpler form in the case of geodesic lines on a given surface. In Section 3 we establish a *new, non-linear* PDE which the potential function $V(u, v)$ has to satisfy in order to admit geodesic lines as trajectories.

We face the problem from the direct and the inverse view point and we give pertinent examples. Sporadic findings are presented in Section 4 (Table 1). Special cases are also examined (Section 5).

Finally, the conclusions are summarized in Section 6.

2. ANALYSIS OF THE PROBLEM

In an Euclidean 3D-space $\mathbb{E}^3$ with an orthonormal Cartesian system of reference $Oxyz$ we assign a smooth surface S:

$$P = P(u, v) \Leftrightarrow \{x_1 = x_1(u, v), x_2 = x_2(u, v), x_3 = x_3(u, v)\} \tag{1}$$

with u, v as curvilinear coordinates on S. On this surface we also consider a mono-parametric family of regular curves given in the solved form

$$f(u, v) = c \tag{2}$$

where c is the parameter of the family (2).

For the given family of orbits we define γ as follows: $\gamma = f_v/f_u$ and the subscripts denote partial differentiation with respect to v or u. The «*slope function*» γ represents the family (2) in the sense that if the family (2) is given, then γ is determined uniquely.

On the other hand, if γ is given, we can obtain a unique family (2). The inverse problem of dynamics consists in finding potentials V for which this family of curves (2) on a given surface (1) are trajectories.

The first fundamental form on the surface S in this system of parameters is given by:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{3}$$

where E, F, G are the coefficients of first fundamental form.

Now, we consider a particle of unit mass which describes any member of the given family (2). Here we have to clear out that trajectories are bound to a given surface by constraints. The kinetic energy (T) of the test particle is given by (Mertens, 1981)

$$T = 1/2\,(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2) \tag{4}$$

where the dot denotes differentiation with respect to time.

Mertens (1981) produced a *linear*, first order partial differential equation for the potential function $V = V(u, v)$ for any pre-assigned dependence $\mathcal{E} = \mathcal{E}(f)$, of the total energy $\mathcal{E}$ of the given family $f = f(u, v)$. This equation reads:

$$(Gf_u - Ff_v)V_u + (Ef_v - Ff_u)V_v = 2W(\mathcal{E}(f) - V) \tag{5}$$

where W is given below:

$$\begin{aligned}
W &= 1/A\,[g(f_v^2 f_{uu} - 2f_u f_v f_{uv} + f_u^2 f_{vv}) - B_1(Gf_u - Ff_v) - B_2(Ef_v - Ff_u)],\\
A &= Ef_v^2 - 2F f_u f_v + Gf_u^2,\\
B_1 &= 1/2\,E_u f_v^2 - E_v f_u f_v + (F_v - 1/2\,G_u)f_u^2,\\
B_2 &= (F_u - 1/2\,E_v)f_v^2 - G_u f_u f_v + 1/2\,G_v f_u^2,\\
g &= EG - F^2.
\end{aligned} \tag{6}$$

The present paper deals with the above equation (5) and especially with the interesting case of geodesic lines on a given surface. At a first stage we shall modify the above equation (5) in a «*geometrical way*» and at a second stage we shall prove that in the case of geodesic lines it is: $W = 0$. Then we shall proceed to the computation of the potential function which produces these curves as trajectories.

2.1. A new formulation of Mertens' equation (1981)

In this paragraph we shall bring the expression of W in (6) in a simpler form. At the beginning we define the following quantities:

$$\Theta_0 = f_v^2 f_{uu} - 2 f_u f_v f_{uv} + f_u^2 f_{vv},$$

$$\Theta_1 = B_1 (Gf_u - Ff_v) + B_2 (Ef_v - Ff_u) \tag{7}$$

The second part Θ_1 is written, after some straightforward algebra, as follows:

$$\Theta_1 = g\Theta_2,$$

$$\Theta_2 = \Gamma^2_{11} f_v^3 + (\Gamma^1_{11} - 2\Gamma^2_{12}) f_v^2 f_u + (\Gamma^2_{22} - 2\Gamma^1_{12}) f_v f_u^2 + \Gamma^1_{22} f_u^3 \tag{8}$$

where the symbols Γ^1_{ij}, Γ^2_{ij} $(i, j = 1, 2)$ are the *Christoffel's symbols of second kind*. So, in view of (7) and (8), the expression of W can be written as follows:

$$W = 1/A\,(g\Theta_0 - \Theta_1) = g/A\,(\Theta_0 - \Theta_2), \tag{9}$$

Moreover, we introduce the notation $Z = \gamma\gamma_u - \gamma_v$ and with the use of «*slope function*» γ, the eq. (5) is written as:

$$(G - \gamma F)V_u + (\gamma E - F)V_v + 2\Delta/A_1(\mathcal{E}(f) - V) = 0 \tag{10}$$

where

$$\Delta = g\,[\Gamma^2_{11}\gamma^3 + (\Gamma^1_{11} - 2\Gamma^2_{12})\gamma^2 + (\Gamma^2_{22} - 2\Gamma^1_{12})\gamma + \Gamma^1_{22} + Z],$$

$$A_1 = E\gamma^2 - 2F\gamma + G,$$

$$W = -f_u \Delta/A_1 \tag{11}$$

2.2. Geodesic lines on a given surface

It is known from the Differential Geometry (Struik, 1961, Lipschutz, 1969) that a regular curve of the surface (1) defined by the system $u = u(s)$, $v = v(s)$ (s is the physical parameter) is a geodesic line if and only if the functions $u(s)$ and $v(s)$ satisfy the following equations:

$$u'' + \Gamma^1_{11} (u')^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} (v')^2 = 0,$$

$$v'' + \Gamma^2_{11} (u')^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} (v')^2 = 0 \qquad (12)$$

where the prime denotes total differentiation with respect to the physical parameter s.

Moreover, if a curve is given in the form $f(u, v) = 0$, then the above two equations turn to be one. It is shown, after some straightforward calculations, that a regular curve $f(u, v) = 0$ is a geodesic line if and only if the following relation holds:

$$\Gamma^2_{11} f_v^3 + (\Gamma^1_{11} - 2\Gamma^2_{12}) f_v^2 f_u + (\Gamma^2_{22} - 2\Gamma^1_{12}) f_v f_u^2 + \Gamma^1_{22} f_u^3 - \qquad (13)$$
$$- (f_v^2 f_{uu} - 2 f_u f_v f_{uv} + f_u^2 f_{vv}) = 0$$

or due to (7) and (8):

$$\Theta_2 - \Theta_0 = 0 \qquad (14)$$

This result coincides with the zeroing of the quantity W defined in (9). Thus, if the mono-parametric family of curves $f(u, v) = c$ represents a family of geodesic lines on a given surface S , then the quantity W becomes equal to zero ($W = 0$) or equivalently $\Delta = 0$ (Δ is defined in (11)). Then the equation (10) reads:

$$(G - \gamma F)V_u + (\gamma E - F)V_v = 0 \qquad (15)$$

From (15) we obtain the following expression for the «*slope function*» γ:

$$\gamma = \frac{FV_v - GV_u}{EV_v - FV_u} \cdot \qquad (16)$$

3. POTENTIALS PRODUCING GEODESIC TRAJECTORIES

In the previous section we showed that in the case of geodesic lines on a given surface it is $\Delta = 0$ (Δ is defined in (11)). In order to find certain potentials creating a mono-parametric family of geodesic lines on a given surface, we insert the expression (16) of the «*slope function*» γ in Δ and then we set $\Delta = 0$. After some straightforward calculations, we obtain a *new, non-linear* PDE of second order in the unknown function $V(u, v)$. This equation reads:

$$p_{30} V_v^3 + p_{21} V_v^2 V_u + p_{12} V_v V_u^2 + p_{03} V_u^3 +$$

$$+ V_{uu} V_v (GV_u - FV_v) + V_{uv} (EV_v^2 - GV_u^2) +$$

$$+ V_{vv} V_u (FV_u - EV_v) = 0 \tag{17}$$

and the coefficients $p_{30}, p_{21}, p_{12}, p_{03}$ are given in the *Appendix I*. Here, it is important to say that the PDE (17) includes derivatives of first and second order in $V(u, v)$ and expressions of $E, F\, G$ and not orbital elements. Moreover, we ascertain that not *any* potentials produce geodesic lines on a given surface but only those which satisfy the eq. (17).

Remark 1: If we consider straight lines on a plane (i.e. $E = 1$, $F = 0$, $G = 1$), we shall use *Cartesian* coordinates x, y instead of the curvilinear ones u, v. Hence, the coefficients $p_{30}, p_{21}, p_{12}, p_{03}$ are identically zero and the (17) takes the form:

$$V_x V_y (V_{xx} - V_{yy}) = V_{xy} (V_x^2 - V_y^2) \tag{18}$$

The equation (18) is the same one with that found by Bozis and Anisiu (2001a) for straight lines produced by planar potentials in inertial frames. Moreover, the same authors (2001b) studied planar potentials which produce families of straight lines in rotating frames. On the other hand, *isolated* straight lines produced by two-dimensional potentials $V = V(x, y)$ in inertial frames Oxy were found by Antonov and Timoshkova (1993) with the aid of Szebehely's *PDE* (1974). In addition to that, Bozis and Kotoulas (2004) found three-dimensional potentials creating *two*-parametric families of straight lines in 3-D space.

Now, we proceed more and we formulate the following:

Theorem 1: *If $w(u, v)$ is a solution of the PDE (17), then all functions $V = \mathcal{B}(w)$, with $\mathcal{B}$ an arbitrary C^2-function, are solutions of (17). For a potential $w(u, v)$ associated with a family of geodesic lines γ, the function $\mathcal{B}(w)$ will be associated with the same family of geodesic lines.*

Proof. Indeed, let $w(u, v)$ be a solution of (17). We prepare the derivatives of first and second order of the potential function $V(u, v)$:

$$V_v = \mathcal{B}'(w) w_v, \quad V_u = \mathcal{B}'(w) w_u,$$

$$V_{vv} = \mathcal{B}''(w)w_v^{\,2} + \mathcal{B}'(w)w_{vv},$$

$$V_{uu} = \mathcal{B}''(w)w_u^{\,2} + \mathcal{B}'(w)w_{uu},$$

$$V_{uv} = \mathcal{B}''(w)w_v\, w_u + \mathcal{B}'(w)w_{uv} \tag{19}$$

and inserting them into (17) we obtain, after some straightforward algebra,

$$p_{30}\, w_v^{\,3} + p_{21} w_v^{\,2} w_u + p_{12} w_v\, w_u^{\,2} + p_{03} w_u^{\,3} +$$

$$+ w_{uu}\, w_v\, (G w_u - F w_v) + w_{uv}\, (E w_v^{\,2} - G w_u^{\,2}) +$$

$$+ w_{vv} w_u\, (F w_u - E w_v) = 0 \tag{20}$$

which is true because $w(u, v)$ was supposed to be a solution of (17). Now, it is obvious, according to (16), that both $w(u, v)$ and $\mathcal{B}(w)$ lead to the same γ. Analogous theorems for potentials creating families of straight lines in plane and in 3-D space were found by Bozis and Anisiu (2001a) and recently by Bozis and Kotoulas (2004).

3.1. The direct view-point of the problem

In this paragraph we shall study the problem from the direct view-point namely if we find an appropriate potential $V(u, v)$ satisfying the *non-linear* PDE (17) for the given metric, then we determine the family of geodesic lines which is produced by this potential. This is done with the aid of the equation (16). Indeed, from (16) we determine the «*slope function*» $\gamma(u, v)$ and due to the fact that the «*slope function*» $\gamma(u, v)$ represents the family of regular curves (1), we find the family of curves (1) from the ODE:

$$\frac{dv}{du} = -\frac{1}{\gamma(u, v)} \tag{21}$$

- *Example* 1: We consider the metric

$$E = 1 + 4u^2, \;\; F = 4uv, \;\; G = 1 + 4v^2 \tag{22}$$

which is realized by the surface S: $\bar{x}\,(u, v) = \{u, v, u^2 + v^2\}$ (elliptic paraboloid) and we search for a potential satisfying (17). Using *the method of determination of constants*, we sought for solutions of the form $V(u, v) = a_1\, u^2 + a_2\, v^2$ where a_1, a_2 are constants and we found

$$V(u, v) = a_1\,(u^2 + v^2) \tag{23}$$

According to the **Theorem 1**, all the functions $V(u, v) = \mathcal{B}(u^2 + v^2)$ are solutions to this problem.

Now, we shall determine the family of curves (2) which is produced by the potential function (23). So, from (16) we obtain:

$$\gamma = -u/v \tag{24}$$

Inserting (24) into (21), we get:

$$f(u, v) = v/u = c \tag{25}$$

3.2. The inverse view-point of the problem

In this paragraph we shall study the problem from the inverse view-point namely if a family of regular curves (2) is considered on a smooth surface (1), we have the possibility:

- 1) to check if this family is indeed a family of geodesic lines. This is so if $\Delta = 0$ from (11) or equivalently $W = 0$ from (9) and
- 2) to find the potential creating the corresponding family. This can be done solving the PDE (15) analytically.

We present the following

- Example 2: We consider the metric

$$E = 1 + (h')^2, \;\; F = 0, \;\; G = u^2, \;\; h = h(u) \tag{26}$$

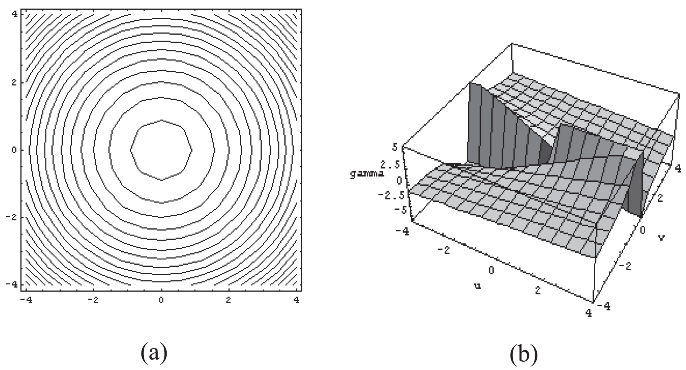

Figure 1. (a) The contour lines of the potential (23) and (b) the trajectories (24) in the 3D-space $u - v - \gamma$ **(the symbol *gamma* stands for** γ**).**

which is realized by the *revolution surface S*: $\bar{x}\ (u, v) = \{u \cos v, u \sin v, h(u)\}$ (the prime denotes total differentiation of the function h with respect to u). It is known (Struik 1961, pp. 134) that the geodesic lines of this surface are given by:

$$v = \pm k_0 \int \frac{\sqrt{1+(h')^2}}{u\sqrt{u^2-k_0^2}}\,du, \quad k_0 = const. \neq 0 \tag{27}$$

In the previous relation (27) we consider the sign (–) and we create the mono-parametric family of regular curves:

$$f(u,v) = v + k_0 \int \frac{\sqrt{1+(h')^2}}{u\sqrt{u^2-k_0^2}}\,du = c \tag{28}$$

where c is the parameter of the family and k_0 has a specific non-zero value (e.g. $k_0 = 1$). We made the analogous calculations and from (11) we obtained $\Delta = 0$. So, this mono-parametric family of regular curves is indeed a family of geodesic lines on the above surface. Hence, we have:

$$\gamma = \frac{u\sqrt{u^2-k_0^2}}{k_0\sqrt{1+(h')^2}} \tag{29}$$

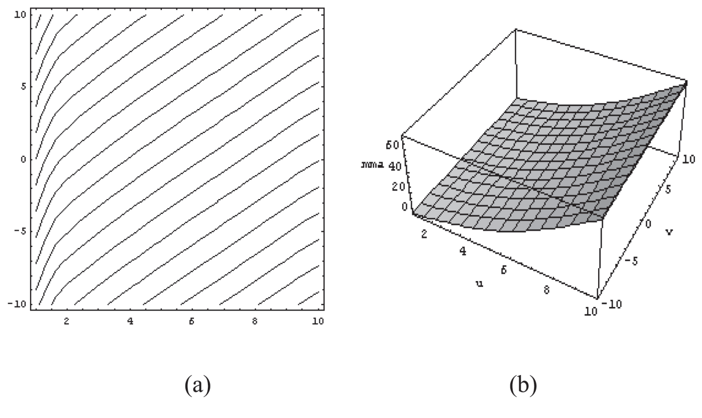

Figure 2. (a) The contour lines of the potential (31) and (b) the trajectories (29) in the 3D-space $u - v - \gamma$ (the symbol *gamma* stands for γ). These diagrams were taken for $h(u) = u$, $k_0 = 1$ and $V(u, v) = \Phi(z) = z$ in (31).

and the equation (16) reads:

$$k_0 u V_u + \sqrt{(u^2 - k_0^2)(1 + (h')^2)}\,V_v = 0 \tag{30}$$

with the general solution

$$V(u,v) = \Phi(z), \quad z = k_0 v - \int \frac{\sqrt{(u^2 - k_0^2)(1 + (h')^2)}}{u}\, du \tag{31}$$

where Φ is an arbitrary C^2-function of its argument z.

4. SPORADIC FINDINGS

In addition to the examples of Section 3, we give in Table 1 certain sporadic findings of compatible pairs of families of geodesic trajectories and potentials. Starting with the PDE (17), we examined many surfaces and we tried to find solutions for the potential function $V(u, v)$ with *the method of determination of constants* (cases 1,2). Then we determined the compatible family of orbits using the relation (16).

On the other hand, having checked that the considered family of curves (2) is indeed a family of geodesic lines on the given surface, we managed to find solutions for the potential function $V(u, v)$ solving the PDE (15) (cases 3, 4, 5, 6). In the last case, i.e. case 7, we sought for solutions of the form $V(u, v) = u + R(v)$ where R is an arbitrary C^2-function. All the results are summarized in Table 1. The examined surfaces are the following ones:

- 1) «*Enneper's surface*»: $\bar{x}(u,v) = \left\{ u - \frac{u^3}{3} + uv^2, \quad -v + \frac{v^3}{3} - vu^2, \quad u^2 - v^2 \right\}$
- 2) Hyperboloid of one sheet: $\bar{x}(u, v) = \{\cos u - v \sin u,\ \sin u + v \cos u,\ v\}$
- 3) Sphere: $\bar{x}(u, v) = \{R \cos u \sin v,\ R \sin u \sin v,\ R \cos v\},\ R > 0$
- 4) Cylinder: $\bar{x}(u, v) = \{u, v, (u + v)^2\}$
- 5) Hyperbolic Paraboloid: $\bar{x}(u, v) = \left\{ u, v, \frac{u+v}{u-v} \right\}$
- 6) Hyperbolic Paraboloid: $\bar{x}(u, v) = \{u, v, uv\}$
- 7) Cylinder: $\bar{x}(u, v) = \{u, v, u + v^2\}$

Especially, for the last case, i.e. case (7), we found that the potential function

$$V(u, v) = 4u + 2v(v + \sqrt{1 + 2v^2}) + \sqrt{2}\, arcsinh\,(\sqrt{2}\, v) \tag{32}$$

Table 1. Families of geodesic trajectories and potentials.

Family: $f(u, v) = c$	*Surface*	*Potential* $V(u, v)$
v/u	(1)	$u^2 + v^2$
u	(2)	$u + 2v$
u	(3)	v
$u + v$	(4)	$v - u$
v/u	(5)	$u^2 + v^2$
u	(6)	$v\sqrt{1+u^2}$

admits the family of geodesic lines:

$$f(u, v) = 4\sqrt{2}\,u + 2\sqrt{2}\,v^2 - v\sqrt{2+4v^2} - arcsinh(\sqrt{2}\,v) = c \tag{33}$$

5. SPECIAL CASES

In this section we shall study some special cases of the problem in which many solutions can be found. First of all, we shall begin with

Case 1: One-dimensional potentials: V = V(u)

Then the equation (17) takes the form:

$$p_{03}\,V_u^{\,3} = 0 \tag{34}$$

or, equivalently, $p_{03} = 0$ (p_{03} is defined in the *Appendix I*). So, we have:

$$F^2G_v - 2GFF_v + G^2E_v = 0 \tag{35}$$

Thus, we can formulate the following:

Proposition 1: *If the coefficients of the first fundamental form of a given surface satisfy the relation (35), then a one-dimensional potential V = V(u) exists and produces a family of geodesic lines given by (16).*

- *Example* 1: We consider the surface $\bar{x}\,(u, v) = \{u\cos v,\ u\sin v,\ u^2 + v\}$. Then we have:

$$E = 1 + 4u^2,\ \ F = 2u,\ \ G = 1 + u^2 \tag{36}$$

We observe that the components of the metric tensor satisfy the relation (35). Thus, a *one*-dimensional potential $V = V(u)$ exists which produces a mono-parametric family of geodesic trajectories. From (16) we obtain:

$$\gamma = \frac{1+u^2}{2u} \tag{37}$$

and from (21) we determine the mono-parametric family of orbits:

$$f(u, v) = v + \log(1 + u^2) = c \tag{38}$$

The case of the potentials $V = V(v)$ is studied in an analogous way.

Case 2: Isothermic system of coordinates on the given surface

Up to now we did not make any assumption about the coordinate system on the given surface. At this point we can assume that the system of coordinates is isothermic, namely

$$F = 0 \text{ and } E = G \tag{39}$$

Surfaces with the above properties (39) are known in the literature as «*Liouville's surfaces*». In this case the equation (17) takes the simpler form:

$$(G_v V_u - G_u V_v)(V_v^2 + V_u^2) + 2GV_u V_v (V_{uu} - V_{vv}) + 2GV_{uv}(V_v^2 - V_u^2) = 0 \tag{40}$$

and we shall seek solutions of the form

$$V = R(a_1 u + a_2 v + a_3) \tag{41}$$

where R is an arbitrary C^2-function of two arguments and a_1, a_2, a_3 are constants. We prepare the derivatives of first and second order of the potential V defined in (41) and we replace them into (40). After some straightforward algebra we end up to the relation:

$$(R')^3(a_1^2 + a_2^2)(G_v a_1 - G_u a_2) = 0 \tag{42}$$

Thus, we have $G_v a_1 - G_u a_2 = 0$ or, equivalently,

$$G_v/G_u = a_2/a_1 = k_1 = const. \tag{43}$$

The general solution of (43) is:

$$G = H(u + k_1 v), \;\; k_1 = const. \tag{44}$$

where H is an arbitrary C^2-function. The corresponding family of geodesic trajectories which are produced by the potential (41) is:

$$f(u, v) = v - k_1 u = c \tag{45}$$

Now, we can formulate the following:

Proposition 2: *For surfaces of «Liouville's type» with the property (44), a two-dimensional potential of the form (41) always exists and produces the family of geodesic lines (45).*

- *Example 2*: We assign the surface of «*Liouville's type*» with the following metric:

$$ds^2 = (u + v)(du^2 + dv^2), \;\; u, v > 0 \tag{46}$$

It is easy to check that the coefficients of the first fundamental form are of the type (45) where the constant k_1 is taken equal to unit. Thus, the potential function is of the form (41), namely:

$$V(u, v) = R(u + v + a_3), \;\; a_3\text{: free constant} \tag{47}$$

and produces the family of geodesic lines:

$$f(u, v) = v - u = c \tag{48}$$

6. CONCLUSIONS

In the present work we dealt with an interesting case of the inverse problem in Lagrangian Dynamics: we considered a mono-parametric family of geodesic lines on a given surface and we sought for potentials producing them.

In section 2 we derived a new formulation of Mertens' equation (1981) for potentials producing mono-parametric families of regular orbits on a given surface. We showed that in the case of geodesic lines it is: $W = 0$ and the above equation takes a simpler form. In section 3 we produced a *new, non-linear* PDE (17) for the potential function $V(u, v)$. This equation is the *basic* result of our study. Generally speaking, it is not expected to find solutions of this PDE on *any* given surface. However, for specific surfaces we found non-trivial solutions and the results were presented totally in Table 1. The problem was studied by two sides: direct and inverse. The examples cover both the general and the special cases.

Furthermore, from the mathematical view point, a potential with a constant value (e.g. $V = 1$) satisfies the equation (17) and many solutions of geodesic lines clearly exist. From the physical view point, this means that the power which acts on the test particle is zero. Thus, in the presence of such potential, we leave the test particle of unit mass in a point and it remains there. No motion can take place on the curve in this case.

In Section 5 we examined some special cases and find solutions of the PDE (17). More precisely, we dealt with *one*- dimensional potentials $V = V(u)$ and we showed that these potentials produce families of geodesic lines as trajectories not to any given surface but only to those ones whose coefficients of the first fundamental form satisfy the relation (35). For surfaces of «*Liouville's type*» with the property (44), we showed that there exist potentials of the form (41) which produce families of geodesic trajectories given by (45). All the computations were aided by *MATHEMATICA 5.1*.

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APPENDIX I (COEFFICIENTS OF PDE (17))

$$p_{30} = -\frac{F^2E_u - 2EFF_u + E^2G_u}{2g},$$

$$p_{21} = -\frac{2EFF_v - E^2G_v + 2(F^2 + EG)F_u - F^2E_v - 2FGE_u - 2EFG_u}{2g},$$

$$p_{12} = \frac{2FGF_u - G^2E_u - 2FGE_v + 2(F^2 + EG)F_v - 2EFG_v - F^2G_u}{2g},$$

$$p_{03} = \frac{F^2G_v - 2GFF_v + G^2E_v}{2g},$$

$$g = EF - G^2$$

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