ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE

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Abstract

It is well-known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space $\boldsymbol{R}_1^3$ have singularities in general. They are both characterized as zero mean curvature surfaces. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. As a continuation of a previous work by the authors, we give the first example of a family of such surfaces which change type across a light-like line. As a corollary, we also obtain a family of zero mean curvature hypersurfaces in $\boldsymbol{R}_1^{n+1}$ that change type across an $(n-1)$-dimensional light-like plane.

Introduction

Many examples of space-like maximal surfaces containing singular curves in the Lorentz–Minkowski 3-space $(\boldsymbol{R}_1^3; t, x, y)$ of signature $(-++)$ have been constructed in [11], [1], [12], [8], [4] and [5].

In this paper, we are interested in the zero mean curvature surfaces in $\boldsymbol{R}_1^3$ changing their causal type: Klyachin [10] showed under a sufficiently weak regularity assumption that a zero mean curvature surface in $\boldsymbol{R}_1^3$ changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (i.e., their projections into the xy-plane are locally convex plane curves), or
- light-like lines, which are degenerate everywhere.

Given a non-degenerate null curve γ in $\boldsymbol{R}_1^3$, there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to

a time-like minimal surface (cf. [6], [10], [9] and [7]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal surfaces. (The reference [3] is an expository article on this subject.) However, if γ is a light-like line, the aforementioned construction fails, since the isothermal coordinates break down at the light-like singular points. Locally, such a surface is the graph of a function $t = f(x, y)$ satisfying

$$(*) \qquad (1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy} = 0,$$

where $f_x = \partial f / \partial x$, $f_{xy} = \partial^2 f / \partial y \partial x$, etc. We call this and its graph the *zero mean curvature equation* and a *zero mean curvature surface*, respectively. Until now, zero mean curvature surfaces which actually change type across a light-like line were unknown. As announced in [2], the main purpose of this paper is to construct such an example. In Section 1, we give a formal power series solution of the zero mean curvature equation describing all zero mean curvature surfaces which contain a light-like line. Using this, we give the precise statement of our main result and show how the statement can be reduced to a proposition (cf. Proposition 1.3). In Section 2, we then prove it. As a consequence, we obtain the first example of (a family of) zero mean curvature surfaces which change type across a light-like line.

1. The main theorem

We discuss solutions of the zero mean curvature equation $(*)$ which have the following form

$$(1.1) \qquad f(x, y) = b_0(y) + \sum_{k=1}^{\infty} \frac{b_k(y)}{k} x^k,$$

where $b_k(y)$ $(k = 1, 2, \ldots)$ are C^∞-functions. When f contains a singular light-like line, we may assume without loss of generality that (cf. [2])

$$(1.2) \qquad b_0(y) = y, \quad b_1(y) = 0.$$

As was pointed out in [2], there exists a real constant μ called the *characteristic* of f such that $b_2(y)$ satisfies the following equation

$$(1.3) \qquad b_2'(y) + b_2(y)^2 + \mu = 0 \quad (' = d/dy).$$

Now we derive the differential equations satisfied by $b_k(y)$ for $k \geq 3$ assuming (1.2). If we set

$$Y := f_y - 1 = \sum_{k=2}^{\infty} \frac{b_k'(y)}{k} x^k$$

and

$$P := 2(Yf_{xx} - f_x f_{xy}), \quad Q := Y^2 f_{xx} - 2f_x f_{xy} Y, \quad R := f_x^2 f_{yy},$$

then, by straightforward calculations, we see that

$$P = -b_2 b_2' x^2 - \frac{4}{3} b_2 b_3' x^3 - \sum_{k=4}^{\infty} \left(P_k + \frac{2(k-1)}{k} b_2 b_k' + (3-k) b_2' b_k \right) x^k,$$

$$Q = -\sum_{k=4}^{\infty} Q_k x^k, \quad R = \sum_{k=4}^{\infty} R_k x^k,$$

where

$$P_k := \sum_{m=3}^{k-1} \frac{2(k - 2m + 3)}{k - m + 2} b_m b_{k-m+2}',$$

(1.4)

$$Q_k := \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n - k + m - 1}{mn} b_m' b_n' b_{k-m-n+2},$$

$$R_k := \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{b_m b_n b_{k-m-n+2}''}{k - m - n + 2}$$

for $k \geq 4$, and that the zero mean curvature equation $(*)$ reduces to

$$\sum_{k=2}^{\infty} \frac{b_k''}{k} x^k = f_{yy} = P + Q + R.$$

It is now immediate, by comparing the coefficients of x^k from both sides, to see that each b_k ($k \geq 3$) satisfies the following ordinary differential equation

(1.5) $$b_k''(y) + 2(k-1)b_2(y)b_k'(y) + k(3-k)b_2'(y)b_k(y) = -k(P_k + Q_k - R_k),$$

where $P_3 = Q_3 = R_3 = 0$ and P_k, Q_k and R_k are as in (1.4) for $k \geq 4$. Note that P_k, Q_k and R_k are written in terms of b_j ($j = 1, \ldots, k-1$) and their derivatives.

Now, we consider the case that $1 - f_x^2 - f_y^2$ changes sign across the light-like line $\{t = y, \ x = 0\}$. This case occurs only when the characteristic μ as in (1.3) of f vanishes [2]. If we set

$$b_2(y) = 0 \quad (y \in \mathbf{R}),$$

then (1.3) holds for $\mu = 0$. So we assume

(1.6) $$b_0(y) = y, \quad b_1(y) = 0, \quad b_2(y) = 0, \quad b_3(y) = 3cy,$$

where c is a non-zero constant. Then $f(x, y)$ in (1.1) can be rewritten as

$$(1.7) \qquad f(x, y) = y + cyx^3 + \sum_{k=4}^{\infty} \frac{b_k(y)}{k} x^k.$$

In this situation, we will find a solution satisfying

$$(1.8) \qquad b_k(0) = b_k'(0) = 0 \quad (k \geq 4).$$

Then (1.5) reduces to

$$(1.9) \qquad b_k''(y) = -k(P_k + Q_k - R_k), \quad b_k(0) = b_k'(0) = 0, \quad (k \geq 4),$$

$$(1.10) \qquad P_k = \sum_{m=3}^{k-1} \frac{2(k - 2m + 3)}{k - m + 2} b_m(y) b_{k-m+2}'(y) \quad (k \geq 4),$$

$$(1.11) \qquad Q_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{3n - k + m - 1}{mn} b_m'(y) b_n'(y) b_{k-m-n+2}(y) \quad (k \geq 7),$$

$$(1.12) \qquad R_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{b_m(y) b_n(y) b_{k-m-n+2}''(y)}{k - m - n + 2} \quad (k \geq 7),$$

and $Q_k = R_k = 0$ for $4 \leq k \leq 6$, where the fact that $b_2(y) = 0$ has been extensively used. For example,

$$b_0 = y, \quad b_1 = b_2 = 0, \quad b_3 = 3cy, \quad b_4 = -4c^2 y^3, \quad b_5 = 9c^3 y^5,$$
$$b_6 = -24c^2 y^7, \quad b_7 = 70c^5 y^9 - 14c^3 y^3, \ldots.$$

In this article, we show the following assertion:

Theorem 1.1. *For each positive number c, the formal power series solution $f(x, y)$ uniquely determined by (1.9), (1.10), (1.11) and (1.12) gives a real analytic zero mean curvature surface on a neighborhood of $(x, y) = (0, 0)$. In particular, there exists a non-trivial 1-parameter family of real analytic zero mean curvature surfaces each of which changes type across a light-like line (see Fig. 1).*

As a consequence, we get the following:

Corollary 1.2. *There exists a family of zero-mean curvature hypersurfaces in Lorentz–Minkowski space $\mathbf{R}_1^{n+1}$ each of which changes type across an $(n - 1)$-dimensional light-like plane.*

Proof.   Let f be as in the theorem. The graph of the function defined by

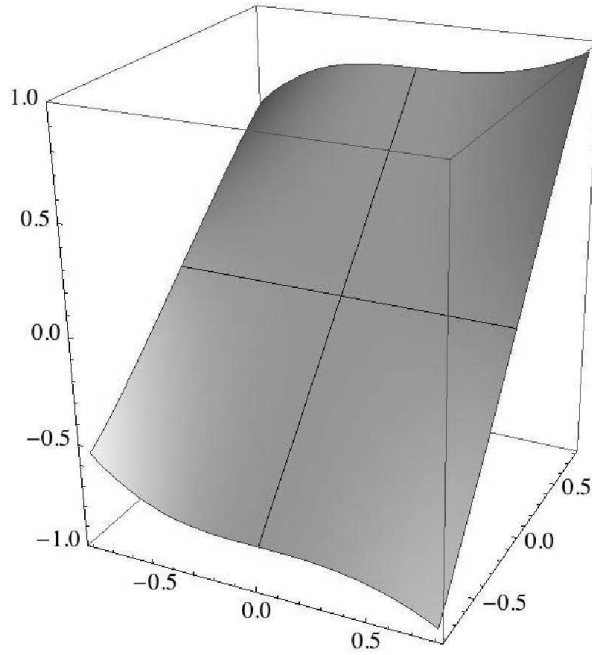$$\mathbf{R}^n \ni (x_1, \ldots, x_n) \mapsto f(x_1, x_2) \in \mathbf{R}$$

Fig. 1. The graph of $t = f(x, y)$ for $c = 1/2$ and $|x|, |y| < 0.8$
(The range of the graph is wider than the range used in our math-
ematical estimation. However, this figure still has a sufficiently
small numerical error term in the Taylor expansion.)

gives the desired hypersurface. In this case, the zero mean curvature equation

$$\left(1 - \sum_{j=1}^{n} f_{x_j}^2\right) \sum_{i=1}^{n} f_{x_i x_i} + \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i} f_{x_j} = 0 \quad (f_{x_i} := \partial f / \partial x_i, \ f_{x_i x_j} := \partial^2 f / \partial x_j \partial x_i)$$

reduces to $(*)$ in the introduction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

To prove Theorem 1.1, it is sufficient to show that for arbitrary positive constants $c > 0$ and $\delta > 0$ there exist positive constants n_0, θ_0, and C such that

$$(1.13) \qquad\qquad\qquad |b_k(y)| \leq \theta_0 C^k \quad (|y| \leq \delta)$$

holds for $k \geq n_0$. In fact, if (1.13) holds, then the series (1.7) converges uniformly over the rectangle $[-C^{-1}, C^{-1}] \times [-\delta, \delta]$.

The key assertion to prove (1.13) is the following

Proposition 1.3. *For each $c > 0$ and $\delta > 0$, we set*

$$(1.14) \qquad M := 3 \max\{144 c\tau |\delta|^{3/2}, \sqrt[4]{192 c^2 \tau}\},$$

where τ is the positive constant given by (A.3) *in the appendix, such that*

$$(1.15) \qquad t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \le \tau \quad (0 < t < 1/2).$$

Then the function $\{b_l(y)\}_{l \ge 3}$ formally determined by the recursive formulas (1.9)–(1.12) *satisfies the inequalities*

$$(1.16) \qquad |b_l''(y)| \le c|y|^{l^*} M^{l-3},$$

$$(1.17) \qquad |b_l'(y)| \le \frac{3c|y|^{l^*+1}}{l^*+2} M^{l-3},$$

$$(1.18) \qquad |b_l(y)| \le \frac{3c|y|^{l^*+2}}{(l^*+2)^2} M^{l-3}$$

for any

$$(1.19) \qquad y \in [-\delta, \delta],$$

where

$$(1.20) \qquad l^* := \frac{1}{2}(l-1) - 2 \quad (l \ge 3).$$

Once this proposition is proven, (1.13) follows immediately. In fact, if we set

$$\theta_0 = \frac{3}{c}(\delta M)^3, \quad C := \delta M$$

and $n_0 \ge 7$, then $1 \le l^* + 2 < l - 3$ and (1.13) follows from

$$\frac{3c|y|^{l^*+2}}{(l^*+2)^2} M^{l-3} \le \theta_0 C^l.$$

2. Proof of Proposition 1.3

We prove the proposition using induction on the number $l \ge 3$. If $l = 3$, then

$$|b_3''(y)| = 0 \le \frac{c}{|y|} = c|y|^{3^*} M^0,$$

$$|b_3'(y)| = 3c = \frac{3c|y|^{3^*+1}}{3^*+2} M^0,$$

$$|b_3(y)| = 3c|y| = \frac{3c|y|^{3^*+2}}{(3^*+2)^2} M^0$$

hold, using that $b_3(y) = 3cy$, $M^0 = 1$ and $3^* = -1$. So we prove the assertion for $l \geq 4$. Since (1.17), (1.18) follow from (1.16) by integration, it is sufficient to show that (1.16) holds for each $l \geq 4$. (In fact, the most delicate case is $l = 4$. In this case $l^* = -1/2$ and we can use the fact that $\int_0^{y_0} 1/\sqrt{y}\, dy$ for $y_0 > 0$ converges.)

The inequality (1.16) follows if one shows that, for each $k \geq 4$

$$(2.1) \qquad |kP_k(y)|, |kQ_k(y)|, |kR_k(y)| \leq \frac{c}{3}|y|^{k^*} M^{k-3} \quad (|y| \leq \delta)$$

under the assumption that (1.16), (1.17) and (1.18) hold for all $3 \leq l \leq k - 1$. In fact, if (2.1) holds, (1.16) for $l = k$ follows immediately. Then by the initial condition (1.9) (cf. (1.8)), we have (1.17) and (1.18) for $l = k$ by integration.

The estimation of $|kP_k|$ for $k \geq 4$. By (1.10) and using the fact that (1.17), (1.18) hold for $l \leq k - 1$, we have for each $|y| < \delta$ that

$$
\begin{aligned}
|kP_k| &\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} |b_m(y)|\, |b'_{k-m+2}(y)| \\
&\leq \sum_{m=3}^{k-1} \frac{2k|k-2m+3|}{k-m+2} \left(\frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right) \left(\frac{3cM^{k-m+2-3}|y|^{(k-m+2)^*+1}}{(k-m+2)^*+2} \right) \\
&= cM^{k-3}|y|^{k^*} \frac{144c|y|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\
&\leq cM^{k-3}|y|^{k^*} \frac{144c|\delta|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)(k-m+2)} \\
&\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k|k-2m+3|}{(m-1)^2(k-m+1)^2}.
\end{aligned}
$$

Here, we used (1.14). Since

$$\max_{m=3,\dots,k-1} |k-2m+3| = \max_{m=3,k-1} |k-2m+3| = \max\{|k-3|,|-k+5|\},$$

by setting $q = m - 1$, we have that

$$
\begin{aligned}
|kP_k| &\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-1} \frac{k^2}{(m-1)^2(k-m+1)^2} = \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \sum_{q=2}^{k-2} \frac{k^3}{q^2(k-q)^2} \\
&\leq \frac{c}{3\tau} M^{k-3}|y|^{k^*} \frac{1}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2(1-u)^2} \leq \frac{c}{3} M^{k-3}|y|^{k^*},
\end{aligned}
$$

where we applied Lemma A.1 and (1.15) at the last step of the estimations. Hence, we get (2.1) for kP_k.

The estimation of $|kQ_k|$ for $k \geq 7$. By (1.11) and the induction assumption, we have that

$$
\begin{aligned}
|kQ_k| &\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} |b'_m(y)| \, |b'_n(y)| \, |b_{k-m-n+2}(y)| \\
&\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} \left(\frac{3cM^{m-3}|y|^{m^*+1}}{m^*+2} \right) \\
&\qquad \times \left(\frac{3cM^{n-3}|y|^{n^*+1}}{n^*+2} \right) \left(\frac{3cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*+2}}{((k-m-n+2)^*+2)^2} \right) \\
&= cM^{k-3}|y|^{k^*} \frac{432c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{(m-1)^2(n-1)^2(k-m-n+2)^2}.
\end{aligned}
$$

Now we apply the inequalities

$$
\max_{\substack{3 \leq m \leq k-4 \\ 3 \leq n \leq k-m-1}} |3n-k+m-1| = \max_{(m,n)=(3,3),(3,k-4),(k-4,3)} |3n-k+m-1|
$$

$$
= \max\{|-k+11|, 4, |2k-10|\} \leq 2k,
$$

and also

$$
\frac{432c^2}{M^4} \leq \frac{1}{36\tau},
$$

which follows from (1.14). Setting $p := m-1$, $q = n-1$, we have that

$$
\begin{aligned}
|kQ_k| &\leq \frac{c}{36\tau} M^{k-3}|y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{2k^2}{(m-1)^2(n-1)^2(k-m-n+2)^2} \\
&= \frac{c}{18\tau} M^{k-3}|y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2}.
\end{aligned}
$$

Now applying Lemma A.2, we have that

$$
|kQ_k| \leq \frac{c}{18\tau} M^{k-3}|y|^{k^*} \times 6\tau \leq \frac{c}{3} M^{k-3}|y|^{k^*},
$$

which proves (2.1) for kQ_k.

The estimation of $|kR_k|$ for $k \geq 7$. As in the case of $|kQ_k|$, we have that

$$
|kR_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|b_m(y)|\,|b_n(y)|\,|b''_{k-m-n+2}(y)|}{k-m-n+2}
$$

$$
\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k-m-n+2} \left(\frac{3cM^{m-3}|y|^{m^*+2}}{(m^*+2)^2} \right)
$$

$$
\times \left(\frac{3cM^{n-3}|y|^{n^*+2}}{(n^*+2)^2} \right) (cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*})
$$

$$
= 144c^3 M^{k-7}|y|^{k^*} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{(k-m-n+2)(m-1)^2(n-1)^2}
$$

$$
= cM^{k-3}|y|^{k^*} \frac{144c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^2}{(k-m-n+2)^2(m-1)^2(n-1)^2}.
$$

Now we set $p = m - 1$, $q = n - 1$, and using the inequality

$$
3^4 \times 144c^2\tau \leq 3^4 \times 192c^2\tau < M^4,
$$

we have that

$$
|kR_k| \leq \frac{c}{3^4\tau} M^{k-3}|y|^{k^*} \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2}.
$$

By applying Lemma A.2, we have that

$$
|kR_k| \leq \frac{c}{3^4\tau} M^{k-3}|y|^{k^*} \times 6\tau < \frac{c}{3} M^{k-3}|y|^{k^*},
$$

which proves (2.1) for kR_k. This completes the proof of Proposition 1.3.

Appendix A. Inequalities used in the proof of Theorem 1.1

For $a > 0$, it holds that

$$
(A.1) \qquad \frac{1}{u^2(a-u)^2} = \frac{1}{a^3}\left(\frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right).
$$

Therefore,

$$
(A.2) \qquad \int_t^{a-t} \frac{du}{u^2(a-u)^2} = \frac{2}{a^3}\left(\frac{a(a-2t)}{t(a-t)} + 2\log\frac{a-t}{t} \right) \quad (0 < t < a/2).
$$

In particular, one can show that there exists a positive constant τ such that

(A.3) $$t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \le \tau \quad (0 < t < 1/2).$$

The following assertion is needed to prove (2.1) for $kP_k(y)$:

Lemma A.1. *Let p be a non-negative integer and k an integer satisfying $k \ge p + 4$. Then the inequality*

$$\sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \le \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \quad (a := 1 - p/k)$$

holds.

Proof.   In fact, if we set $a := 1 - p/k$, then (A.1) yields that

$$\frac{k^3}{q^2(k-p-q)^2} = \frac{1}{k} \frac{1}{(q/k)^2(a-q/k)^2}$$
$$= \frac{1}{a^3}\left[\frac{1}{k}\left(\frac{a}{(q/k)^2} + \frac{2}{q/k}\right) + \frac{1}{k}\left(\frac{a}{(a-q/k)^2} + \frac{2}{a-q/k}\right)\right].$$

Since $x \mapsto (a + 2x)/x^2$ is a monotone decreasing function and the function $x \mapsto (a + 2(a-x))/(a-x)^2$ is monotone increasing on the interval $(0, a/2)$, we have that

$$\frac{k^3}{q^2(k-p-q)^2} \le \frac{1}{a^3}\left[\int_{(q-1)/k}^{q/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{q/k}^{(q+1)/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du\right],$$

which yields that

$$\sum_{q=2}^{k-p-2} \frac{a^3 k^3}{q^2(k-p-q)^2} \le \int_{1/k}^{a-2/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{2/k}^{a-1/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du$$

$$\le \int_{1/k}^{a-1/k}\left(\frac{a}{u^2} + \frac{2}{u}\right) du + \int_{1/k}^{a-1/k}\left(\frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du$$

$$\le \int_{1/k}^{a-1/k}\left(\frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u}\right) du$$

$$= \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2}.$$

This proves the assertion. □

The following assertion is needed to prove (2.1) for $kQ_k(y)$ and $kR_k(y)$:

Lemma A.2. *For any integer $k \geq 7$, the following inequalities holds*:

$$\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2 (1-u)^2} \leq 6\tau,$$

where τ is a constant satisfying (A.3).

Proof. We set $a = a(p) := 1 - (p/k)$. Applying Lemma A.1 and the identity (A.2), we have that

$$\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} = \sum_{p=2}^{k-5} \left[\frac{1}{kp^2} \sum_{q=2}^{k-p-2} \frac{k^3}{q^2 (k-p-q)^2} \right]$$

$$\leq \sum_{p=2}^{k-5} \left[\frac{1}{kp^2} \int_{1/k}^{a-1/k} \frac{du}{u^2 (a-u)^2} \right] = \sum_{p=2}^{k-5} \left[\frac{1}{p^2} \frac{2}{a^2} \left(\frac{a-2/k}{a-1/k} + 2\frac{\log(ka-1)}{ka} \right) \right]$$

$$\leq \sum_{p=2}^{k-5} \left[\frac{2}{p^2 a^2} \left(1 + 2\frac{\log ka}{ka} \right) \right] \leq \sum_{p=2}^{k-5} \frac{6}{p^2 a^2},$$

where we used the fact that $(\log ka)/(ka) < 1$. By applying Lemma A.1 and by using the property (A.3) of the constant τ, it holds that

$$\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k-p-q)^2} \leq 6 \sum_{p=2}^{k-5} \frac{1}{p^2 (1-p/k)^2} = \frac{6}{k} \sum_{p=2}^{k-5} \frac{k^3}{p^2 (k-p)^2}$$

$$\leq \frac{6}{k} \sum_{p=2}^{k-2} \frac{k^3}{p^2 (k-p)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2 (1-u)^2} < 6\tau,$$

which proves the assertion. □

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