Energy requirement for a working dynamo

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Abstract

There has for many years been interest in finding necessary conditions for dynamo action. These are usually expressed in terms of bounds on integrated properties of the flow. The bounds can clearly be improved when the flow structure can be taken into account. Recent research presents techniques for finding optimised dynamos (that is with the lowest dynamo threshold) subject to constraints, (for example, with fixed mean square *vorticity*). It is natural to ask if such an optimum solution can exist when the mean square *velocity* is fixed. The aim of this note is to show that this is not the case and in fact that a steady or periodic dynamo can exist in a bounded conductor with an arbitrarily small value of the kinetic energy.

Keywords: Dynamo Theory; Magnetic fields; Anti-dynamo theorems

1 Introduction

In this paper we consider the problem of finding optimal conditions for non-decaying (steady or time-periodic) solutions to the dimensionless induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = R_m \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \nabla^2 \boldsymbol{B} \tag{1}$$

in a fixed spherical conductor $\mathcal{D}_0$ of radius R_0, and volume $V_0 = 4\pi R_0^3/3$ surrounded by insulator, with a given velocity field $\boldsymbol{u}$. We can take $R_0 = 1$ without loss of generality. The velocity may in principle depend on space and time; in this note we suppose that $\boldsymbol{u}$ is steady. If such solutions exist we say that we have dynamo action. There is a long history of research on necessary conditions for dynamo action, beginning with the lower bounds of Backus (1958) (see also Proctor, 1977) on the maximum strain rate, Childress (1969) on the maximum speed, and more recently Proctor (1979) on the rms dissipation. These conditions all rely on the equation

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for the rate of change of the total magnetic energy. Because of the non-normality of the induction equation it is not possible to give sufficient conditions for dynamo action based on the growth of the magnetic energy, as energy can grow transiently for some initial conditions even for flows which are not dynamos . However, if the threshold for dynamo action can be optimised for all flows in a given class, and the optimising flow can be given explicitly, then we can assert that dynamo action will occur for sufficiently vigorous flows of that nature, and this amounts to a sufficient condition.

A programme of this type has recently been undertaken by Willis (2012) and Chen *et al.* (2015) who have given explicit optimised steady dynamos as measured by the rms dissipation, in a periodic domain and a cube respectively. Though the task has not been completed for a conducting sphere, it seems clear that such a programme is feasible using spherical harmonics rather than the Fourier decomposition appropriate to a cubic geometry. We can easily imagine a generalisation to other bounded conductor shapes.

2 Analysis

While the above authors appear to have successfully found optimised steady velocity fields obeying the above constraint, with a bound necessarily rather greater than would emerge from the necessary condition of Proctor (1979), they noted that problems arose when considering instead flows obeying a constraint on the energy of the flow. Thus we revisit the problem for this constraint. We shall show that there is in fact no smooth optimising field and that - perhaps counterintuitively - the minimum energy for a working dynamo in a conducting sphere is zero.

As a class of fields we choose those that are normalised with respect to the total energy, so that $E = \int_{\mathcal{D}_0} |\boldsymbol{u}|^2 \, \mathrm{d}^3\boldsymbol{x} = 1$. We suppose that for a particular flow $\boldsymbol{u}$ in this class there is a *critical value* of R_m corresponding to a steady solution or a periodic solution (with period T) of equation (1). We shall show that we can find a sequence of flows in this class, which are non-zero in smaller and smaller subregions of $\mathcal{D}_0$, corresponding to a sequence of critical values of R_m which become arbitrarily small.

First we consider a comparison problem; we choose a velocity field $\boldsymbol{u}_0 = \boldsymbol{f}(\boldsymbol{x})$ which vanishes outside $\mathcal{D}_0$ but with an external stationary conductor, with the same diffusivity as the internal fluid, extending to infinity. We then solve this modified dynamo problem, confining velocities to normalised ones normalised so that $E = 1$, and suppose that for $R_m = R_{mc}$ equation (1) is solved for a steady magnetic field $\boldsymbol{B}_0(\boldsymbol{x})$ or a periodic field with period T_0.

Now consider a velocity field $\boldsymbol{u}_1$ of the form

$$\boldsymbol{u}_1 = \delta^{-3/2}\boldsymbol{f}(\boldsymbol{y}) \,, \qquad \boldsymbol{y} = \boldsymbol{x}/\delta \,, \tag{2}$$

for some constant $\delta < 1$. Because $\boldsymbol{u}_1$ vanishes outside the sphere $r = \delta R_0$ it can be verified that $\boldsymbol{u}_1$ is in the class $E = V_0$. If we rescale $\boldsymbol{x}$ in equation (1), and write $\boldsymbol{\nabla} = \delta^{-1}\boldsymbol{\nabla}_{\boldsymbol{y}}$, $\partial/\partial t = \delta^{-2}\partial/\partial\tau$, where $\boldsymbol{\nabla}_{\boldsymbol{y}}$ denotes derivatives with respect to $\boldsymbol{y}$, then we can rewrite the induction equation as

$$\delta^{-2}\frac{\partial \boldsymbol{B}}{\partial \tau} = R_m\delta^{-1}\boldsymbol{\nabla}_{\boldsymbol{y}} \times (\boldsymbol{u}_1 \times \boldsymbol{B}) + \delta^{-2}\nabla^2_{\boldsymbol{y}}\boldsymbol{B} \,, \tag{3}$$

and substituting for $\boldsymbol{u}_1$ we get

$$\frac{\partial \boldsymbol{B}}{\partial \tau} = R_m\delta^{-1/2}\boldsymbol{\nabla}_{\boldsymbol{y}} \times (\boldsymbol{f}(\boldsymbol{y}) \times \boldsymbol{B}) + \nabla^2_{\boldsymbol{y}}\boldsymbol{B} \tag{4}$$

and this equation has solution $\boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{y})$, either steady or with period T_0 in τ, i.e. period $\delta^2 T_0$ in t, provided that $R_m = R_{mc}\delta^{1/2}$.

It is then clear that for the comparison problem we may find a sequence of velocity fields for $\delta \to 0$ satisfying $E = 1$ for which the corresponding values of R_m for marginal dynamo action tend to zero. Thus there is no lower bound on the critical R_m for a dynamo when normalised with respect to the energy of the flow.

We now replace the material outside $r = R_0$ by insulator, and consider how this changes the critical R_m. For very small δ the magnetic field at $r = R_0$ for the comparison problem will be very small relative to the the field near the origin. To see this note that in the outer region $\boldsymbol{u} = 0$, and so the magnetic field at large $|\boldsymbol{y}|$ will obey the equation $\partial \boldsymbol{B}/\partial \tau = \nabla^2_{\boldsymbol{y}} \boldsymbol{B}$. The far field solutions of this equation, vanishing at infinity, are exponentially small for time-periodic solutions and fall off no slower than δ^2 for $|\boldsymbol{y}| \sim \delta^{-1}$. Thus the relative change in the critical value of R_m due to imposing the insulating boundary condition at $r = R_0$ will be no greater than $\delta^2 \ll \delta^{1/2}$ and thus in the limit $\delta \to 0$ the critical value of R_m will be the same, that is arbitrarily small.

It is clear intuitively that a very similar argument will work in a periodic box or a cubical domain, as the comparison problem does not depend on the shape of the domain as long as the region of non-zero $\boldsymbol{u}$ lies within the domain. In the periodic case the flow $\boldsymbol{u}_1$ will be repeated in each periodic box, but other aspects of the argument are unchanged.

Note that as the limit is approached the velocity field is zero almost everywhere, very large in a tiny volume and with very large gradients. Thus the final limit is not part of the class of smooth fields originally posited. It is perhaps this that leads to difficulties for the numerical minimisation scheme.

3 Discussion

It has been shown above that the critical value of R_m for marginal dynamos with a given energy is zero. This can be stated alternatively as showing that for a given value of the magnetic diffusivity the critical value of the energy is arbitrarily small. We can generalise this result in a simple way. Choose a number p, and define $F_p = E^p D^{1-p}$ where $D = \int_{\mathcal{D}_0} |\boldsymbol{\nabla u}|^2 \, \mathrm{d}^3\boldsymbol{x}$ is the dissipation. In a similar way to the above we can seek an optimised flow with rminimal R_m subject to $F_p = 1$. Applying the same transformations as in equation (2) we can define a flow $\boldsymbol{u}_2 = \delta^{-q} \boldsymbol{f}(\boldsymbol{y})$, where $q = (2p+1)/2$ and so by following on in the same way as above we find that the critical value of R_m scales as $\delta^{q-1} = \delta^{p-1/2}$. Thus normalising with F_p for $p > 1/2$ will yield a zero critical R_m, but not otherwise. The case $p = 1/2$ is interesting since for the infinite conductor at least there is an infinity of possible flows, connected by a scale transformation, that has the same value of R_m normalised with $F_{1/2}$.

Finally it should be noted that although there is no minimum energy bound, we can find bounds for higher order norms of $\boldsymbol{u}$. For example, we can show from the induction equation (1) that, if $\mathcal{D}_0$ is surrounded by insulator, and M is the total magnetic energy $\frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{B}|^2 \, \mathrm{d}^3\boldsymbol{x}$,

$$\frac{\mathrm{d}M}{\mathrm{d}t} \le R_m \left(\int_{\mathcal{D}_0} |\boldsymbol{u}|^4 \, \mathrm{d}^3\boldsymbol{x} \right)^{1/4} \left(\int_{\mathcal{D}_0} |\boldsymbol{B}|^4 \, \mathrm{d}^3\boldsymbol{x} \right)^{1/4} \left(\int_{\mathcal{D}_0} |\nabla \boldsymbol{B}|^2 \, \mathrm{d}^3\boldsymbol{x} \right)^{1/2} - \int_{\mathcal{D}_0} |\boldsymbol{\nabla B}|^2 \, \mathrm{d}^3\boldsymbol{x} \,. \tag{5}$$

We can also show

$$\int_{\mathcal{D}_0} |\boldsymbol{B}|^4 \, \mathrm{d}^3\boldsymbol{x} \;\le\; c_1^2 (2M)^{1/2} \left(\int_{\mathcal{D}_0} |\boldsymbol{\nabla B}|^2 \, \mathrm{d}^3\boldsymbol{x} \right)^{3/2} , \tag{6}$$

where c_1 is a dimensionless constant (Proctor, 1979, 2007). in addition we have the well-known Poincaré inequality

$$\frac{\int_{\mathcal{D}_0} |\boldsymbol{\nabla B}|^2 \, \mathrm{d}^3\boldsymbol{x}}{2M} \;\ge\; \frac{\pi^2}{R_0^2} . \tag{7}$$

Thus among velocity fields normalised so that $\int_{\mathcal{D}_0} |\boldsymbol{u}|^4 \, \mathrm{d}^3\boldsymbol{x} = 1$ we require for dynamo action

$$R_m^2 \;\ge\; \frac{\int_{\mathcal{D}_0} |\boldsymbol{\nabla B}|^2 \, \mathrm{d}^3\boldsymbol{x}}{\left(\int_{\mathcal{D}_0} |\boldsymbol{B}|^4 \, \mathrm{d}^3\boldsymbol{x} \right)^{1/2}} \;\ge\; \frac{1}{c_1} \left[\frac{\int_{\mathcal{D}_0} |\boldsymbol{\nabla B}|^2 \, \mathrm{d}^3\boldsymbol{x}}{2M} \right]^{1/4} \;\ge\; \frac{1}{c_1} \sqrt{\frac{\pi}{R_0}} . \tag{8}$$

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