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The Probability of Condorcet Cycles and Super Majority Rules*

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Majority voting aggregates individual preference profiles into a binary relation on the set of alternatives. Condorcet cycles are cycles of the aggregated binary relation. We show that the relative volume of the subset of the $(n!-1)$-simplex that represents profile distributions such that the aggregated preferences display Condorcet cycles is a decreasing function of the super majority level τ bounded by the expression

$$n!\left(\frac{1-\tau}{0.4714}\right)^{n!}.$$

This expression shows that Condorcet cycles become rare events for super majority rules larger than 53%. *Journal of Economic Literature* Classification Number: D71.

1. INTRODUCTION

It is known since Condorcet that the aggregation of individual preference preorderings by the simple majority rule may yield cycles. Condorcet's original example involves three alternatives a, b, and c and three agents with the rankings $a \succ b \succ c$, $b \succ c \succ a$, and $c \succ a \succ b$. Pairwise comparisons of these alternatives under majority rule show that there is always a majority to prefer a to b, b to c, and c to a. Such a cycle occurs not only for the simple majority rule but, more generally, for any super majority

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rule τ smaller than 2/3. Condorcet's observation can easily be extended to n alternatives by considering the circular permutations of an arbitrary ordering of the n alternatives, which shows that cycles cannot be avoided by super majority rules smaller than $1-1/n$: see [9]. Since super majority rules are unable by themselves to prevent the occurrence of cycles, an alternative line of research that dates back to Arrow in [1] has been to weaken the unanimity condition. A continuity argument suggests that individual preferences that are not too far apart within the spectrum of all possible individual preference profiles must aggregate by majority voting into binary relations that display no cycles. Results by Black, Caplin and Nalebuff, Grandmont, Kramer, Plott, Tullock, in [3, 4, 8, 13, 15, 19] give sufficient conditions on individual preferences (e.g., the famous single peakedness condition and its multi-dimensional extensions) to guarantee that aggregation by simple or super majority voting does not lead to cycles. In contrast with these sufficient conditions of increasing levels of generality, other results show that the probability of observing cycles is close to one for many preference profiles and alternatives. Such a viewpoint is stressed in particular by Arrow and Raynaud in [2] following previous work by DeMeyer, Fishburn, Gehrlein, Guilbaud, Kelly, Maskin, May, and Plott in [5–7, 10, 11, 14]. Note, however, that almost all the latter results are proved under the assumption that "all alternatives are intrinsically equally favored" [14, loc.cit.], a significant restriction as we will see in a moment.

The research reported in this paper first defines a model where the two different lines of study that we have briefly described—the first approach placing restrictions on preferences; the second one assuming that all preferences are equally probable—can fit together. Our own research fits in between these two approaches by considering the distribution of individual preferences as a parameter of the aggregation problem and by studying the relationships between the properties of the aggregated preferences (defined by simple majority and super majority rules) and the distribution of individual preferences.[1] The second line is equivalent to imposing a fixed distribution of preference profiles (with a variable number of alternatives) while the first amounts to finding regions of the set of distribution profiles for which there are no Condorcet cycles.

An important feature of the parametric approach is to enable one to characterize sets of parameters for which a given property is satisfied. In our case, this becomes the characterization, i.e., the description, of preference distributions for which a given super majority rule yields Condorcet cycles. Unfortunately, such description becomes practically impossible as soon as

[1] When the final version of this paper was almost completed, we became aware that the same setup is described in the recent book [17] by Saari; nevertheless, there is no overlap between Saari's results and ours.

the number of alternatives becomes larger than 5. A weaker form of the characterization problem is therefore to study the (relative) volume or Lebesgue measure of the set of parameters that satisfy a given property. The volume is a poor substitute for a complete geometrical characterization of a set. Nevertheless, the information provided by the volume becomes almost sufficient in itself when the volume gets close to zero. We therefore estimate in this paper an upper bound $Y(n, \tau)$ of the relative volume $V(n, \tau)$ of the subset of the $(n!-1)$-simplex that consists of distributions for which the aggregated preferences for the τ-super majority rule display Condorcet cycles. We find an expression of the form

$$Y(n, \tau) = n! \left(\frac{1-\tau}{0.4714} \right)^{n!}.$$

For $\tau = 54\%$ and $n \geqslant 7$, for example, this implies $Y(n, \tau) < 10^{-52}$, an extremely small number indeed. This upper bound is certainly not the best one, but we did not place much effort in refining it given its remarkably low value when either the number of alternatives tends to infinity or the super majority level defined by τ increases above the threshold value of 53%. Within our setup, this makes the Condorcet cycles a theoretical curiosity without any practical bearing for super majority rules that exceed the threshold value of 53%.

The paper is organized as follows. The main assumptions and definitions are contained in section two, where we also introduce some tools of combinatorics. In section three, we define and state the main properties of the upper bound $Y(n, \tau)$ of the volume $V(n, \tau)$ and show how remarkably small this bound (and the volume henceforth) is as a function of the number of alternatives n and of the level τ of the super majority. The mathematical proof of the central theorem is given in the Appendix.

2. DEFINITIONS, ASSUMPTIONS, AND NOTATION

Consider the set $A_n = \{1, ..., n\}$ of n elements (identified with the n first natural integers) that represent n alternatives. Any permutation σ of the set A_n defines a complete ordering of these alternatives and can be identified with an individual preference relation. We therefore denote by $\mathfrak{S}_n$ the set of individual preference relations. The preferences of a collection of m agents are then defined by a map from the set $\{1, 2, ..., m\}$ into $\mathfrak{S}_n$. The set of such maps is finite, and therefore discrete. This makes it a poor candidate as a set of parameters because infinitesimal variations of the parameters are not possible. A better parameter space is obtained by considering the distribution of preferences among agents. Let us associate with every individual preference ordering the number of agents whose preferences are defined by that ordering. Since any majority rule is based on the relative proportions of voters, only the ratio of agents having a given ordering to the total number of agents matters. The distribution of individual preference ordering can then be represented by an element of the $(n!-1)$-simplex. Conversely, any element of the $(n!-1)$-simplex can be interpreted as representing a distribution of individual preferences provided the number of agents is allowed to tend to infinity. This makes the $(n!-1)$-simplex a continuum and, therefore, a parameter space suitable for the application of Calculus. Restrictive assumptions on individual preferences then translate into assumptions on the location of the distribution profile in the $(n!-1)$-simplex. For example, assuming an infinite number of agents, the commonly used assumption that "all alternatives are intrinsically equally favored," also known in the literature as "impartial culture," amounts to considering distribution profiles located at the center of the $(n!-1)$-simplex. Similarly, the assumption that all preferences are identical translates into distribution profiles located at one of the $n!$ vertices of the $(n!-1)$-simplex.

2.1. *Miscellanies about Preference Profiles, Binary Relations, and Permutations*

Cycles of Alternatives for an Arbitrary Binary Relation

Let us consider an arbitrary binary relation $\succ$ on the set of alternatives A_n. A cycle for the binary relation $\succ$ is an ordered collection of elements $\mathfrak{a} = (a_1, a_2, ..., a_q)$ that satisfy

$$a_1 \succ a_2 \succ \cdots \succ a_q \succ a_1.$$

The cycle $\mathfrak{a}$ is then said to have length q. The length of a cycle is necessarily $\geqslant 3$. Note that the image by a circular permutation of the ordered set $\mathfrak{a}$ represents the same cycle for the binary relation $\succ$. Therefore, a cycle of $\succ$ can be identified with an equivalence class of ordered sets, two ordered sets being equivalent if one is obtained from the other by a circular permutation, i.e., a permutation without subcycles.

The following lemma gives an upper bound of the number of cycles for an arbitrary binary relation on the set A_n of n alternatives.

LEMMA 1. *The total number of cycles for an arbitrary binary relation on the set A_n is $\leqslant n!$.*

Proof. The number of ordered q-tuples of the set A_n is equal to the product of $\binom{n}{q}$, the number of ways to choose q elements among n elements, and of the number $q!$ of permutations of the q elements. Since each equivalence class of cycles contains exactly q elements, the number of classes corresponding to the cycles of length q is therefore equal to

The number of different cycles of the set A_n is therefore equal to

$$(q-1)!\binom{n}{q}=\frac{n!}{q(n-q)!}.$$

$$\sum_{q=3}^{q=n}\frac{n!}{q(n-q)!}=n!\sum_{q=3}^{q=n}\frac{1}{q(n-q)!} \tag{1}$$

$$=n!\left(\frac{1}{3(n-3)!}+\cdots+\frac{1}{(n-2)\,2!}+\frac{1}{(n-1)\,1!}+\frac{1}{n}\right) \tag{2}$$

$$<n!\underbrace{\left(\frac{1}{n-1}+\cdots+\frac{1}{n-1}\right)}_{(n-2)\text{ times}}=n!\,\frac{n-2}{n-1}<n!. \tag{3}$$

2.2. *Readings and Eulerian Numbers*

In this subsection, we review some properties of permutations that will be needed in the next section. For further details, we refer the reader to [12, 16].

Readings

A permutation is said to require k *readings* if we must scan it k times from left to right in order to read off its elements in increasing order. For example, the permutation

$$(491825367)$$

requires four readings that are obtained as follows: the first reading consists of the elements 1, 2, and 3; the second reading consists of 4, 5, 6, and 7; the third and fourth readings consist of 8 and 9, respectively.

Eulerian Numbers

By definition, the number of permutations of P elements featuring r readings is the *Eulerian number* $\langle {p \atop r} \rangle$. By convention, one defines $\langle {0 \atop 0} \rangle = 1$ and $\langle {0 \atop r} \rangle = 0$ for $r \neq 0$.

The first values of the Eulerian numbers are shown in the following table:

p, r	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	20416	1191	120	1

The Eulerian numbers satisfy the following recurrence relation:

$$\left\langle {p \atop r} \right\rangle = r \left\langle {p-1 \atop r} \right\rangle + (p+1-r) \left\langle {p-1 \atop r-1} \right\rangle.$$

The Eulerian numbers satisfy the following almost obvious properties:

$$\left\langle {p \atop 0} \right\rangle = \left\langle {p \atop 1} \right\rangle + \cdots + \left\langle {p \atop p} \right\rangle = p! \tag{4}$$

$$\left\langle {p \atop r} \right\rangle = \left\langle {p \atop p+1-r} \right\rangle, \qquad p \geqslant 1. \tag{5}$$

2.3. *Preference, majority voting, and a partition of the* $\mathfrak{S}_n$

The Setup

Let σ be a permutation of the finite set of alternatives $A_n = \{1, ..., n\}$, a permutation that represents an individual preference profile $\succ_\sigma$, by which it is understood that we have the following ordering of alternatives:

$$\sigma(1) \succ_\sigma \sigma(2) \succ_\sigma \cdots \succ_\sigma \sigma(n).$$

In order to represent in a mathematically convenient way the outcome of aggregate pairwise comparisons of alternatives, we introduce the set

$$P_\sigma = \{(i, j) \in A_n \times A_n \mid i \succ_\sigma j, i \neq j\}$$

and we let χ_σ denote its characteristic function, i.e.,

$$\chi_\sigma(i, j) = \begin{cases} 1 & \text{for } \quad i \neq j \quad \text{ and } \quad i \succ_\sigma j \\ 0 & \text{otherwise.} \end{cases}$$

The following simple lemma will turn out to be useful later.

LEMMA 2.

$$\chi_\sigma(i, j) = \chi_{\mathrm{id}}(\sigma^{-1}(i), \sigma^{-1}(j)).$$

Proof. The statement of the lemma follows from $i \succ_\sigma j$ equivalent to $\sigma^{-1}(i) < \sigma^{-1}(j)$, itself equivalent to $\sigma^{-1}(i) \succ_{\mathrm{id}} \sigma^{-1}(j)$. ∎

Let m_σ denote the number of agents whose preference profile is σ, $m = \sum_{\sigma \in \mathfrak{S}_n} m_\sigma$ the total number of people, $\lambda_\sigma = m_\sigma/m$ the relative number of people whose preferences profile is $\sigma \in \mathfrak{S}_n$. The vector $\lambda = (\lambda_\sigma)_{\sigma \in \mathfrak{S}_n}$ in the $(n! - 1)$-simplex then represents the distribution of preference profiles.

When the alternatives i and j are compared, the total number of people that prefer i to j is equal to

$$\sum_{\sigma \in \mathfrak{S}_n} m_\sigma \chi_\sigma(i, j)$$

and their relative number is therefore equal to

$$\sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \chi_\sigma(i, j).$$

Let us introduce τ, the level of the super majority rule. The following lemma is obvious:

LEMMA 3. *The alternative i is preferred to the alternative j under the super majority rule τ if and only if*

$$\sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \chi_\sigma(i, j) > \tau. \tag{6}$$

This condition can readily be applied to yield a necessary condition for the existence of cycles. More specifically, let $\mathfrak{a} = (a_1, a_2, ..., a_q)$ be a cycle of length q for the binary relation defined on the set A_n by super majority voting. Define $a_{q+1} = a_1$. In the remaining part of this section, we are going to partition the set of individual preference profiles depending on how "close" they are to the cycle $\mathfrak{a}$. This will lead us to compute the number of elements of related sets. We will eventually show that the numbers of elements of the most important sets will not depend on the actual cycle $\mathfrak{a}$ but only on the length q of that cycle; these numbers will also be related to the Eulerian numbers that have already been introduced. With a fixed cycle of alternatives $\mathfrak{a}$, we have:

LEMMA 4. *The inequality*

$$\sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \left(\sum_{i=1}^{q} \chi_\sigma(a_{i+1}, a_i) \right) < q(1-\tau)$$

is satisfied for the cycle $\mathfrak{a} = (a_1, a_2, ..., a_q)$.

Proof. The addition of the q inequalities implied by Lemma 3, namely

$$\sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \chi_\sigma(a_i, a_{i+1}) > \tau$$

for $i = 1, ..., q$, yields the inequality

$$\sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \left(\sum_{i=1}^{q} \chi_\sigma(a_i, a_{i+1}) \right) > q\tau.$$

We also have the equality

$$q = \sum_{\sigma \in \mathfrak{S}_n} \lambda_\sigma \left(\sum_{i=1}^{q} \underbrace{\chi_\sigma(a_i, a_{i+1}) + \chi_\sigma(a_{i+1}, a_i)}_{=1} \right)$$

that follows readily from the equality to 1 of the sum $\chi_\sigma(a_i, a_{i+1}) + \chi_\sigma(a_{i+1}, a_i)$. Subtracting from this equality the above inequality yields the inequality of the lemma. ∎

It follows from Lemma 4 that it will be helpful to have the following integer valued function associated with every ordered subset $\mathfrak{a}$ of A_n:

$$X_\sigma(\mathfrak{a}) = \sum_{i=1}^{q} \chi_\sigma(a_{i+1}, a_i)$$

with the convention $a_{q+1} = a_1$. Though the function $X_\sigma(\mathfrak{a})$ is defined for any subset $\mathfrak{a}$ of A_n, its definition is related to the interpretation of these subsets as cycles.

LEMMA 5. *The expression* $X_\sigma(\mathfrak{a})$ *is an integer that satisfies the inequalities*

$$1 \leqslant X_\sigma(\mathfrak{a}) \leqslant q-1.$$

Proof. The values 0 and q are obviously excluded because the permutation σ defines an ordering of A_n, which is a transitive binary relation. ∎

LEMMA 6. *Let s be a permutation of the set of alternatives A_n. We then have*

$$X_\sigma(\mathfrak{a}) = X_{s \circ \sigma}(s(\mathfrak{a})).$$

Proof. By definition, we have

$$\begin{aligned} X_{s\circ\sigma}(s(\mathfrak{a})) &= \sum_{i=1}^{q} \chi_{s\circ\sigma}(s(a_{i+1}), s(a_i)) \\ &= \sum_{i=1}^{q} \chi_{\mathrm{id}}(\sigma^{-1} \circ s^{-1} \circ s(a_{i+1}), \sigma^{-1} \circ s^{-1} \circ s(a_i)) \\ &= \sum_{i=1}^{q} \chi_{\mathrm{id}}(\sigma^{-1}(a_{i+1}), \sigma^{-1}(a_i)) \\ &= X_\sigma(\mathfrak{a}) \end{aligned}$$

as follows from two successive applications of Lemma 2. ▮

Partition of $\mathfrak{S}_n$ by the Function $X_\sigma(\mathfrak{a})$

Let us partition the set of permutations or, in our context, of individual preference profiles $\mathfrak{S}_n$ into the subsets $B_{n,r}(\mathfrak{a})$ defined for $r = 1, ..., q-1$ and $q \leqslant n$ by

$$B_{n,r}(\mathfrak{a}) = \{\sigma \in \mathfrak{S}_n \mid X_\sigma(\mathfrak{a}) = r\}.$$

A first step is to compute the number of elements $b_{n,r}(\mathfrak{a})$ of these sets $B_{n,r}(\mathfrak{a})$. We observe that this number does not depend on the choice of a specific cycle $\mathfrak{a}$ but only on its length q as follows from:

LEMMA 7. *Let s be a permutation of the set A_n. Then we have*

$$b_{n,r}(\mathfrak{a}) = b_{n,r}(s(\mathfrak{a})).$$

Proof. Follows readily from the definition of the set $B_{n,r}(\mathfrak{a})$ combined with Lemma 6. ▮

This result enables us to assume from now on that the cycle $\mathfrak{a}$ is the set $A_q = \{1, 2, ..., q\}$ of the first q alternatives. We therefore denote by $b_{n,r}(q)$ the number of elements of the set $B_{n,r}(\mathfrak{a})$, number that we now know does not depend on the actual cycle but only on its length.

The computation of $b_{n,r}(q)$ proceeds by iteration on n.

LEMMA 8. *Let r, n, and q be fixed and satisfy* $1 \leqslant r \leqslant q-1$; *we then have*

$$b_{n+1,r}(q) = (n+1)\, b_{n,r}(q).$$

Proof. Consider a preference profile σ in $B_{n,r}(\mathfrak{a})$. From the definition, this profile has r pairwise comparisons that are identical with the q pairwise comparisons that define the cycle $\mathfrak{a}$. Let us add one more alternative denoted by $(n+1)$; this addition has no effect on the cycle $\mathfrak{a}$. On the other hand, the preference profile σ defines $(n+1)$ new preference profiles on the $(n+1)$ alternatives because there exist $(n+1)$ places for the new alternative. All these preference profiles belong to the set $B_{n+1,r}(\mathfrak{a})$, which implies that the number of elements of $B_{n+1,r}(\mathfrak{a})$ is equal to $(n+1)$ times the number of elements of $B_{n,r}(\mathfrak{a})$. ∎

The following corollary is then straightforward.

COROLLARY 9. *For all n, k, and q such that* $3 \leqslant q \leqslant n$ *and* $0 \leqslant k \leqslant n-q$, *we have*

$$b_{n,r}(q) = \frac{n!}{(n-k)!}\, b_{n-k,r}(q),$$

and

$$b_{n,r}(q) = \frac{n!}{q!}\, b_{q,r}(q).$$

The only numbers to compute are therefore the coefficients $(b_{r,q}(q))_{1 \leqslant r \leqslant q-1}$. The following proposition relates them to the Eulerian numbers. First, we need a technical result where σ is a permutation of A_q

LEMMA 10. *The number of readings of the permutation* σ *is equal to*

$$1 + \sum_{i=1}^{q-1} \chi_\sigma(i+1, i).$$

Proof. The proof proceeds by induction on the number q of elements of the permutation.

For $q=2$, there are only two possible permutations σ, either (12) or (21), and the number of readings has to be compared with

$$1 + \chi_\sigma(2, 1).$$

The permutation (12) requires only one reading while $\chi_\sigma(2,1)$ is equal to 0. The permutation (21) requires two readings but then $\chi_\sigma(2,1)$ is equal to 1.

Therefore, in both cases, the expression giving the number of readings is correct.

Assume now that the formula has been established up to $q-1$. Let σ denote an arbitrary permutation of the q elements $1, 2, ..., q$. Let $\sigma_{\hat{q}}$ be the permutation of the first $q-1$ elements $1, 2, ..., q-1$ obtained by deleting the element q from the permutation σ. Let us compare the number of readings of $\sigma_{\hat{q}}$ with the number of readings of σ. The scans for these two permutations that involve the numbers up to $q-2$ are identical. The scan of $\sigma_{\hat{q}}$ that contains $q-1$ can be extended to read q in σ if and only if $q-1 \succ_\sigma q$. On the other hand, for $q \succ_\sigma q-1$, an additional scan (that consists only of the element q) is necessary to read the permutation σ. The induction assumption is that the number of readings of the permutation $\sigma_{\hat{q}}$ is equal to

$$1+\sum_{i=1}^{q-2} \chi_{\sigma_{\hat{q}}}(i+1, i).$$

It follows from the definition of $\sigma_{\hat{q}}$ that $\chi_{\sigma_{\hat{q}}}(i+1, i)=\chi_\sigma(i+1, i)$ for $i \leqslant q-2$. The additional term $\chi_\sigma(q, q-1)$ is equal to 0 or 1 depending on whether we have $q-1 \succ_\sigma q$ or $q \succ_\sigma q-1$. This therefore shows that the number of readings of the permutation σ having q elements is indeed given by the formula of the lemma. ∎

Proposition 11. *For all q and r, $1 \leqslant r \leqslant q-1$, we have*

$$b_{q,r}(q)=q\left\langle {q-1 \atop r} \right\rangle.$$

Proof. First, we note that, if σ belongs to the set $B_{q,r}(A_q)$, then the q permutations obtained by composing σ with the q circular permutation of A_q also belong to $B_{q,r}(A_q)$. Therefore, we can assume that we have $\sigma(q)=q$. The second step consists of computing $X_\sigma(A_q)$. By definition, this number is equal to

$$X_\sigma(A_q)=\sum_{i=1}^{q} \chi_\sigma(i+1, i)$$

with the convention that $\chi_\sigma(q+1, q)$ is equal to $\chi_\sigma(1, q)=1$. One notices that $\chi_\sigma(q, q-1)$ is equal to 0. This implies that

$$X_\sigma(A_q)=1+\sum_{i=1}^{q-2} \chi_\sigma(i+1, i),$$

itself equal to the number of readings of the permutation $\sigma_{\hat{q}}$ by Lemma 10. ∎

2.4. *From the* $(n!-1)$*-Simplex to the* $(q-1)$*-Simplex.*

In this section, the cycle $\mathfrak{a}$ is again arbitrary with length q. Let us introduce the vector $\mu(\mathfrak{a})=(\mu_1(\mathfrak{a}), ..., \mu_{q-1}(\mathfrak{a}))$, where

$$\mu_r(\mathfrak{a})=\sum_{\sigma\in B_{n,r}(\mathfrak{a})}\lambda_\sigma$$

will to play an important role in the forthcoming developments. Note also that the following equality is obviously satisfied,

$$\sum_{r=1}^{q-1}\mu_r(\mathfrak{a})=1,$$

which that implies that the vector $\mu(\mathfrak{a})$ actually belongs to the $(q-2)$-simplex. We readily use the vector $\mu(\mathfrak{a})$ to reformulate Lemma 4 in a form that will be easier to exploit:

LEMMA 12. *The inequality*

$$\sum_{r=1}^{q-1}r\mu_r(\mathfrak{a})<q(1-\tau)$$

is satisfied for the (aggregated) cycle $\mathfrak{a}=(a_1, a_2, ..., a_q)$.

Proof. The sum

$$\sum_{\sigma\in\mathfrak{S}_n}\lambda_\sigma X_\sigma(\mathfrak{a})$$

can be rewritten as

$$\sum_{r=1}^{q-1}\left(\sum_{\sigma\in B_{n,r}(\mathfrak{a})}\lambda_\sigma X_\sigma(\mathfrak{a})\right)$$

and is therefore equal to

$$\sum_{r=1}^{q-1}\left(\sum_{\sigma\in B_{n,r}(\mathfrak{a})}\lambda_\sigma\right)r=\sum_{r=1}^{q-1}r\mu_r(\mathfrak{a}).\quad\blacksquare$$

This necessary condition enables us to give a necessary condition for the existence of cycles of length q for a given super majority rule τ. Compare with [9].

LEMMA 13. *The super majority rule* $\tau\geqslant 1-1/q$ *is necessary and sufficient to exclude cycles of length equal to q. It is sufficient to exclude cycles of length* $\leqslant q$.

Proof. The condition is obviously necessary: it suffices to follow the lines of Condorcet's example. Consider the collection of q agents having respectively the q preference profiles $(a_i, a_{i+1}, ..., a_q, a_1, a_2, ..., a_{i-1}, \tilde{\sigma})$ for $i = 1, ..., q$ where $\tilde{\sigma}$ is a fixed permutation of the complement $A_n \setminus \{a_1, a_2, ..., a_q\}$. Then the set $(a_1, a_2, ..., a_q)$ is a cycle for $\tau \leqslant 1 - 1/q$.

It follows from Lemma 12 that, if $\mathfrak{a}$ is a cycle, then the following inequalities are satisfied,

$$1 = \sum_{r=1}^{q-1} \mu_r(\mathfrak{a}) \leqslant \sum_{r=1}^{q-1} r\mu_r(\mathfrak{a}) < q(1-\tau)$$

which implies the inequality $1 < q(1-\tau)$ equivalent to $\tau < 1 - 1/q$. Therefore, having $\tau \geqslant 1 - 1/q$ is incompatible with the existence of a cycle of length q. ∎

3. THE COMPUTATION OF THE UPPER BOUND $Y(n, \tau)$

Working in the $(q-1)$-simplex by way of the map $\lambda \to \mu$ greatly simplifies the analysis of the geometry of the sets $V(n, \tau)$ for which cycles are observed for super majority rules. The proof of Proposition 13 has already exploited the dimension reduction brought by that map. The computation of an upper bound of the volume $V(n, \tau)$ makes crucial use of the smaller dimension given by the map.

3.1. *Volume of the Set*

$$\Lambda_{N, \varepsilon} = \left\{ (\lambda_i)_{1 \leqslant i \leqslant N} \in \mathbb{R}_+^N \;\middle|\; \sum_{i=1}^{i=N} \lambda_i = \varepsilon \right\}.$$

We first compute the volume of the set

$$\Lambda'_{N, \varepsilon} = \left\{ (\lambda_i)_{1 \leqslant i \leqslant N-1} \in \mathbb{R}_+^{N-1} \;\middle|\; \sum_{i=1}^{i=N-1} \lambda_i < \varepsilon \right\},$$

which is the projection of $\Lambda_{N, \varepsilon}$ on the space defined by the $(N-1)$ first coordinates along the direction of λ_N.

The volume is equal to the multiple integral

$$\int_0^{\varepsilon} d\lambda_1 \int_0^{\varepsilon - \lambda_1} d\lambda_2 \cdots \int_0^{\varepsilon - \lambda_1 - \lambda_2 - \cdots - \lambda_{N-2}} d\lambda_{N-1},$$

and is therefore equal to

$$V(\Lambda'_{N,\varepsilon}) = \frac{\varepsilon^{(N-1)}}{(N-1)!}.$$

From now on, $b_{n,r}$ stands for $b_{n,r}(\mathfrak{a})$, where the cycle $\mathfrak{a}$ is fixed of length q.

LEMMA 14. *The volume of the set defined in Lemma* 12 *projected in the hyperplane* $\lambda_\sigma = 0$ ($\sigma \in B_{n,1}(\mathfrak{a})$ *fixed) is given by the multiple integral*

$$\int_0^{\alpha_{q-1}} \frac{\mu_{q-1}^{b_{n,q-1}-1}}{(b_{n,q-1}-1)!} d\mu_{q-1} \int_0^{\alpha_{q-2}} \cdots \int_0^{\alpha_1} \frac{\mu_1^{b_{n,1}-2}}{(b_{n,1}-2)!} d\mu_1,$$

where

$$\alpha_r = q(1-\tau) - \sum_{j=r+1}^{j=q-1} j\mu_j,$$

itself equal to

$$\left(\frac{1}{2}\right)^{b_{n,2}} \left(\frac{1}{3}\right)^{b_{n,3}} \cdots \left(\frac{1}{q-2}\right)^{b_{n,q-2}} \left(\frac{1}{q-1}\right)^{b_{n,q-1}} \frac{(q(1-\tau))^{n!-1}}{(n!-1)!}.$$

Proof. The first assertion follows from a computation by successive integrations of the volume of the set $\Lambda'_{N,\varepsilon}$ defined by the inequality

$$\sum_{r=1}^{q-1} \Lambda_r < \varepsilon$$

using a partition of the coordinates into q components made of the subsets B_r with $1 \leqslant r \leqslant q-1$ and $\# B_r = b_r$, and the auxiliary integration variables $\Lambda_r = \sum_{h \in B_r} \lambda_h$. The volume of the projection on $(b_r - 1)$ coordinates of the set defined by the equation $\sum_{h \in B_r} \lambda_h = \Lambda_r$ is equal to $\Lambda_r^{b_r-1}/(b_r-1)!$. Therefore, the volume of $\Lambda'_{N,\varepsilon}$ is given by the multiple integral

$$\int_0^{\varepsilon} \frac{\Lambda_{q-1}^{b_{q-1}-1}}{(b_{q-1}-1)!} d\Lambda_{q-1} \int_0^{\varepsilon - \Lambda_{q-1}} \frac{\Lambda_{q-2}^{b_{q-2}-1}}{(b_{q-2}-1)!}$$

$$\times d\Lambda_{q-2} \cdots \int_0^{\varepsilon - \Lambda_{q-1} - \cdots - \Lambda_2} \frac{\Lambda_1^{b_1-2}}{(b_1-2)!} d\Lambda_1. \quad \blacksquare$$

Proposition 15. *An upper bound of the relative volume of the set of parameters for which the cycle* $\mathfrak{a}$ *of length* q *exists is given by* $(V(q-1)(1-\tau))^{n!}$, *where*

$$V(q-1)=\frac{q}{2^{\left\langle {q-1 \atop 2} \right\rangle/(q-1)!}3^{\left\langle {q-1 \atop 3} \right\rangle/(q-1)!}\cdots(q-1)^{\left\langle {q-1 \atop q-1} \right\rangle/(q-1)!}}$$

Proof. The expression of the volume given in Lemma 14 can be rewritten as

$$\frac{1}{(n!-1)!}\frac{1}{q(1-\tau)}\left[\frac{q(1-\tau)}{1^{b_{n,1}/n!}2^{b_{n,2}/n!}\cdots(q-1)\,b^{n,q-1/n!}}\right]^{n!}.$$

The volume of the projection of the $(n!-1)$-simplex on the hyperplane $\lambda_\sigma=0$ is equal to $1/(n!-1)!$. Recall from Proposition 11 combined with Corollary 9 that we have

$$\frac{b_{n,r}}{n!}=\frac{\left\langle {q-1 \atop r} \right\rangle}{(q-1)!}.$$

(These numbers are known in the literature as the normalized Eulerian numbers.) Combined with the inequality $q(1-\tau)\geqslant 1$, this yields an upper bound for the relative volume. ∎

The sequence $V(q)$ plays a crucial role in determining the bound we are looking for. Its first values are

$$V(2)=2.12132\ldots$$

$$V(3)=2.09823\ldots$$

$$V(4)=2.07601\ldots$$

$$V(5)=2.06177\ldots$$

$$V(6)=2.05207\ldots$$

$$V(7)=2.04502\ldots.$$

Numerical computations seem to suggest that this sequence is decreasing, which we have not been able to prove. This property of the sequence is similar to a property of monotonicity of the convergence on the interval $[0, 1/2)$ of the distribution function of the mean value of uniformly distributed random variables defined on $[0, 1]$; for details, see the Appendix.

The latter property[2] is not satisfied in general; direct computations suggest that the property is for uniformly distributed random variables.

Nevertheless, we do not need these strong properties for establishing our upper bound. It suffices that we show that $V(2)$ bounds all the other elements of the sequence $V(q)$. The following property regarding the sequence $V(q)$ is all that we need:

THEOREM 16. *The sequence* $V(q)$ *tends toward* 2 *and all its elements are smaller than* $V(2)$ *for* $q \geqslant 2$, *with*

$$V(2) = \frac{3}{2^{1/2}} = \frac{1}{0.4714...}. \quad \blacksquare$$

Proof. See the Appendix.

Remark. We also show in the Appendix that our proof would be significantly simplified had we had a proof of the monotone convergence (over the interval $[0, 1/2)$ of the distribution function of the mean value of uniformly distributed random variables).

From the theorem and from Lemma 1, we get the following upper bound $Y(n, \tau)$ of the relative measure of $V(n, \tau)$:

THEOREM 17.

$$Y(n, \tau) = n! \left(\frac{1 - \tau}{0.4714} \right)^{n!}.$$

Proof. Obvious from the previous developments. ∎

4. CONCLUDING COMMENTS

This paper can be viewed as another illustration of the power of the parametric approach, this time within the setup of social choice theory. A key ingredient in the success of the parametric approach in the current setup is the identification of individual preference orderings with permutations, an identification that enables us to relate the existence of Condorcet cycles to combinatorial properties of permutations—through the Eulerian numbers—not to mention the far-reaching probabilistic interpretation of these Eulerian numbers through the uniform distribution.

[2] It is well known by the central limit theorem that the distribution function converges to the function equal to 0 on the interval $[0, 1/2)$; it is the monotone convergence that is at stake here.

The main conclusion of this paper is that the relative size of the set of parameters for which a Condorcet cycle occurs is incredibly small under the suitable circumstances. We prove this result by computing an upper bound that is itself very small so that, in order to make our point, we do not have to and, therefore, we do not try to find the least upper bound of this volume. Our results can easily be improved to yield better bounds. Such bounds will actually be necessary if one wants to compare the importance of acyclicity (the lack of Condorcet cycles) with the importance of incompleteness for aggregated preferences obtained through super majority rules, a point raised to us by Karl Shell. A referee contributed a bound that can be used in dealing with the comparison problem. The referee's results are to be published in this Journal.

APPENDIX. THE PROOFS

Proof of Theorem 16. Define

$$A(q) = \log V(q) = \log(q+1) - \sum_{r=1}^{r=q} b_r(q) \log r,$$

where

$$b_r(q) = \frac{\left\langle \begin{matrix} q \\ r \end{matrix} \right\rangle}{q!}.$$

The first assertion of the theorem will be proved if we can show that the sequence $A(q)$ tends toward $\log 2$.

We first rewrite $A(q)$ using

$$\sum_{r=1}^{r=q} b_r(q) = 1$$

and obtain

$$A(q) = \left(\sum_{r=1}^{r=q} b_r(q) \right) \log(q+1) - \sum_{r=1}^{r=q} b_r(q) \log r;$$

hence,

$$A(q) = -\sum_{r=1}^{r=q} b_r(q) \log\left(\frac{r}{q+1}\right). \tag{7}$$

The proof will go through the computation of an upper and a lower bound of $-A(q)$. We start with an upper bound.

PROPOSITION 18. *For all* $q \geqslant 2$,

$$-A(q) \leqslant -\log 2.$$

Proof. Given symmetry:

$$b_r(q) = b_{q+1-r}(q),$$

if q is even,

$$-A(q) = \sum_{r=1}^{r=q/2} b_r(q) \left[\log\left(\frac{r}{q+1}\right) + \log\left(1 - \frac{r}{q+1}\right)\right],$$

and if q is odd,

$$-A(q) = b_{(q+1)/2} \log\left(\frac{1}{2}\right) + \sum_{r=1}^{r=(q-1)/2} b_r(q) \left[\log\left(\frac{r}{q+1}\right) + \log\left(\frac{r}{q+1}\right)\right].$$

But in each case, the convexity of the logarithm implies that

$$\log\left(\frac{r}{q+1}\right) + \log\left(1 - \frac{r}{q+1}\right) \leqslant 2 \log\left(\frac{1}{2}\right).$$

One concludes by observing that $\sum_{r=1}^{r=q} b_r(q) = 1$. ∎

Let Y_q be the sum of q independent uniform random variables on $[0, 1]$ and let F_q be the distribution function of Y_q. One knows from [18] that

$$F_q(x) = \frac{1}{q!} \sum_{j=0}^{j=[x]} (-1)^j \binom{q}{j} (x-j)^q,$$

where $[x]$ is the greatest integer $\leqslant x$. One also knows that, for any integer r such that $1 \leqslant r \leqslant q$,

$$F_q(r) - F_q(r-1) = \frac{1}{q!} \left\langle {q \atop r} \right\rangle = b_r(q).$$

Define G_q as the distribution function of the random variable Y_q/q; we then have the following lower bound:

PROPOSITION 19. *For all $q \geqslant 2$,*

$$-A(q) \geqslant -\frac{2}{q} - \int_0^1 \frac{G_q(t)}{t}\, dt.$$

Proof. Given Eq. (7), $-A(q)$ is the mathematical expectation of the random variable

$$\log\left(\frac{[Y_q]+1}{q+1}\right).$$

One has

$$\frac{[Y_q]+1}{q+1} \geqslant \frac{Y_q}{q+1} = \left(1 - \frac{1}{q+1}\right)\frac{Y_q}{q}.$$

This implies the inequality

$$\log\left(\frac{[Y_q]+1}{q+1}\right) \geqslant \log\left(1 - \frac{1}{q+1}\right) - \log\left(\frac{Y_q}{q}\right).$$

This yields for the expected value the inequality

$$-A(q) \geqslant \log\left(1 - \frac{1}{q+1}\right) - \int_0^1 \log t\, dG_q(t).$$

Using integration by parts, we get

$$\int_0^1 \log t\, dG_q(t) = [\log tG_q(t)]_0^1 - \int_0^1 \frac{G_q(t)}{t}\, dt,$$

and one checks readily that

$$[\log tG_q(t)]_0^1$$

is equal to 0. This readily implies the inequality

$$-A(q) \geqslant \log\left(1 - \frac{1}{q+1}\right) - \int_0^1 \frac{G_q(t)}{t}\, dt.$$

But, for $q \geqslant 2$, we have

$$\log\left(1 - \frac{1}{q+1}\right) \geqslant -\frac{2}{q+1} \geqslant -\frac{2}{q},$$

hence the proposition. ∎

A sufficient condition for Theorem 16 to hold true would be that, for all fixed $t \in [0, 1/2)$, the sequence $(G_q(t))_{q \geqslant 2}$ is decreasing (a symmetry argument then implies that the sequence is increasing for $t \in (1/2, 1]$.) We conjecture that this property is true; it would imply that the sequence $(-2/q - \int_0^1 G_q(t)/t\, dt)_{q \geqslant 2}$ is increasing and tends toward log 2. Numerical computations yield the inequality $-2/q - \int_0^1 (G_q(t)/t)\, dt \geqslant -A(2)$ for $q \geqslant 20$, and $-A(q) \geqslant -A(2)$ for $q \leqslant 20$.

Attempting to prove this conjecture has led us into challenging mathematical problems that we have not been able to solve. We therefore have had to use an indirect route to prove Theorem 16. We have proceeded using a sequence of functions ϕ_q with the property that $G_q(t) \leqslant \phi_q(t)$ on $[0, 1/2)$ for all q and $(-\int_0^1 (\phi_q(t)/t)\, dt)_{q \geqslant 2}$ is increasing and tends toward log 2.

Consider the integers $q \geqslant 100$, and define $\alpha(q) = q/2 - \sqrt{q}$ and $x(q) = q/2 - 3\sqrt{q}/2$. Define the functions Φ_q^1 and Φ_q^2 on $[0, q/2]$ by

$$\Phi_q^1(x) = \frac{1}{2}\left(\frac{x}{\alpha(q)}\right)^{\sqrt{q}},$$

and

$$\Phi_q^2(x) = a(q)\left(x - \frac{q}{2}\right) + \frac{1}{2},$$

where

$$a(q) = \frac{1}{3\sqrt{q}}\left(1 - \left(\frac{x(q)}{\alpha(q)}\right)^{\sqrt{q}}\right).$$

PROPOSITION 20. *For all* $q \geqslant 100$,

$$\begin{cases} 0 \leqslant F_q(x) \leqslant \Phi_q^1(x) & \text{if } 0 \leqslant x \leqslant x(q), \\ 0 \leqslant F_q(x) \leqslant \Phi_q^2(x) & \text{if } x(q) \leqslant x \leqslant q/2. \end{cases}$$

COROLLARY 21. *For all* $q \geqslant 324$,

$$-\log 2 - \frac{2}{q} - \frac{2}{\sqrt{q}} \leqslant -A(q) \leqslant -\log 2.$$

It follows from this corollary that for q large enough, actually $q \geqslant 1000$, $V(q)$ is smaller than $V(2)$. A computer-aided computation of the terms $V(q)$ for $q \leqslant 999$ that shows that the inequality $V(q) \leqslant V(2)$ is satisfied completes the proof of Theorem 16. ∎

Proof of Corollary 21. For all t in $[0, 1/2)$, $G_q(t) = F_q(qt)$; define in the same way $\phi^i_q(t) = \Phi^i_q(qt)$ for $i = 1, 2$. From Proposition 20 and from the inequality $G_q(t) \leqslant 1$, for all t in $(1/2, 1]$, it follows that

$$\int_0^1 \frac{G_q(t)}{t}\,dt \leqslant \int_0^{x(q)/q} \frac{\phi^1_q(t)}{t}\,dt + \int_{x(q)/q}^{1/2} \frac{\phi^2_q(t)}{t}\,dt + \int_{1/2}^1 \frac{dt}{t}.$$

But

$$\int_0^{x(q)/q} \frac{\phi^1_q(t)}{t}\,dt = \frac{1}{2\sqrt{q}} \left(\frac{x(q)}{\alpha(q)}\right)^{\sqrt{q}}$$

and

$$\left(\frac{x(q)}{\alpha(q)}\right)^{\sqrt{q}} = \left(1 - \frac{1}{\sqrt{q} - 2}\right)^{\sqrt{q}},$$

which increases monotonically towards $1/e$. Then, for all $q \geqslant 4$,

$$\left(\frac{x(q)}{\alpha(q)}\right)^{\sqrt{q}} \leqslant \frac{1}{e}$$

and

$$\int_0^{x(q)/q} \frac{\phi^1_q(t)}{t}\,dt \leqslant \frac{1}{2e\sqrt{q}}.$$

From $\phi^2_q(t) \leqslant 1/2$ for $t \leqslant 1/2$ combined with $q \geqslant 1$, one also gets

$$\int_{x(q)/q}^{1/2} \frac{\phi^2_q(t)}{t}\,dt \leqslant \frac{1}{2}\left(\frac{1}{2} - \frac{x(q)}{q}\right) \frac{q}{x(q)} = \frac{1}{2} \frac{3}{2\sqrt{q}} \frac{q}{x(q)}.$$

From

$$\frac{q}{x(q)} = \frac{2}{1 - 3/\sqrt{q}} \leqslant \frac{2}{1 - 3/18} = \frac{12}{5}$$

for $q \geqslant 18^2 = 324$ it follows that

$$\int_{x(q)/q}^{1/2} \frac{\phi^2_q(t)}{t}\,dt \leqslant \frac{9}{5\sqrt{q}}.$$

This implies the inequality

$$\int_0^1 \frac{G_q(t)}{t}\,dt \leqslant \log 2 + \frac{9}{5\sqrt{q}} + \frac{1}{2e\sqrt{q}} \leqslant \log 2 + \frac{2}{\sqrt{q}}. \quad \blacksquare$$

Proof of Proposition 20. One first needs a property of the distribution functions $(F_q)_{q \in \mathbf{N}}$. By definition, F_q is the distribution function of $Y_q = Y_{q-1} + X_q$ where Y_{q-1} has F_{q-1} as a distribution function, and X_q is a uniformly distributed random variable on $[0, 1]$. A convolution product yields the following induction formula.

Lemma 22. *For all q,*

$$F_q(x) = \int_{x-1}^{x} F_{q-1}(t)\,dt.$$

Corollary 23. *For all q, F_q is symmetrical with respect to the point $(q/2, 1/2)$, i.e., for all $x \in [0, q]$,*

$$F_q(x) + F_q(q-x) = 1.$$

Moreover, the function F_q is increasing and convex on $[0, q/2]$.

Proof. We prove the corollary by induction on q. The symmetry property is obvious for $q = 1$. Let us compute

$$F_{q+1}(x) + F_{q+1}(q+1-x) = \int_{x-1}^{x} F_q(t)\,dt + \int_{q+1-x-1}^{q+1-x} F_q(t)\,dt$$

which, by the change of variable $u = q - t$ in the second integral, is equal to

$$\int_{x-1}^{x} F_q(t)\,dt - \int_{x}^{x-1} F_q(q-t)\,dt = \int_{x-1}^{x} (F_q(t) + F_q(q-t))\,dt = \int_{x-1}^{x} dt = 1,$$

which proves the symmetry property.

The property that the function F_q is increasing is obvious for any q. Convexity is obvious for $q = 1$. Again, let us use induction on q. The second derivative of F_{q+1} is

$$F''_{q+1}(x) = F'_q(x) - F'_q(x-1).$$

If F_q is convex on $[0, q/2]$, F'_q is increasing, so that on $[0, q/2]$ F''_{q+1} is positive. Now take $x \in [q/2, (q+1)/2]$; thanks to $F'_q(x) = F'_q(q-x)$, which follows from the symmetry of F_q, one has

$$F''_{q+1}(x) = F'_q(q-x) - F'_q(x-1).$$

The conclusion follows from $x - 1 \leqslant (q-1)/2 \leqslant q - x \leqslant q/2$, combined with F'_q increasing on $[0, q/2]$. ∎

We are going to prove Proposition 20 through the analysis of the following five cases: (1) $x \in [0, 1]$; (2) $x \in [1, x(q)]$; (3) $x \in [1 + x(q), q/2]$; (4) $x \in [x(q), 1 + x(q)]$; (5) $x \in [q/2, (q+1)/2]$. We use induction except for the last case.

First Case: $x \in [0, 1]$. A simple induction argument yields $F_q(x) = x^q/q$. For the inequality $F_q(x) \leqslant \Phi^1_q(x)$ to hold, since for all $q \geqslant 1$ and $t \in [0, 1]$, $t^{\sqrt{q}} \geqslant t^q$, it is sufficient to prove that $F_q(1) \leqslant \Phi^1_q(1)$, which reduces here to proving that

$$\eta(q) = \frac{1}{q!} \leqslant \frac{1}{2} \left(\frac{1}{\alpha(q)} \right)^{\sqrt{q}} = \gamma(q).$$

We prove in fact that $\gamma(q)/\eta(q)$ is $\geqslant 1$ using the inequality

$$\rho(q) = \frac{\gamma(q+1)}{\eta(q+1)} \frac{\eta(q)}{\gamma(q)} \geqslant 1.$$

One has

$$\frac{\gamma(q+1)}{\gamma(q)} = \left(1 - \frac{1}{q+1}\right)^{\sqrt{q}} \left(\frac{2}{q+1}\right)^{\sqrt{q+1} - \sqrt{q}} \frac{(1 - 2/\sqrt{q})^{\sqrt{q}}}{(1 - 2/\sqrt{q+1})^{\sqrt{q+1}}}.$$

One can check that

$$\left(1 - \frac{1}{q+1}\right)^{\sqrt{q}} \geqslant 1 - \frac{\sqrt{q}}{q+1} \geqslant 1 - \frac{1}{\sqrt{q}}$$

and

$$\left(\frac{2}{q+1}\right)^{\sqrt{q+1} - \sqrt{q}} \geqslant \left(\frac{2}{q+1}\right)^{1/(2\sqrt{q+1})} \geqslant 0.77.$$

(Indeed, let $c = \sqrt{2/(q+1)}$; the last expression can be rewritten $c^{c/\sqrt{2}}$. The function $f(x) = x \log x$ reaches its minimum at $x = 1/e$, hence the inequality.)

LEMMA 24. *For* $q \geqslant 10$,

$$\frac{(1-2/\sqrt{q})^{\sqrt{q}}}{(1-2/\sqrt{q+1})^{\sqrt{q+1}}} \geqslant 1 - \frac{1}{q\sqrt{q}} - \frac{36}{q^2}.$$

Proof. Take the logarithm of the expression

$$\sqrt{q}\left(\log\left(1-\frac{2}{\sqrt{q}}\right) - \left(1+\frac{1}{q}\right)^{1/2} \log\left(1-\frac{2}{\sqrt{q}}\left(1+\frac{1}{q}\right)^{-1/2}\right)\right).$$

For all q,

$$\left(1+\frac{1}{q}\right)^{-1/2} \geqslant 1-\frac{1}{2q},$$

so that

$$\log\left(1-\frac{2}{\sqrt{q}}\left(1+\frac{1}{q}\right)^{-1/2}\right) \leqslant \log\left(1-\left(\frac{2}{\sqrt{q}}-\frac{1}{q\sqrt{q}}\right)\right)$$

and, for $\varepsilon(q)$ smaller than 1,

$$\log(1-\varepsilon(q)) \leqslant -\varepsilon(q) - \frac{\varepsilon(q)^2}{2} - \frac{\varepsilon(q)^3}{3} - \frac{\varepsilon(q)^4}{4},$$

which yields

$$\log\left(1-\left(\frac{2}{\sqrt{q}}-\frac{1}{q\sqrt{q}}\right)\right) \leqslant -\frac{2}{\sqrt{q}} - \frac{2}{q} - \frac{5}{3q\sqrt{q}} - \frac{2}{q^2} + \underbrace{\left(\frac{4}{q^2\sqrt{q}} + \frac{1}{3q^4\sqrt{q}} + \frac{8}{q^3} + \frac{2}{q^5}\right)}_{\leqslant 15/q^2\sqrt{q}}$$

(We delete the negative terms from the right-hand side of the inequality, the same thing will be done later to get rid of similarly complex terms.) One also has for all q

$$\left(1+\frac{1}{q}\right)^{1/2} \leqslant 1+\frac{1}{2q},$$

so that one obtains, through the same simplification,

$$\left(1+\frac{1}{q}\right)^{1/2}\log\left(1-\frac{2}{\sqrt{q}}\left(1+\frac{1}{q}\right)^{-1/2}\right)\leqslant -\frac{2}{\sqrt{q}}-\frac{2}{q}-\frac{8}{3q\sqrt{q}}-\frac{3}{q^2}+\frac{23}{q^2\sqrt{q}}.$$

A simple computation shows that, for $q\geqslant 10$,

$$\log\left(1-\frac{2}{\sqrt{q}}\right)\geqslant -\frac{2}{\sqrt{q}}-\frac{2}{q}-\frac{8}{3q\sqrt{q}}-\frac{4}{q^2}-\frac{13}{q^2\sqrt{q}},$$

and then

$$\log\left(\frac{(1-2/\sqrt{q})^{\sqrt{q}}}{(1-2/\sqrt{q+1})^{\sqrt{q+1}}}\right)\geqslant -\frac{1}{q\sqrt{q}}-\frac{36}{q^2}.$$

One concludes by using the inequality $e^x\geqslant 1+x$. ∎

The inequality

$$\rho(q)=\frac{\gamma(q+1)}{\gamma(q)}\,\frac{\eta(q)}{\eta(q+1)}\geqslant 0.77(q+1)\left(1-\frac{1}{\sqrt{q}}\right)\left(1-\frac{1}{q\sqrt{q}}-\frac{36}{q^2}\right)$$
$$\geqslant 0.77(q+1)\left(1-\frac{1}{\sqrt{q}}-\frac{1}{q\sqrt{q}}-\frac{35}{q^2}\right).$$

can then be derived. The right hand side term is obviously bigger than 1 when $q\geqslant 10$. Given

$$\frac{\gamma(q+1)}{\eta(q+1)}=\rho(q)\,\frac{\gamma(q)}{\eta(q)},$$

if $\gamma(q)/\eta(q)\geqslant 1$, we have $\gamma(q+1)/\eta(q+1)$. Also, $\gamma(5)/\eta(5)\geqslant 1$. Hence the first case.

Second case: $x\in[1, x(q)]$. The proof proceeds by induction on q. We suppose that Proposition 20 holds true for some q. One then has

$$F_{q+1}(x)=\int_{x-1}^{x} F_q(t)\,dt\leqslant\int_{x-1}^{x}\Phi_q^1(t)\,dt.$$

The inequality

$$\int_{x-1}^{x}\Phi_q^1(t)\,dt\leqslant\Phi_{q+1}^1(x)$$

holds if and only if we have

$$\frac{1}{2}\left(\frac{1}{\alpha(q)}\right)^{\sqrt{q}}\left(\frac{1}{1+\sqrt{q}}\right)(x^{1+\sqrt{q}}-(x-1)^{1+\sqrt{q}})\leqslant\frac{1}{2}\left(\frac{x}{\alpha(q+1)}\right)^{\sqrt{q+1}},$$

which is equivalent to the inequality

$$h_q(x)=x^{1+\sqrt{q}-\sqrt{q+1}}\left(1-\left(1-\frac{1}{x}\right)^{1+\sqrt{q}}\right)\leqslant(1+\sqrt{q})\,\frac{\alpha(q)^{\sqrt{q}}}{\alpha(q+1)^{\sqrt{q+1}}}. \quad (8)$$

LEMMA 25. *The function h_q is increasing on $[1, q/2]$.*

Proof. Compute the derivative of h_q:

$$\begin{aligned}h'_q(x)&=(1+\sqrt{q}-\sqrt{q+1})\,x^{\sqrt{q}-\sqrt{q+1}}\left[1-\left(1-\frac{1}{x}\right)^{1+\sqrt{q}}\right]\\&\quad+x^{1+\sqrt{q}-\sqrt{q+1}}\left(-(1+\sqrt{q})\left(1-\frac{1}{x}\right)^{\sqrt{q}}\frac{1}{x^2}\right)\\&=x^{\sqrt{q}-\sqrt{q+1}}\left((1+\sqrt{q}-\sqrt{q+1})\left[1-\left(1-\frac{1}{x}\right)^{1+\sqrt{q}}\right]\right.\\&\quad\left.-(1+\sqrt{q})\left(1-\frac{1}{x}\right)^{\sqrt{q}}\frac{1}{x}\right).\end{aligned}$$

Let $y=1-1/x$; $h'_q(x)$ is positive for $x\in[1, q/2]$ whenever

$$(\sqrt{q+1})\,y^{1+\sqrt{q}}-(1+\sqrt{q})\,y^{\sqrt{q}}+(1+\sqrt{q}-\sqrt{q+1})$$

is positive for $y\in[1, 1-2/q]$. Let $z=y^{\sqrt{q}}$ and define

$$\tilde{h}_q(z)=(\sqrt{q+1})\,z^{(1+\sqrt{q})/\sqrt{q}}-(1+\sqrt{q})\,z+(1+\sqrt{q}-\sqrt{q+1});$$

the derivative of $\tilde{h}_q$,

$$\tilde{h}'_q(z)=\sqrt{q+1}\left(\frac{1+\sqrt{q}}{\sqrt{q}}\right)z^{1/\sqrt{q}}-(1+\sqrt{q})$$

is negative when $y=z^{1/\sqrt{q}}\leqslant\sqrt{q/(q+1)}=(1+1/q)^{-1/2}$ which is always true for $y\leqslant 1-2/q$ since for all q,

$$\left(1+\frac{1}{q}\right)^{-1/2}\geqslant 1-\frac{1}{2q}\geqslant 1-\frac{2}{q}.$$

Then, $\tilde{h}_q$ being decreasing with respect to z, the only thing that remains to be shown for the lemma to hold is that the value at $(1-2/q)^{\sqrt{q}}$ of $\tilde{h}_q$ is positive. This is equivalent to proving that

$$\frac{\tilde{h}_q((1-2/q)^{\sqrt{q}})}{\sqrt{q+1}} = \left(1-\frac{2}{q}\right)^{\sqrt{q}} \left(1-\frac{2}{q}-\frac{1+\sqrt{q}}{\sqrt{q+1}}\right)+\frac{1+\sqrt{q}}{\sqrt{q+1}}-1 \geqslant 0.$$

A simple computation shows that for $x \in [0, 1]$ and $\alpha \geqslant 2$, $(1-x)^{\alpha} \leqslant 1-\alpha x+\alpha(\alpha-1)\, x^2/2$ which implies

$$\left(1-\frac{2}{q}\right)^{\sqrt{q}} \leqslant 1-\frac{2}{\sqrt{q}}+\frac{2}{q}-\frac{2}{q\sqrt{q}};$$

moreover, $1-1/(2q) \leqslant (1+1/q)^{-1/2} \leqslant 1-1/(2q)+3/(8q^2)$, so that

$$\left(1+\frac{1}{\sqrt{q}}\right)\left(1-\frac{1}{2q}\right) \leqslant \frac{1+\sqrt{q}}{\sqrt{q+1}} = \left(1+\frac{1}{\sqrt{q}}\right)\left(1+\frac{1}{q}\right)^{-1/2}$$
$$\leqslant \left(1+\frac{1}{\sqrt{q}}\right)\left(1-\frac{1}{2q}+\frac{3}{8q^2}\right),$$

and, therefore,

$$\frac{1+\sqrt{q}}{\sqrt{q+1}}-1 \geqslant \frac{1}{\sqrt{q}}-\frac{1}{2q}-\frac{1}{2q\sqrt{q}}$$

and

$$\frac{1+\sqrt{q}}{\sqrt{q+1}}-1+\frac{2}{q} \leqslant \frac{1}{\sqrt{q}}+\frac{3}{2q}-\frac{1}{2q\sqrt{q}}+\frac{3}{8q^2}\left(1+\frac{1}{\sqrt{q}}\right)$$

which yields

$$\left(1-\frac{2}{q}\right)^{\sqrt{q}}\left(\frac{2}{q}+\frac{1+\sqrt{q}}{\sqrt{q+1}}-1\right) \leqslant \frac{1}{\sqrt{q}}-\frac{1}{2q}-\frac{3}{2q\sqrt{q}}+\frac{19}{8q^2}+\frac{1}{q^3},$$

the latter inequality being obtained by deleting in the right-hand side expression all the negative terms we are not interested in. Then the lemma holds true as soon as

$$\frac{1}{q\sqrt{q}} \geqslant \frac{19}{8q^2}+\frac{1}{q^3},$$

which is satisfied for $q \geqslant 8$.

Hence, h_q being increasing, the second case will be established if we can show

$$h_q(x(q)) \leqslant (1+\sqrt{q})\, \alpha(q)^{\sqrt{q}}/\alpha(q+1)^{\sqrt{q+1}},$$

which will prove inequality (8). This follows from:

LEMMA 26. *For all* $q \geqslant 20$,

$$h_q(1+x(q)) \leqslant (1+\sqrt{q})\, \frac{\alpha(q)^{\sqrt{q}}}{\alpha(q+1)^{\sqrt{q+1}}}.$$

Proof. Simple computations reduce the problem to showing the inequality

$$\frac{q}{2}\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{1+\sqrt{q}-\sqrt{q+1}}\left(1-\left(1-\frac{2}{q}\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{-1}\right)^{1+\sqrt{q}}\right)$$
$$\leqslant (1+\sqrt{q})\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} \frac{(1-2/\sqrt{q})^{\sqrt{q}}}{(1-2/\sqrt{q+1})^{\sqrt{q+1}}}.$$

Consider the quantity

$$-\log\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} = \sqrt{q}\left(1+\frac{1}{q}\right)^{1/2} \log\left(1+\frac{1}{q}\right);$$

given that $(1+1/q)^{1/2} \leqslant 1+1/(2q)$ and $\log(1+1/q) \leqslant 1/q$, the former expression is smaller than $(1/\sqrt{q})+1/(2q\sqrt{q})$. Then we have

$$\log\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} \geqslant -\frac{1}{\sqrt{q}}-\frac{1}{2q\sqrt{q}}.$$

For x negative, we have $e^x \geqslant 1+x+x^2/2+x^3/6$, which, by taking the exponential of the previous inequality, yields

$$\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} \geqslant 1-\frac{1}{\sqrt{q}}-\frac{1}{2q\sqrt{q}}+\frac{1}{2q}-\frac{1}{6q\sqrt{q}}-\frac{5}{2q^2\sqrt{q}}$$

and then

$$(1+\sqrt{q})\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} \geqslant \sqrt{q}-\frac{1}{2\sqrt{q}}-\frac{1}{6q}-\frac{2}{3q\sqrt{q}}-\frac{5}{q^2}.$$

Combined with Lemma 24, this yields the inequality

$$(1+\sqrt{q})\left(\frac{q}{q+1}\right)^{\sqrt{q+1}} \frac{(1-2/\sqrt{q})^{\sqrt{q}}}{(1-2/\sqrt{q+1})^{\sqrt{q+1}}}$$

$$\geqslant \sqrt{q}-\frac{1}{2\sqrt{q}}-\frac{7}{6q}-\frac{2}{3q\sqrt{q}}-\frac{41}{q^2}.$$

Starting from $1+\sqrt{q}-\sqrt{q+1}=1-1/(\sqrt{q}+\sqrt{q+1})\geqslant 1-1/(2\sqrt{q})$, we get

$$\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{1+\sqrt{q}-\sqrt{q+1}} \leqslant \left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{1-1/(2\sqrt{q})}.$$

The logarithm of the right-hand side term is smaller than

$$\left(1-\frac{1}{2\sqrt{q}}\right)\left(-\frac{3}{\sqrt{q}}-\frac{5}{2q}+\frac{6}{q\sqrt{q}}-\frac{2}{q^2}\right),$$

and an upper bound for $q\geqslant 2$ is

$$-\frac{3}{\sqrt{q}}-\frac{1}{q}+\frac{9}{q\sqrt{q}}$$

as follows from $\varepsilon(q)<1$, and $\log(1-\varepsilon(q))\leqslant -\varepsilon(q)-\varepsilon(q)^2/2$. For x negative, one knows that $e^x\leqslant 1+x+x^2/2$ so that, after simplifications, one gets

$$\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{1-1/(2\sqrt{q})} \leqslant 1-\frac{3}{\sqrt{q}}+\frac{7}{2q}+\frac{12}{q\sqrt{q}}+\frac{11}{q^2}.$$

Moreover, for $\varepsilon\leqslant 1-0.5^{1/3}$, the inequality $1/(1-\varepsilon)\leqslant 1+\varepsilon+2\varepsilon^2$ yields, for $q\geqslant 300$,

$$\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{-1} \leqslant 1+\frac{3}{\sqrt{q}}+\frac{16}{q}+\frac{24}{q^2},$$

which, denoting $\varepsilon(q)=2/q+6/(q\sqrt{q})+32/q^2+48/q^3$, implies that

$$\left(1-\frac{2}{q}\left(1-\left(\frac{3}{\sqrt{q}}-\frac{2}{q}\right)\right)^{-1}\right)^{1+\sqrt{q}} \geqslant (1-\varepsilon(q))^{1+\sqrt{q}}.$$

For $q \geqslant 400$, $\varepsilon(q) \leqslant \tilde{\varepsilon}(q) = 2/q + 8/(q\sqrt{q})$. Simple computations show that for $x \in [0, 1]$ and $\alpha \geqslant 3$, $(1-x)^{\alpha} \geqslant 1 - \alpha x + \alpha(\alpha-1)\,x^2/2 - \alpha(\alpha-1)(\alpha-2)\,x^3/6$ so that $(1-\tilde{\varepsilon}(q))^{1+\sqrt{q}} \geqslant 1 - \sqrt{q}\,\tilde{\varepsilon}(q) - \tilde{\varepsilon}(q) + q\tilde{\varepsilon}(q)^2/2 + \sqrt{q}\,\tilde{\varepsilon}(q)^2/2 - q\sqrt{q}\,\tilde{\varepsilon}(q)^3/6$. This yields

$$(1-\tilde{\varepsilon}(q))^{1+\sqrt{q}} \geqslant 1 - \frac{2}{\sqrt{q}} - \frac{6}{q} - \frac{22}{3q\sqrt{q}} - \frac{34}{q^2} - \frac{64}{q^2\sqrt{q}} - \frac{88}{q^3}$$
$$\geqslant 1 - \frac{2}{\sqrt{q}} - \frac{6}{q} - \frac{22}{3q\sqrt{q}} - \frac{42}{q^2}$$

for $q \geqslant 400$.

Then, we get

$$\frac{q}{2}\left[1 - \left(1 - \frac{2}{q}\left(1 - \left(\frac{3}{\sqrt{q}} - \frac{2}{q}\right)\right)^{-1}\right)^{1+\sqrt{q}}\right] \geqslant \sqrt{q} + 3 + \frac{11}{3\sqrt{q}} + \frac{21}{q}$$

which yields the following inequality:

$$\frac{q}{2}\left(1 - \left(\frac{3}{\sqrt{q}} - \frac{2}{q}\right)\right)^{1-1/(2\sqrt{q})}\left[1 - \left(1 - \frac{2}{q}\left(1 - \left(\frac{3}{\sqrt{q}} - \frac{2}{q}\right)\right)^{-1}\right)^{1+\sqrt{q}}\right]$$
$$\leqslant \sqrt{q} - \frac{11}{6\sqrt{q}} + \frac{63}{2q} + \frac{28}{q\sqrt{q}} + \frac{246}{q^2} + \frac{714}{q^2\sqrt{q}} + \frac{882}{q^3}.$$

These bounds enable us to prove that the lemma holds for $q \geqslant 1000$, because, then

$$-\frac{4}{3\sqrt{q}} + \frac{66}{2q} + \frac{40}{q\sqrt{q}} + \frac{288}{q^2} + \frac{714}{q^2\sqrt{q}} + \frac{882}{q^3} \leqslant 0.$$

It follows from direct computations that the lemma also holds true for $100 \leqslant q \leqslant 1000$. ∎

Third Case: $x \in [1 + x(q), q/2]$. The proof proceeds by induction on q. Let us assume that Proposition 20 holds true for q, then we get

$$F_{q+1}(x) = \int_{x-1}^{x} F_q(t)\,dt \leqslant \int_{x-1}^{x} \Phi_q^2(t)\,dt.$$

The inequality

$$\int_{x-1}^{x} \Phi_q^2(t)\,dt \leqslant \Phi_{q+1}^2(x)$$

holds true if and only if

$$\frac{1}{2}+a(q)\left(\left(x-\frac{1}{2}\right)-\frac{q}{2}\right)\leqslant\frac{1}{2}+a(q+1)\left(x-\frac{q+1}{2}\right)$$

which is equivalent to

$$(a(q)-a(q+1))\left(x-\frac{1}{2}\right)\leqslant(a(q)-a(q+1))\,\frac{q}{2}.$$

But $a(q)$ is decreasing: it is the product of the decreasing quantity $1/(3\sqrt{q})$ by the other decreasing quantity $1-(x(q)/\alpha(q))^{\sqrt{q}}$; indeed, in the proof of Corollary 21, $(x(q)/\alpha(q))^{\sqrt{q}}$ has already been shown to be increasing. Then the above inequality holds true as soon as $x-1/2\leqslant q/2$; and here we have $x\leqslant q/2$. Hence the third case.

Fourth Case: $x\in[x(q), 1+x(q)]$. The proof proceeds again by induction on q. If Proposition 20 holds true for q, given $x-1\leqslant x(q)\leqslant x\leqslant x(q)+1$, one has

$$F_{q+1}(x)=\int_{x-1}^{x}F_q(t)\,dt\leqslant\int_{x-1}^{x(q)}\Phi_q^1(t)\,dt+\int_{x(q)}^{x}\Phi_q^2(t)\,dt.$$

But Φ_q^1 and Φ_q^2 have been defined with the same value at $x(q)$ while the slope on $[-1+x(q), 1+x(q)]$ of Φ_q^1 is always bigger than $a(q)$, which is the slope of Φ_q^2. Indeed, since the slope of Φ_q^1 is increasing and the graph of Φ_q^2 is a line, the inequality is satisfied if the slope of Φ_q^1 at $(-1+x(q))$ is bigger than $a(q)$, as it is stated in the following lemma.

LEMMA 27. *For all* $q\geqslant 100$,

$$\Phi_q^{1\prime}(-1+x(q))\geqslant a(q).$$

Proof. Recall the definition of $a(q)$:

$$a(q)=\frac{1}{3\sqrt{q}}\left(1-\left(\frac{1-3/\sqrt{q}}{1-2/\sqrt{q}}\right)^{\sqrt{q}}\right).$$

One has

$$\left(1-\frac{2}{\sqrt{q}}\right)^{-1}\geqslant 1+\frac{2}{\sqrt{q}},$$

then

$$\left(\frac{1-3/\sqrt{q}}{1-2/\sqrt{q}}\right) \geqslant 1 - \frac{1}{\sqrt{q}} - \frac{6}{q}.$$

Simple computations show that, for $\varepsilon(q) \leqslant 0.3$, $\log(1-\varepsilon(q)) \geqslant -\varepsilon(q) - \varepsilon(q)^2$, so that for $q \geqslant 20$, we get

$$\sqrt{q}\log\left(\frac{1-3/\sqrt{q}}{1-2/\sqrt{q}}\right) \geqslant -1 - \frac{7}{\sqrt{q}} - \frac{12}{q} - \frac{36}{q\sqrt{q}}.$$

Then, since $e^x \geqslant 1+x$, we get

$$\left(\frac{1-3/\sqrt{q}}{1-2/\sqrt{q}}\right)^{\sqrt{q}} \geqslant \frac{1}{e}\left(1 - \frac{7}{\sqrt{q}} - \frac{12}{q} - \frac{36}{q\sqrt{q}}\right)$$

and

$$a(q) \leqslant \frac{1-1/e}{3\sqrt{q}} + \frac{7}{3eq} + \frac{4}{eq\sqrt{q}} - \frac{12}{eq^2}. \tag{9}$$

Similarly, we have

$$\Phi_q^{1\prime}(-1+x(q)) = \frac{1}{\sqrt{q}(1-2/\sqrt{q})}\left(\frac{1-(3/\sqrt{q}+2/q)}{1-2/\sqrt{q}}\right)^{\sqrt{q}-1}.$$

We also have

$$\begin{aligned}\frac{1-(3/\sqrt{q}+2/q)}{1-2/\sqrt{q}} &\geqslant \left(1 - \frac{3}{\sqrt{q}} - \frac{2}{q}\right)\left(1 + \frac{2}{\sqrt{q}}\right)\\ &\geqslant 1 - \frac{1}{\sqrt{q}} - \frac{8}{q} - \frac{4}{q\sqrt{q}}\\ &\geqslant 1 - \frac{1}{\sqrt{q}} - \frac{12}{q}.\end{aligned}$$

Then, for $q \geqslant 20$, the inequality

$$\log\left(\frac{1-(3/\sqrt{q}+2/q)}{1-2/\sqrt{q}}\right) \geqslant -\frac{1}{\sqrt{q}} - \frac{13}{q} - \frac{24}{q\sqrt{q}} - \frac{144}{q^2}$$

holds thanks to $\varepsilon(q) \leqslant 0.3$, $\log(1-\varepsilon(q)) \geqslant -\varepsilon(q)-(q)^2$. This yields

$$(\sqrt{q}-1)\left(\frac{1-(3/\sqrt{q}+2/q)}{1-2/\sqrt{q}}\right) \geqslant -1-\frac{12}{\sqrt{q}}-\frac{11}{q}-\frac{120}{q\sqrt{q}}.$$

Since $e^x \geqslant 1+x$, we have

$$\left(\frac{1-(3/\sqrt{q}+2/q)}{1-2/\sqrt{q}}\right)^{\sqrt{q}-1} \geqslant \frac{1}{e}\left(1-\frac{12}{\sqrt{q}}-\frac{11}{q}-\frac{120}{q\sqrt{q}}\right)$$

and

$$\Phi_q^{1\prime}(-1+x(q)) \geqslant \frac{1}{\sqrt{q}}\left(1+\frac{2}{\sqrt{q}}\right)\frac{1}{e}\left(1-\frac{12}{\sqrt{q}}-\frac{11}{q}-\frac{120}{q\sqrt{q}}\right)$$
$$\geqslant \frac{1}{e\sqrt{q}}\left(1-\frac{10}{\sqrt{q}}-\frac{35}{q}-\frac{142}{q\sqrt{q}}-\frac{240}{q^2}\right).$$

This last inequality, compared to inequality (9), yields the lemma for $q \geqslant 1050$. The lemma is shown to be true for $70 \leqslant q \leqslant 1050$ by direct computations. ∎

Then the graph of Φ_q^1 lies under the graph of Φ_q^2 for $x < x(q)$, and above for $x > x(q)$. This implies that

$$\int_{x-1}^{x(q)} \Phi_q^1(t)\,dt \leqslant \int_{x-1}^{x(q)} \Phi_q^2(t)\,dt$$

and

$$\int_{x(q)}^{x} \Phi_q^2(t)\,dt \leqslant \int_{x(q)}^{x} \Phi_q^1(t)\,dt.$$

This implies, for $i=1, 2$, that

$$F_{q+1}(x) \leqslant \int_{x-1}^{x} \Phi_q^i(t)\,dt$$

and we have seen that, when x is $\leqslant 1+x(q)$, the two inequalities

$$\int_{x-1}^{x} \Phi_q^i(t)\,dt \leqslant \Phi_{q+1}^i(x)$$

are satisfied.

Fifth Case: $x \in [q/2, (q+1)/2]$. A direct line of reasoning through Corollary 23 enables us to conclude immediately. Indeed, we have

$F_{q+1}((q+1)/2) = \Phi^2_{q+1}((q+1)/2)$ and $F_{q+1}(q/2) \leqslant \Phi^2_{q+1}(q/2)$, thanks to the preceding case. In addition, the graph of Φ^2_{q+1} is a line whereas the graph of F_{q+1} is convex on $[q/2, (q+1)/2]$. Then $F_{q+1}(x) \leqslant \Phi^2_{q+1}(x)$ for $x \in [q/2, (q+1)/2]$. ▮

The proof of Proposition 20 is therefore completed once we show that we can start the induction. Though numerical computations show that the proposition is already true for $q = 20$, we start the induction argument only at $q = 100$ because we have proved various inequalities (see in particular Lemma 27) only for $q \geqslant 100$. ▮

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