Aggregation of Coarse Preferences *

Hervé Crès†

February 22, 2000

Abstract

We consider weak preference orderings over a set A_n of n alternatives. An individual preference is of *refinement* $\ell \leq n$ if it first partitions A_n into ℓ subsets of 'tied' alternatives, and then ranks these subsets within a linear ordering. When $\ell < n$, preferences are *coarse*. It is shown that, if the refinement of preferences does not exceed ℓ, a super majority rule (within non-abstaining voters) with rate $1 - 1/\ell$ is necessary and sufficient to rule out Condorcet cycles of any length. It is argued moreover how the coarser the individual preferences, (1) the smaller the rate of super majority necessary to rule out cycles 'in probability'; (2) the more probable the pairwise comparisons of alternatives, for any given super majority rule.

Keywords: Super majority voting, condorcet cycles, weak preference orderings.
JEL Classification: C7, D7.

*I am grateful to Dominique Lepelley, Matthieu Martin, Vincent Merlin, Hervé Moulin and Maurice Salles for helpful comments and suggestions.

†HEC School of Management, 78 351 Jouy-en-Josas, France. Tel: 33 1 39 67 94 12 ; fax: 33 1 39 67 70 85 ; cres@hec.fr

1 Introduction

The aggregation of individual preferences through majority voting may yield Condorcet cycles. The simplest example of such a cycle involves three collective choices 1, 2, and 3 and three agents whose individual preferences are respectively $1 \succ 2 \succ 3$, $2 \succ 3 \succ 1$ and $3 \succ 1 \succ 2$. Pairwise comparisons of these alternatives under majority rule show that there is always a two-third majority to prefer 1 to 2, 2 to 3, and 3 to 1.

Condorcet cycles are the main obstacle to the ability of majority voting to aggregate individual preferences. Moreover, it is a simple exercise to construct preference profiles in which some cycle appears. To what extent Condorcet cycles really jeopardize the efficiency of majority voting as a collective decision-making device is a question that has been addressed from the early fifties with the pioneering work of Guilbaud (1952), who computes the probability of the occurrence of such cycles when there are three alternatives and a large number of voters. His work was followed by a line of others, among them: Niemi and Weisberg (1968); DeMeyer and Plott (1970); Kelly (1974); Gehrlein and Fishburn (1976). They all tend show that, under the simple majority rule, the probability of occurrence of Condorcet cycles converges to one when both the numbers of alternatives and voters become large. All these results are obtained through the assumption of *impartial culture*, which states that voters draw at random their preferences in the set of all possible orderings of the n alternatives, i.e., the group of all permutations of n elements, $\mathcal{S}_n$.

The oldest problem of the theory of Social Choice is to determine conditions under which such cycles are ruled out. This question has been solved along two main lines. The first one, initiated by Black (1958) through the concept of single-peakedness gives sufficient conditions on individual preferences that rule out cycles. The family of spatial voting models globally pertains to this line: Alternatives are taken on some underlying continuous space (usually taken to be Euclidean); individual preferences over the set of alternatives are 'structured' by classical assumptions (like continuity and convexity or symmetry); and/or the profiles of individual preferences show some features which can be given natural interpretations in terms of 'consensus' of the society; see, e.g., Plott (1967), Kramer (1973), Grandmont (1978). This line usually suffers from restricted assumptions on the 'domain' of such preferences: the set of allowed preferences is sometimes drastically shrunk. The second strand of research focuses on super majority rules. It was shown that for super majority rules of rates higher than $1 - 1/q$, there cannot exist cycles of length smaller than q (Ferejohn and Grether, 1974). But, if alternatives are numerous, such an argument guarantees the aggregation process to be devoid of cycles only for the unanimity rule, which is highly intractable: the acyclicity of the aggregated preference is obtained at the expense of a lot of incompleteness in this collective preference. Caplin and Nalebuff (1988) yields a very beautiful result along both lines.

Balasko and Crès (1997, 1998) introduce, along these two lines, in a mostly discrete setup, milder assumptions than usual (i.e., respectively, super majority rules with a low rate and a domain restriction —based on some consensus of individual preferences— for which a large ratio of all rankings are admissible) which rule out Condorcet cycles, if not completely, at least with a high probability. The same simple framework is used in the present paper. Let there be n alternatives. There are $n!$ ways of ordering them: one can identify the preference orderings with the permutations of the n alternatives. To each preference σ is associated the percentage λ_σ of the voting population having this particular individual preference. The vector $(\lambda_\sigma)_{\sigma \in \mathcal{S}}$ (where $\mathcal{S}$ is the domain of individual preferences; e.g. $\mathcal{S} = \mathcal{S}_n$, the whole symmetric group of cardinality $n!$, if there are no restrictions) is then a point in the $(\sharp\mathcal{S} - 1)$-simplex; it represents a *profile* of individual preferences, i.e., a particular distribution of voters among preferences. It appears that for certain profiles, at least one Condorcet cycle obtains. Hence some regions give rise to Condorcet cycles, and some others do not. The authors compute a brutal upper bound of the relative Lebesgue measure of the regions in which cycles appear: these measures are very small.

But one could argue that it is difficult, even for a very clever and rational voter, to completely order a large set of social alternatives. Reasons could be the difficulty of gathering full information about the alternatives or the lack of time as well as the lack of interest to compute this complete ranking. It makes sense to assume that people usually have their subset of most preferred alternatives, their subset of least preferred alternatives and some intermediate subsets, the number of which depends on the concern of the voter about the poll, his ability to make up his mind in a limited amount of time about what is at stake and which alternative is truly better. Hence it is not unreasonable to claim that voters' preferences, although transitive, may exhibit a lot of *ties* between the alternatives: individual preferences are *coarse*. A tie between two alternatives is not meant to be a real indifference. It only means that the voter finds it too costly either to figure out what is his real preference between the two alternatives, or to express this preference through voting: he is *undecided*. He then prefers to abstain from voting. A preference is said to be of *refinement* $\ell \leq n$ when voters cannot distinguish, and then order, more than ℓ groups of alternatives. Hence, in the case of three alternatives, the only preference of refinement

1 is the fully indecisive ranking: $1 \sim 2 \sim 3$. It is of refinement 2, or *dichotomous*, if the individual can distinguish only its most preferred alternatives and its least preferred ones; in the case of three alternatives: $1 \succ (2 \sim 3)$; $(2 \sim 3) \succ 1$; $2 \succ (1 \sim 3)$; $(1 \sim 3) \succ 2$; $3 \succ (1 \sim 2)$ and $(1 \sim 2) \succ 3$.

There has been in recent years a strand of research supporting the idea that ties within individual preferences makes their aggregation easier and surer: beyond the early work of Inada (1964) and Fishburn and Gehrlein (1980), one can refer to Gehrlein (1997), Gehrlein and Valognes (1998), Jones et alii (1995), Lepelley and Martin (1998), Saari (1995), Tataru and Merlin (1997) and Van Deemen (1997).

It is shown in the present paper that, if the refinement of preferences does not exceed ℓ, a super majority rule with rate $1-1/\ell$ is necessary and sufficient to rule out Condorcet cycles of any length, a generalization of Ferejohn and Grether (1974). Hence the key factor to rule out cycles is not the number of alternatives, but the maximal degree of refinement of the preferences expressed in the society. This is the object of Section 3. It is moreover argued (in Section 4) how the coarser the individual preferences, the lower the rate of super majority necessary to reduce to almost zero the relative size of the set of profiles for which Condorcet cycles appear, hence their probability under various cultures. The drawback of super majority rules is that it makes it rather improbable that pairwise comparisons of alternatives be possible. It is shown that, for a fixed super majority rule, the probability of a pairwise comparison between two alternatives increases toward 1 when the number of preferences exhibiting a strong ranking between these two alternatives decreases. Hence the coarser the individual preferences, the more likely pairwise comparisons, for a fixed super majority rule.

2 The Setup

2.1 Individual Preferences

We consider a set of n political alternatives $A_n = \{1, 2, \ldots, n\}$. Let us first define individual preferences over A_n. A *fine* individual preference is a complete strict and transitive ranking over A_n, hence can be identified with a permutation of $\{1, 2, \ldots, n\}$: to each alternative is associated its rank in the individual preference; e.g., the permutation 231 of A_3 stands for the individual preference $3 \succ 1 \succ 2$.

A voter v associates a rank to each alternative in A_n. Weak preference orderings are allowed: two different alternatives can be given the same rank, which means that in the voter's opinion, there is a tie between them. Hence an individual preference over the set of alternatives can be conceptualized through the following two-stages process. A voter will be said to have a preference of *refinement* ℓ if it can first partition the set of alternatives A_n into ℓ subsets, and then rank the subsets within a linear ordering. His preference is coarse to the extent that he might not exhibit a strong preference ordering between a pair of alternatives taken in the same subset. But it is transitive:

$$\left.\begin{array}{c} i \succ_v \text{ or } \sim_v j \\ \text{and} \\ j \succ_v \text{ or } \sim_v k \end{array}\right) \Longrightarrow i \succ_v \text{ or } \sim_v k \ ,$$

and at least one strict preference in the left-hand side of the assertion entails the strict preference in the right-hand side.

To illustrate this point, take $n = 7$ and consider an individual having the following preference: his first-best alternatives are $\{3, 5\}$, but he cannot make up his mind whether alternative 3 is better or worse than alternative 5, and decides to stick to a tie; his second-best choices consist in the subset of alternatives $\{1, 4, 7\}$, and finally his third-best alternatives are $\{2, 6\}$; he ends up exhibiting the preference of refinement 3:

$$(3 \sim 5) \succ (1 \sim 4 \sim 7) \succ (2 \sim 6).$$

In this paper individual preferences will be represented as *sequences* (see below): the preference $(3 \sim 5) \succ (1 \sim 4 \sim 7) \succ (2 \sim 6)$ is identified to the sequence

$$2\ 3\ 1\ 2\ 1\ 3\ 2\ ,$$

listing the *ranks* given to each alternative: alternative 3 and 5 are ranked first, therefore the third and fifth element of the sequence is 1, etc. A preference is represented as a sequence $\sigma\ :\ \sigma(1)\ \sigma(2)\ \ldots\ \sigma(n)$. The integer $\sigma(i) \in \{1, 2, \ldots, \ell\}$ is the rank the voter gives to alternative i.

Sequences

Take an integer ℓ and let $(e_1, e_2, \cdots, e_\ell)$ be strictly positive integers such that

$$e_1 + e_2 + \cdots + e_\ell = n\ .$$

Consider the sequences

$$\sigma\ :\ \sigma(1)\ \sigma(2)\ \ldots\ \sigma(n)\ ,$$

where for all i, $1 \leq i \leq n$, $\sigma(i) \in \{1, 2, \ldots, \ell\}$ and σ reaches the value m exactly e_m times, $1 \leq m \leq \ell$. The sequence σ is said to be of *refinement* ℓ and *specification* $(e_1, e_2, \ldots, e_\ell)$ (for technical points see Carlitz, 1972).

Let $\mathcal{S}_n^\ell$ denote the set of sequences of refinement ℓ and $\mathcal{S}_n^{\leq \ell} = \bigcup_{m=1}^{m=\ell} \mathcal{S}_n^m$ the set of sequences of refinement at most ℓ; $\mathcal{S}_n = \mathcal{S}_n^{\leq n}$ is the set of all sequences on A_n.

A sequence σ of refinement ℓ is merely a surjective mapping from A_n into $\{1, \ldots, \ell\}$. Then the number of sequences of refinement exactly ℓ on a set of n elements is the number of such surjective mappings: it is well known that $\sharp\mathcal{S}_n^\ell = l! S_{n,\ell}$, where $S_{n,\ell}$ the Stirling number of the second kind at entry (n, ℓ). Notice that sequences are also permutations of the multiset

$$\{1^{e_1}, 2^{e_2}, \ldots, \ell^{e_\ell}\} = \{ \underbrace{1, \ldots, 1}_{e_1 \text{ times}}, \underbrace{2, \ldots, 2}_{e_2 \text{ times}}, \ldots, \underbrace{\ell, \ldots, \ell}_{e_\ell \text{ times}} \}.$$

2.2 The voting procedure

Let σ be a sequence of refinement ℓ; denote $\mathcal{S}$ the domain of individual preferences (it is the set $\mathcal{S}_n$ of all sequences if there is no domain restriction). Let m_σ denote the number of voters whose preference is σ. Then $m = \sum_{\sigma \in \mathcal{S}} m_\sigma$ is the total number of voters and $\lambda_\sigma = m_\sigma / m$ is the proportion of voters whose preference is σ. The parameter space considered in this paper in the $(b_\mathcal{S} - 1)$–simplex (where $b_\mathcal{S} = \sharp\mathcal{S}$, the cardinal of $\mathcal{S}$) of *profiles*:

$$\Delta_\mathcal{S} = \left\{ (\lambda_\sigma)_{\sigma \in \mathcal{S}} \mid \sum_{\sigma \in \mathcal{S}} \lambda_\sigma = 1 \right\} .$$

Any profile of individual preferences can be represented by a point in $\Delta_\mathcal{S}$. Conversely, any point in $\Delta_\mathcal{S}$ can represent a profile of individual preferences provided the number of agents is allowed to tend to infinity or even be infinite. This makes $\Delta_\mathcal{S}$ a suitable parameter space.

Define the two following characteristic functions on the pairs (i, j) of alternatives, $i \neq j$:

$$\chi_\sigma(i, j) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_\sigma(i, j) = \begin{cases} 1 & \text{if } \sigma(i) = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

It is obviously the case that for all σ and pair (i, j), one has:

$$\chi_\sigma(i,j) + \chi_\sigma(j,i) + \delta_\sigma(i,j) = 1.$$

When comparing the pair of alternatives (i, j) through the voting procedure, the proportion of the voting population which prefers alternative i to alternative j, and then votes for i against j, is

$$\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \chi_\sigma(i,j).$$

Definition 1 *Consider the super majority rule with rate τ, $0.5 \leq \tau < 1$. Alternative i is collectively preferred to alternative j (denoted $i \succ j$) for the super majority rule τ if and only if*

$$\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \chi_\sigma(i,j) > \mu \sum_{\sigma \in \mathcal{S}} \lambda_\sigma \chi_\sigma(j,i),$$

where $\mu = \dfrac{\tau}{1-\tau}$.

Voters who tie i and j abstain from voting. Alternative i defeats alternative j if it rallies a proportion higher than τ of the *expressed votes*. The simple majority rule, $\tau = 0.5$ corresponds to $\mu = 1$. The majority rule will be called indifferently τ or μ, with $\tau = \dfrac{\mu}{1+\mu}$.

Lemma 1 *Alternative i defeats alternative j for the majority rule μ if and only if*

$$\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \left(1 - \chi_\sigma(i,j) + \mu \chi_\sigma(j,i)\right) < 1 \ . \quad (1)$$

Proof: Straightforwardly obtained from the inequality of definition 1 by adding the quantity $\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \left(1 - \chi_\sigma(i,j)\right)$ to both sides. $\square$

2.3 Cycles of alternatives

The voting procedure defines a binary relation $(\succ)$ on the set of alternatives. A q-cycle of alternatives, or cycle of length q, for the binary relation $(\succ)$ is an ordered q-tuple $a = (a_1, a_2, \ldots, a_q)$ of the set A_n, that satisfy

$$a_1 \succ a_2 \succ \ldots \succ a_q \succ a_1 \ .$$

A cycle of alternatives is clearly defined up to a circular permutation of the q-tuple. Therefore, a cycle for $(\succ)$ can be identified with an equivalence class of ordered sets, two ordered sets being equivalent if one is obtained from the other by a circular permutation.

Inequality (1) is applied to get a necessary condition for the existence of cycles. Define $a_{q+1} = a_1$.

Lemma 2 *The inequality*

$$\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \zeta_\sigma(a) < q \; , \tag{2}$$

where

$$\zeta_\sigma(a) = q - \underbrace{\sum_{i=1}^{q} \chi_\sigma(a_i, a_{i+1})}_{s_\sigma(a)} + \mu \underbrace{\sum_{i=1}^{q} \chi_\sigma(a_{i+1}, a_i)}_{t_\sigma(a)} \; ,$$

is a necessary condition for the occurrence of the cycle a.

Proof: Obtained readily by adding the q inequalities (1) (one for each pair (a_i, a_{i+1})) defining the cycle a. □

Remark: The quantity $s_\sigma(a)$ measures how many of the q pairwise comparisons defining the cycle a the sequence σ strongly agrees with. The quantity $t_\sigma(a)$ measures how many times the sequence σ strongly disagrees with the cycle. Hence $\zeta_\sigma(a)$ can be interpreted as a sort of *distance* of the preference σ to the cycle.

3 The $\left(1 - \frac{1}{\ell}\right)$–super majority rule

It is shown in this section that, if the refinement of preferences does not exceed ℓ, a super majority rule with rate $1 - 1/\ell$ is necessary and sufficient to rule out Condorcet cycles of any length (see Corollary 1 below). Hence the key factor to rule out cycles is not the number of alternatives, but the maximal degree of refinement of the preferences expressed in the society. This will be done by proving the following result.

Proposition 1 *When the refinement of individual preferences does not exceed ℓ, the super majority rule with rate*

$$1 - \frac{1}{\min\,(q, \ell)}$$

is necessary and sufficient to rule out Condorcet cycles of length q.

This result is well known in the case where there are only fine individual preferences, in which case $\ell = n$ and $\min\,(q, \ell) = q$ (cf. Ferejohn and Grether, 1974). Two immediate corollaries obtain.

Corollary 1 *When the refinement of individual preferences does not exceed ℓ, the super majority rule with rate $1 - 1/\ell$ is necessary and sufficient to rule out Condorcet cycles of any length.*

Corollary 2 *When the refinement of individual preferences does not exceed* 2 *(individual preferences are dichotomous), there is no Condorcet cycles through the simple majority rule.*

This last corollary is identical to the result of aggregation obtained with *approval voting* (Brams and Fishburn, 1983). The difference is that approval voting is a mechanism that forces voters to express only dichotomous preferences.

Before proving Proposition 1, it is useful to introduce preliminary comments and notation. The strong properties of symmetry of the setup allows us, without loss of generality, to the study to the cycle $\iota = (1, 2, \ldots, q)$: only the length matters (see Balasko and Crès, lemma 7). Let us introduce a useful concept. In the sequence $\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n)$, a pair of consecutive elements $(\sigma(i), \sigma(i+1))$ is called

1. a *rise* if $\sigma(i) < \sigma(i+1)$ (we denote it $\sigma(i) \uparrow \sigma(i+1)$),
2. a *fall* if $\sigma(i) > \sigma(i+1)$ (we denote it $\sigma(i) \downarrow \sigma(i+1)$),
3. a *level* if $\sigma(i) = \sigma(i+1)$ (we denote it $\sigma(i) \rightarrow \sigma(i+1)$).

If s, t, and u denote the number of rises, falls and levels in σ respectively, then $s+t+u = n-1$, and the following obvious lemma holds.

Lemma 3 *Let σ be a sequence of length n. Let $\bar{s}_q$, $\bar{t}_q$ and $\bar{u}_q$ denote respectively the numbers of rises, falls and levels of the sequence of length $q+1$:*

$$\bar{\sigma} : \quad \sigma(1) \ldots \sigma(q)\sigma(1) \ .$$

Then $\bar{s}_q + \bar{t}_q + \bar{u}_q = q$ and

$$t_\sigma(\iota) = \bar{t}_q \qquad \text{and} \qquad s_\sigma(\iota) = \bar{s}_q \ .$$

Proof of Proposition 1: Consider the case $\ell < q$; for the other case, cf. Balasko and Crès (1997), lemma 13. To prove that the $\left(1 - \dfrac{1}{\ell}\right)$-super majority rule is sufficient to rule out Condorcet cycles, it is sufficient to show that any preference σ of refinement smaller or equal to ℓ satisfies: $\zeta_\sigma(\iota) \geq q$ when $\mu \geq \ell - 1$. Indeed, then

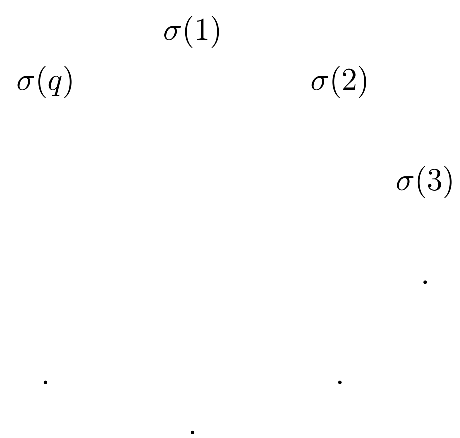
$$\sum_{\sigma \in \mathcal{S}} \lambda_\sigma \zeta_\sigma(\iota) \geq q \sum_{\sigma \in \mathcal{S}} \lambda_\sigma = q \ ,$$

an inequality that rules out the cycle ι as lemma 2 states.

For $\mu \geq \ell - 1$, one has $\zeta_\sigma(\iota) \geq (q - \bar{s}_q) + (\ell - 1)\bar{t}_q$. Let us first cope with the case where $\bar{u}_q = q$, $\bar{s}_q = \bar{t}_q = 0$, which means that the preference σ is totally indecisive between the alternatives $\{1, \ldots, q\}$. Then $\zeta_\sigma(\iota) = q$ and we are home. Otherwise, then $\bar{s}_q$ and $\bar{t}_q$ are both bigger or equal to 1 (since, in the sequence $\bar{\sigma}$, we depart from and arrive to $\sigma(1)$, if there is a rise, there must be a fall, and reciprocally). And a sufficient condition for the property to hold is $(q - \bar{s}_q) + (\ell - 1)\bar{t}_q \geq q$ which is equivalent to proving that

$$\frac{\bar{s}_q}{\bar{t}_q} \leq \ell - 1 \ .$$

Obviously, if this last inequality does not hold, the relative proportions of rises versus falls is higher than $\ell - 1$, then one would have, along the loop



at least one series of ℓ consecutive rises (ignoring levels); by consecutive is meant: uninterrupted by any fall. This would entail that the refinement of the preference σ is bigger or equal to $\ell + 1$, which cannot be. Hence the sufficiency.

As far as necessity is concerned, the standard example of a profile composed with ℓ circular individual preferences la Condorcet immediately allows to conclude. □

4 The probability of Condorcet cycles

This section provides rates of super majority ruling out Condorcet cycles 'in probability', i.e., rates of super majority for which the relative volume, in the simplex $\Delta_\mathcal{S}$, of profiles giving rise to Condorcet cycles is very small. A probabilistic interpretation of the Lebesgue measure on the simplex of profiles can be constructed: it is the classical Impartial Anonymous Culture (IAC) for an infinite number of agents; see Gehrlein (1997). In this section, wherever an interpretation in probabilistic terms is given, this culture (or any other resulting in a probability distribution that is absolutely continuous with respect to the Lebesgue measure) is implicitly assumed. It is argued that the coarser the individual preferences, the lower the required rates.

4.1 An upper bound

Only an upper bound of these volumes will be provided here, and a very brutal one: it is the volume of the set described by inequality[1] (2). But yet an additional argument is then given that reinforces the idea that the coarser the preferences, the easier and surer it is to aggregate them through majority voting.

The key feature to measure the relative volume of the set of profiles (λ_σ) such that $\sum_{\sigma\in\mathcal{S}} \lambda_\sigma \zeta_\sigma(\iota) < q$ is to rewrite this inequality by aggregating the weights λ_σ having the same coefficient $\zeta_\sigma(\iota)$; the dimension of the simplex is thus reduced. To do that, let us partition the set of individual preferences $\mathcal{S}$ by means of the characteristic function $\zeta_\sigma(\iota)$, into the subsets:

$$\mathcal{B}_{\mathcal{S}}(r,t;q) = \{\sigma \in \mathcal{S} \mid \zeta_\sigma(\iota) = r + \mu t\},$$

where $(r,t) \in \{q,0\} \times \{(r,t) \mid 1 \leq t \leq r \leq q-1\}$, therefore defining $1 + \dfrac{q(q-1)}{2}$ such subsets. Hence the new simplex of aggregated weights is of dimension $\dfrac{q(q-1)}{2}$, and is endowed with this measure that is the projection of the Lebesgue measure on the original simplex.

The leading example: $n = 3$. The only cycles are of length 3, so that $q = 3$ is omitted. The domain $\mathcal{S}$ contains 13 individual preferences: the six fine preferences

$$(1):\ 123 \quad (2):\ 132 \quad (3):\ 213 \quad (4):\ 231 \quad (5):\ 312 \quad (6):\ 321\ ,$$

then six dichotomous:

$$(7):\ 112 \quad (8):\ 121 \quad (9):\ 211 \quad (10):\ 221 \quad (11):\ 212 \quad (12):\ 122\ ,$$

and the totally indecisive one, $(13):\ 111$. The partition of $\mathcal{S}$ through the characteristic function ζ gives four subsets:

$$\mathcal{B}_{\mathcal{S}}(1,1) = \{(1),(4),(5)\} \qquad \mathcal{B}_{\mathcal{S}}(2,2) = \{(2),(3),(6)\}$$

$$\mathcal{B}_{\mathcal{S}}(2,1) = \{(7),(8),(9),(10),(11),(12)\} \qquad \mathcal{B}_{\mathcal{S}}(3,0) = \{(13)\}\ .$$

[1]This inequality is obtained by addition of the q inequalities defining the cycle, therefore a lot of information is lost in the process.

Back to the general case, let $b_{\mathcal{S}}(r,t;q)$ be the cardinal of $\mathcal{B}_{\mathcal{S}}(r,t;q)$; $b_{\mathcal{S}} = \sum_{r,t} b_{\mathcal{S}}(r,t;q)$ be the total number of individual preferences. Define finally $\bar{b}_{\mathcal{S}}(r,t;q) = \dfrac{b_{\mathcal{S}}(r,t;q)}{b_{\mathcal{S}}}$.

Proposition 2 *Fix the q-cycle* $\iota = (1, \ldots q)$. *Define the continuous function* $F_{\mathcal{S}}(q;\mu)$ *from* $\mathbf{N} \times \mathbf{R}_+$ *into* $\mathbf{R}_+$*:*

$$F_{\mathcal{S}}(q;\mu) = \prod_{r,t} \left(\frac{r + \mu t}{q} \right)^{\bar{b}_{\mathcal{S}}(r,t;q)} .$$

The quantity

$$\left(\frac{1}{F_{\mathcal{S}}(q,\mu)} \right)^{b_{\mathcal{S}}}$$

is an upper bound of the relative Lebesgue measure, in the $(b_{\mathcal{S}} - 1)$*-simplex, of the set of profiles giving rise to the cycle* ι.

Proof: Standard argument reproduced from Balasko and Crès (1997), proposition (15). □

$F_{\mathcal{S}}(q;\mu)$ is an increasing function of the rate μ and $\lim_{\mu\to\infty} F_{\mathcal{S}}(q;\mu) = +\infty$ readily obtains. Furthermore, $F_{\mathcal{S}}(q;0) \leq 1$ comes from $r \leq q$. As soon as $F_{\mathcal{S}}(q;\mu) > 1$, proposition 2 guarantees, for a big enough number of admissible preferences $b_{\mathcal{S}}$, the volume of 'bad' profiles to be very small (even when multiplied by the number of possible cycles).

Definition 2 *The function* $F_{\mathcal{S}}(q;\cdot)$ *is the* characteristic function *of the set of preferences* $\mathcal{S}$ *with respect to q-cycles,* $3 \leq q \leq n$. *The (unique) positive real* $\mu_{\mathcal{S}}(q)$ *such that* $F_{\mathcal{S}}(q;\mu_{\mathcal{S}}(q)) = 1$ *is the* critical rate *of the set of preferences* $\mathcal{S}$ *with respect to q-cycles.*

It is said critical because it is a threshold: for $\mu > \mu_{\mathcal{S}}(q)$, the probability of the cycle ι is very small as soon as there are enough preferences in $\mathcal{S}$. In the case of the leading example ($n = 3$), the characteristic function is

$$F_{\mathcal{S}}(\mu) = \left[\left(\frac{1+\mu}{3} \right)^3 \left(\frac{2(1+\mu)}{3} \right)^3 \left(\frac{2+\mu}{3} \right)^6 \right]^{\frac{1}{13}} ,$$

and the critical value is $\mu = 1.0718$, corresponding to a super majority rule with rate $\tau = 51.73\%$.

The sequel aims at showing that *the coarser the individual preferences, the lower the critical rate.* Therefore, if individual preferences are coarse, one needs a lower rate of super majority than for fine preferences to rule out cycles in probability.

4.2 Computations of the critical rate of $\mathcal{S}$

The present subsection is rather technical and can be skipped at first reading. It aims at showing that the whole problem reduces to compute the critical rates, with respect to q-cycles, of the set of preferences $\mathcal{S}_q^\ell$ (of refinement ℓ on q alternatives). The argument goes through two steps:

1. by showing that one can study the behavior of the critical rate of $\mathcal{S}$, $\mu_{\mathcal{S}}$, through the behavior of the critical rates of any partition of $\mathcal{S}$: this is done in lemma 4; and the chosen partition is $(\mathcal{S}_n^\ell)_n$ whose critical rates with respect to q-cycles are denoted $(\mu_n^\ell(q))_n$;

2. then by showing that one can study the behavior the $(\mu_n^\ell(q))_{n,q}$ through the behavior of the 'diagonal' critical rates $\mu_q^\ell(q)$ (denoted μ_q^ℓ): this is done in corollary 3.

The following lemma allows us to focus only on a well-chosen partition of $\mathcal{S}$, namely the subsets $(\mathcal{S}_n^\ell)_n$ of preferences of refinement exactly ℓ.

Lemma 4 *Consider two subsets of preferences $\mathcal{S}$ and $\mathcal{S}'$, whose critical rates with respect to q-cycles are $\mu_{\mathcal{S}}(q)$ and $\mu_{\mathcal{S}'}(q)$ respectively, with $\mu_{\mathcal{S}}(q) \leq \mu_{\mathcal{S}'}(q)$. Let $\mu_{\mathcal{S}\cup\mathcal{S}'}(q)$ denote the critical rate, with respect to q-cycles, of the set of preferences $\mathcal{S}\cup\mathcal{S}'$. Then*

$$\mu_{\mathcal{S}}(q) \leq \mu_{\mathcal{S}\cup\mathcal{S}'}(q) \leq \mu_{\mathcal{S}'}(q).$$

Proof: Let $F_{\mathcal{S}}(q;\cdot)$, $F_{\mathcal{S}'}(q;\cdot)$ and $F_{\mathcal{S}\cup\mathcal{S}'}(q;\cdot)$ denote the characteristic functions, with respect to q-cycles, of the set of preferences $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}\cup\mathcal{S}'$ respectively. Then one has

$$F_{\mathcal{S}\cup\mathcal{S}'}(q;\mu) = [F_{\mathcal{S}}(q;\mu)]^{\frac{b_{\mathcal{S}}}{b_{\mathcal{S}}+b_{\mathcal{S}'}}}\,[F_{\mathcal{S}'}(q;\mu)]^{\frac{b_{\mathcal{S}'}}{b_{\mathcal{S}}+b_{\mathcal{S}'}}}.$$

We have

$$F_{\mathcal{S}\cup\mathcal{S}'}(q;\mu_{\mathcal{S}}(q)) = [F_{\mathcal{S}'}(q;\mu_{\mathcal{S}}(q))]^{\frac{b_{\mathcal{S}'}}{b_{\mathcal{S}}+b_{\mathcal{S}'}}}$$

which is ≤ 1 because $\mu_{\mathcal{S}}(q) \leq \mu_{\mathcal{S}'}(q)$ and $F_{\mathcal{S}'}(q;\dot{})$ is strictly increasing. Symmetrically, we have

$$F_{\mathcal{S}\cup\mathcal{S}'}(q;\mu_{\mathcal{S}'}(q)) = [F_{\mathcal{S}\cup\mathcal{S}'}(q;\mu_{\mathcal{S}'}(q))]^{\frac{b_{\mathcal{S}}}{b_{\mathcal{S}}+b_{\mathcal{S}'}}}$$

which is ≥ 1. Hence the result. □

Let us now focus on the set of preferences $\mathcal{S}_n^\ell$ of refinement exactly ℓ, whose critical rate with respect to is denoted $\mu_n^l(q)$. The corresponding partition through the function $\zeta_\sigma(\iota)$

will be denoted $(\mathcal{B}_n^\ell(r,t;q))_{r,t}$ and the related coefficient $(b_n^\ell(r,t;q))_{r,t}$; the total number of elements of $\mathcal{S}_n^\ell$ is b_n^ℓ. The following lemma and its corollary allow us to restrict the study to the cycles of full length: $n = q$.

Lemma 5 *We have:*

$$b_{n+1}^\ell(r,t;q) = \ell b_n^\ell(r,t;q) + \ell b_n^{\ell-1}(r,t;q) \ .$$

Proof: The proof goes as in Balasko and Crès (1997), lemma 8. Adding the $(n+1)$-th alternative has no effect on the property of a given individual preference σ with respect the studied cycle ι, i.e., with respect to the value of $s_\sigma(\iota)$ and $t_\sigma(\iota)$. On the other hand, an individual preference of $\mathcal{B}_n^\ell(r,t;q)$ defines ℓ preferences of $\mathcal{B}_{n+1}^\ell(r,t;q)$, as many as there are different subsets of tied alternatives, and an individual preference of $\mathcal{B}_n^{\ell-1}(r,t;q)$ defines ℓ preferences of $\mathcal{B}_{n+1}^\ell(r,t;q)$, as many as there are places between (and at both ends of) different subsets of tied alternatives. □

Corollary 3 *The critical rate $\mu_{n+1}^\ell(q)$ is contained between $\mu_n^{\ell-1}(q)$ and $\mu_n^\ell(q)$.*

Proof: The proof goes exactly as in lemma (4) and exploits the preceding proposition. □

Thanks to the results of this subsection, we know that we have all the required information about the rates $(\mu_n^\ell(q))_{\ell,n,q}$ (e.g., whether the sequence $(\mu_n^\ell(q))_\ell$ decreases with ℓ) as soon as we can compute the rates $(\mu_q^\ell(q))_{\ell,q}$; denoted from now on $(\mu_q^\ell)_{\ell,q}$.

4.3 Results on critical rates

The only critical rate that are computed are μ_q^q and μ_q^2, the rates of the sets $\mathcal{S}_q^q$ of fine and dichotomous individual preferences.

Proposition 3 *The critical rate of the set of complete preferences $\mathcal{S}_q^q$ with respect to q-cycles is*

$$\mu_q^q = V(q-1) - 1 \ ,$$

where[2] $V(q) = \dfrac{q+1}{1^{\frac{\left\langle {q \atop 1} \right\rangle}{q!}} 2^{\frac{\left\langle {q \atop 2} \right\rangle}{q!}} \ldots q^{\frac{\left\langle {q \atop q} \right\rangle}{q!}}}$.

[2]The quantity $\left\langle {q \atop p} \right\rangle$ is the Eulerian number at entry (q,p).

Proof: The preferences of $\mathcal{S}_q^q$ are fine; therefore, for $r \neq t$, $\bar{b}_q^q(r,t;q) = 0$. Then the characteristic function of $\mathcal{S}_q^q$ becomes

$$F_q^q(q;\mu) = \prod_{t=1}^{q-1} \left(\frac{t(1+\mu)}{q} \right)^{\bar{b}_q^q(t,t;q)} = \frac{(1+\mu)^{\bar{b}_q^q(q)}}{q} \prod_{t=1}^{q-1} t^{\bar{b}_q^q(t,t;q)},$$

where $\bar{b}_q^q(q) = 1$, and $\bar{b}_q^q(t,t;q) = \left\langle \begin{array}{c} q-1 \\ t \end{array} \right\rangle / (q-1)!$ thanks to proposition (11) in Balasko and Crès (1997). Then the characteristic function $F_q^q(q;\cdot)$ can be rewritten

$$F_q^q(q;\mu) = \frac{1+\mu}{V(q-1)};$$

hence the result. $\square$

Proposition 4 *For all $n \geq q$, the critical rate with respect to q-cycles of the set $\mathcal{S}_n^2$ of preferences of refinement 2 is 1:*

$$\mu_n^2(q) = 1.$$

Proof. This is an immediate corollary of proposition 2. $\square$

The following table gives the values of μ_q^ℓ, and corresponding rate τ_q^ℓ, for $3 \leq q \leq 6$ and $3 \leq \ell \leq q$.

$q\backslash l$	2	3	4	5	6
3	1 (50)	1.1213 (52.8)			
4	1 (50)	1.0529 (51.3)	1.0982 (52.3)		
5	1 (50)	1.0327 (50.8)	1.056 (51.3)	1.076 (51.8)	
6	1 (50)	1.0237 (50.6)	1.0395 (51)	1.0516 (51.2)	1.0618 (51.5)

These first values suggest the following conjecture.

Conjecture: *For $\ell \leq q$, $\mu_q^{\ell-1} \leq \mu_q^\ell$*

Hence the coarser the individual preferences, the lower the rate of super-majority necessary to rule them out in probability. We can prove the conjecture for $\ell = q$.

Proposition 5 *For all q s. t.*[3] $V(q) < \sqrt{1 + \frac{2}{q-1}}$ *, then $\mu_q^{q-1} \leq \mu_q^q$.*

Proof: See Appendix. $\square$

[3]This is a stronger property than the one obtained by Balasko and Crès (1997), theorem 16. This is nevertheless checked on computers up to $q = 1000$, which is way above the highest values that are interesting in the framework of this paper. These computations suggest that in fact $V(q)$ can be approximated by $2 + 1/(3q) - 5/(36q^2)$.

4.4 Probabilities of pairwise comparisons

The major weakness of the results provided in this section was raised by Tovey (1997): a rate of super majority, along with ruling out cycles, makes it also highly improbable that two alternatives can be compared. In fact, one should compare the probability of a cycle with the probability of a pairwise comparison.

Here is provided the relative measure of the subset of parameters for which alternatives i and j are comparable through simple majority voting, i.e., the probability that the pairwise comparisons $i \succ j$ or $j \succ i$ occur. Define $b_{\mathcal{S}}^{i \succ j} = \sum_{\sigma \in \mathcal{S}} \chi_\sigma(i,j)$ (resp. $b_{\mathcal{S}}^{i \sim j} = \sum_{\sigma \in \mathcal{S}} \delta_\sigma(i,j)$), i.e. the number of preferences in $\mathcal{S}$ for which alternative i is better than (resp. as good as) alternative j. Of course, for all (i,j),

$$b_{\mathcal{S}}^{i \succ j} + b_{\mathcal{S}}^{i \sim j} + b_{\mathcal{S}}^{j \succ i} = b_{\mathcal{S}} \ .$$

Proposition 6 *The relative Lebesgue measure, in $\Delta_{\mathcal{S}}$, of the subset of parameters for which there is a pairwise comparison between i and j through simple majority voting is*

$$2 \left(\frac{1}{\mu+1} \right)^{b_{\mathcal{S}}^{i \succ j} + b_{\mathcal{S}}^{j \succ i} - 1} \sum_{k=1}^{b_{\mathcal{S}}^{j \succ i}} \begin{pmatrix} b_{\mathcal{S}}^{i \succ j} + b_{\mathcal{S}}^{j \succ i} - 1 \\ b_{\mathcal{S}}^{j \succ i} - k \end{pmatrix} \mu^{b_{\mathcal{S}}^{j \succ i} - k} \ .$$

Proof: Following the principles exposed in Balasko and Crès (1997, 1998), the relative volume of profiles for which $i \succ j$ is:

$$(b_{\mathcal{S}} - 1)! \int_{t=0}^{\frac{1}{\mu+1}} \frac{t^{b_{\mathcal{S}}^{i \sim j} - 1}}{(b_{\mathcal{S}}^{i \sim j} - 1)!} \frac{(1-t)^{b_{\mathcal{S}}^{i \succ j} - 1}}{(b_{\mathcal{S}}^{i \succ j} - 1)!} \, dt \ .$$

It is the same for the profiles for which $j \succ i$, hence the coefficient 2. □

The important fact is that this probability does not depend on $b_{\mathcal{S}}^{i \sim j}$, the number of undecided preferences on the pair (i,j). For values of μ close to 1 and small values of $b_{\mathcal{S}}^{i \succ j} + b_{\mathcal{S}}^{j \succ i}$, it remains non-trivial, whereas the probability of a cycle involving i and j is shown to be very small when $b_{\mathcal{S}}^{i \sim j}$ is big.

Example: Consider $n = 3$ and assume that dichotomous preferences are 9 times more frequent than fine preferences (it amounts to assume the impartial anonymous culture with an initial urn where dichotomous preferences are 9 times more likely to be chosen than fine preferences); then the characteristic rate is $\tau = 50.4\%$, and for a super majority rule with rate 53.5%, the upper bound gives 0.066 for the probability of a cycle (the actual probability is certainly much smaller), whereas the probability of a pairwise comparison is 0.65.

For a fixed super majority rule, the probability of a pairwise comparison between two alternatives increases toward 1 when the number of preferences exhibiting a strong ranking between these two alternatives decreases. Hence the coarser the individual preferences, the more likely pairwise comparisons, for a fixed super majority rule.

5 Concluding comments

The paper gives several arguments tending to show how the coarser the individual preferences, the surer it is that simple or super majority voting rules can aggregate them.

The approach chosen in Section 4 is probabilistic and based on the assumption of impartial anonymous culture (i.e. the Lebesgue measure on the simplex of profiles). A simple artifact allows to include much more distributions on the simplex of profiles: simply by allowing the domain $\mathcal{S}$ to include the same individual preference more than once[4].

As far as the logic of Section 3 is concerned (condition securing the total elimination of cycles), another line could be investigated: this is the impact of the assumption that profiles of preferences contain an irreducible weight of voters with coarse preference. This can be simply illustrated by the leading example where $n = 3$. Denote $\Lambda_1 = \lambda_1 + \lambda_4 + \lambda_5$, $\Lambda_2 = \lambda_2 + \lambda_3 + \lambda_6$ and $\Lambda_3 = \sum_{i=7}^{13} \lambda_i$; one easily gets, as a necessary condition for the cycle $(1, 2, 3)$ to occur:

$$1 + \Lambda_1 > \mu(1 + \Lambda_2) \ . \qquad (3)$$

Under the assumption that at least a ratio ν of the voting population exhibits coarse preferences (which translates in $\Lambda_3 \geq \nu \iff \Lambda_1 + \Lambda_2 < 1 - \nu$), then inequality (3) is impossible if $\mu \geq 2 - \nu$. One of course gets back the result that for dichotomous preferences (i.e., $\nu = 1$) cycles do not occur under the simple majority rule. But moreover

[4]In the example at the end of Section 4, where $n = 3$, $\mathcal{S}$ contains 60 preferences: the 6 fine ones and 9 times the 6 dichotomous ones; the Lebesgue measure on the simplex of dimension 60 then projects into a measure on the usual simplex of dimension 12 with a heavier density on the dichotomous preferences:

$$35! \prod_{i=7}^{12} \frac{\lambda_i^8}{8!} \ .$$

See Balasko and Crès (1997) for details.

if at least half of the voting population exhibits coarse preferences (i.e., $\nu = 0.5$), then $\tau = 60\%$ is enough to exclude cycle, instead of 66% in the general case.

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Appendix

Proposition 7 *For all q and l such that $\ell \leq q$, then we have*

$$F_q^{\ell-1}(1) \geq F_q^{\ell}(1) .$$

It is a step toward proving the conjecture since if, for example, $F_q^{\ell-1}(1) \geq F_q^{\ell}(1)$, and the two curves do not cross too early, then $F_q^{\ell-1}$ crosses the value 1 before F_q^{ℓ} .

Proof: Fix k, $1 \leq k \leq \ell$, $\ell < q$, and consider the two sets $\mathcal{S}$ and $\mathcal{S}'$ of sequences of respective specification $(e_1, \ldots, e_\ell)$ and $(e_1, \ldots, e_{k-1}, e_k - 1, 1, e_{k+1}, \ldots, e_\ell)$, $e_1 + \ldots + e_\ell = q$.

Take a sequence σ of $\mathcal{S}$; it contains e_k times the integer k: $\sigma^{-1}(k) = \{i_1, \ldots, i_{e_k}\}$ is a subset of $\{1, \ldots, \ell\}$. If we replace, in σ, *the* integer $\sigma(i_1) = k$ by the integer $k+1$, and then *all* the integers k', $k+1 \leq k' \leq \ell$, by their immediate successor $k'+1$, then we obtain a sequence of $\mathcal{S}'$: denote it $\hat{\sigma}_{i_1}$. Such a process associates with each sequence of $\mathcal{S}$ e_k sequences of $\mathcal{S}'$. We get then a correspondence Γ_k from $\mathcal{S}$ into $\mathcal{S}'$:

$$\begin{array}{rcccc} \Gamma_k & : & \mathcal{S} & \longrightarrow & \mathcal{S}' \\ & & \sigma & \longrightarrow & \{\hat{\sigma}_{i_1}, \ldots, \hat{\sigma}_{i_l}\} \end{array}$$

It is easy to check that all sequences of $\mathcal{S}'$ can be constructed from the right sequence of $\mathcal{S}$ through this operation. Recall that $\sharp\mathcal{S}' = q!/e_1! \ldots (e_k-1)! \ldots e_\ell!$ and $\sharp\mathcal{S} = q!/e_1! \ldots e_\ell!$ which entails $\sharp\mathcal{S}' = e_k \times \sharp\mathcal{S}$.

We want to look at the effect of such transformations through Γ_k on the characteristic functions of $\mathcal{S}$ and $\mathcal{S}'$ respectively. We need compare the respective values of $\zeta_\sigma(\iota)$ and $\zeta_{\hat{\sigma}_{i_1}}(\iota)$. To do so, thanks to lemma (3) we only have to check how Γ changes the properties of $\bar{\sigma}$ in terms of rises, falls and levels.

Consider an integer $i \in \sigma^{-1}(k)$, σ being an element of the set $B_{\mathcal{S}}(r, p; q)$. There can be nine configurations in terms of rises, falls and levels around $\sigma(i) = k$. We only care about those configurations that are changed by the operation Γ_k described above. They are:

1. $\sigma(i-1) \rightarrow \sigma(i) (= k) \rightarrow \sigma(i+1)$ which becomes $\hat{\sigma}_i(i-1) \uparrow \hat{\sigma}_i(i) (= k+1) \downarrow \hat{\sigma}_i(i+1)$; then $\hat{\sigma}_i \in B_{\mathcal{S}'}(r-1, t+1; q)$,
2. $\sigma(i-1) \rightarrow \sigma(i) \uparrow \sigma(i+1)$ which becomes $\hat{\sigma}_i(i-1) \uparrow \hat{\sigma}_i(i) \uparrow \hat{\sigma}_i(i+1)$; then $\hat{\sigma}_i \in B_{\mathcal{S}'}(r-1, t; q)$,
3. $\sigma(i-1) \rightarrow \sigma(i) \downarrow \sigma(i+1)$ which becomes $\hat{\sigma}_i(i-1) \uparrow \hat{\sigma}_i(i) \downarrow \hat{\sigma}_i(i+1)$; then $\hat{\sigma}_i \in B_{\mathcal{S}'}(r-1, t; q)$,
4. $\sigma(i-1) \uparrow \sigma(i) \rightarrow \sigma(i+1)$ which becomes $\hat{\sigma}_i(i-1) \uparrow \hat{\sigma}_i(i) \downarrow \hat{\sigma}_i(i+1)$; then $\hat{\sigma}_i \in B_{\mathcal{S}'}(r, t+1; q)$,
5. $\sigma(i-1) \downarrow \sigma(i) \rightarrow \sigma(i+1)$ which becomes $\hat{\sigma}_i(i-1) \downarrow \hat{\sigma}_i(i) \downarrow \hat{\sigma}_i(i+1)$; then $\hat{\sigma}_i \in B_{\mathcal{S}'}(r, t+1; q)$,

Denote $\hat{\mathcal{S}}_i$ the set $\mathcal{S}$ after transformation of the sequence σ as described above. Then we have:

- $F_{\hat{\mathcal{S}}_i} = F_{\mathcal{S}} \times \left(\frac{(r-1)+(t+1)\mu}{r+t\mu}\right)^{1/\sharp\mathcal{S}}$ in case 1,
- $F_{\hat{\mathcal{S}}_i} = F_{\mathcal{S}} \times \left(\frac{(r-1)+t\mu}{r+t\mu}\right)^{1/\sharp\mathcal{S}}$ in cases 2 and 3,
- $F_{\hat{\mathcal{S}}_i} = F_{\mathcal{S}} \times \left(\frac{r+(t+1)\mu}{r+t\mu}\right)^{1/\sharp\mathcal{S}}$ in cases 4 and 5.

It is clear that for $\mu = 1$, case 1 is neutral with respect to the characteristic function. In order to get the proposition, it is sufficient to prove that to each transformation of type 4 or 5 corresponds a transformation of type 2 or 3. Indeed, the association of a case of type 2 or 3 with a case of type 4 or 5 amounts to multiply the characteristic function $F_{\mathcal{S}}$ by

$$\left(\frac{(r+t\mu)^2 + (r+t\mu)(\mu-1) - \mu}{(r+t\mu)^2}\right)^{1/\sharp\mathcal{S}}$$

which is strictly smaller than 1 for $\mu = 1$. The final argument to get the proposition is that the full transformation of $\mathcal{S}$ through Γ_k gives e_k times $\mathcal{S}'$, which has the same characteristic function as $\mathcal{S}$.

Assume that case 4 arises, i.e., we start from a sequence $\sigma \in B_{\mathcal{S}}(r,p;q)$ such that $k = \sigma(i)$ whose environment is $\sigma(i-1) \uparrow \sigma(i) \to \sigma(i+1)$. It entails in particular that $\sigma(i+1) = k$. Consider the ordered set $\{i, i+1, \ldots, q, 1, 2, \ldots, i-1\}$. Reading this set from left to right, pick the first integer (call it j) such that $\sigma(j) \neq k$. That such a j exists is obvious, otherwise σ is not a surjection on $\{1, 2, \ldots, \ell\}$. We then know that the environment of $k = \sigma(j)$ is $\sigma(j-1) \to \sigma(j) \downarrow \sigma(j+1)$ or $\sigma(j-1) \to \sigma(j) \uparrow \sigma(j+1)$ depending on whether $\sigma(j+1)$ is smaller or bigger than k, meaning that case 2 or 3 automatically follows case 4. The same line of reasoning holds if case 5 arises. $\square$

Proof of proposition (5): We reproduce the preceding proof in the case where $e_j = 1$ for all $1 \leq j \neq k \leq q-1$ and $e_k = 2$. For this set, case 1 cannot occur, because an integer appears at most twice in a sequence. On top of that if σ is a sequence of $\mathcal{S}$ included in $B_{\mathcal{S}}(r,t;q)$, then necessarily, since there is at most one level, i.e., we have: $t \leq r \leq t+1$. Only when $r = t+1$ the transformation through Γ_k as described in the preceding proof is not neutral with respect to $F_{\mathcal{S}}$: we know that it then amounts to multiply $F_{\mathcal{S}}$ by the quantity

$$\left(\frac{t(t+1)(1+\mu)^2}{[(t+1)+t\mu]^2}\right)^{1/\sharp\mathcal{S}}.$$

Associate to $\sigma \in B_{\mathcal{S}}(t+1,t;q)$ the reverse sequence $Opp(\sigma):\ \sigma(n)\ldots\sigma(1)$. It has also one level and as many falls as σ has rises, i.e., $q-t-1$, so that we have $Opp(\sigma) \in B_{\mathcal{S}}(q-t-1, q-t; q)$. And the transformation Γ_k applied to $Opp(\sigma)$ amounts to multiply $F_{\mathcal{S}}$ by the quantity

$$\left(\frac{(q-t-1)(q-t)(1+\mu)^2}{[(q-t)+(q-t-1)\mu]^2}\right)^{1/\sharp\mathcal{S}}.$$

If we conjugate the transformations of σ and $Opp(\sigma)$, we obtain a multiplication by the quantity $[f_q(t;\mu)]^{1/\sharp\mathcal{S}}$, where the function f_q is given by:

$$f_q(x;\mu) = \frac{x(x+1)(q-x-1)(q-x)(1+\mu)^4}{[(x+1)+x\mu]^2[(q-x)+(q-x-1)\mu]^2}.$$

It can be easily proved that $f_q(\cdot;\mu)$ reaches its maximum for $x = (q-1)/2$, and that $f_q((q-1)/2;\mu)$ is smaller than 1 when $\mu \leq \sqrt{1+2/(q-1)}$. Indeed

$$f_q((q-1)/2;\mu) = \left(\frac{\sqrt{(q+1)(q-1)}(1+\mu)}{(q+1)+(q-1)\mu}\right)^4,$$

so that it is smaller than 1 if and only if $[\sqrt{(q+1)(q-1)} - (q-1)]\mu \leq (q+1) - \sqrt{(q+1)(q-1)}$ which is equivalent to $\sqrt{q-1}(\sqrt{q+1} - \sqrt{q-1})\mu \leq \sqrt{q+1}(\sqrt{q+1} - \sqrt{q-1})$ hence $\mu \leq \sqrt{\frac{q+1}{q-1}} = \sqrt{1+2/(q-1)}$.

The proposition holds true, first because the argument can be reproduced for all k, $1 \leq k \leq q-1$— we obtain the whole set $\mathcal{S}_q^q$ for each k—, and second because for $\mu = V(q-1) \leq \sqrt{1+2/(q-1)}$, $F_{\mathcal{S}_q^q}(q; V(q-1)) = 1$ and then $F_{\mathcal{S}_q^{q-1}}(q; V(q-1)) > 1$ which implies $\mu_q^{q-1} < \mu_q^q = V(q-1)$. $\square$