Hai, D.D.
Osaka J. Math.
**52** (2015), 393–408

# A COMPARISON PRINCIPLE AND APPLICATIONS TO ASYMPTOTICALLY $p$-LINEAR BOUNDARY VALUE PROBLEMS

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(Received January 16, 2013, revised November 18, 2013)

**Abstract**

Consider the problems

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega, \\ -\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$. We prove a strong comparison principle that allows $f - g$ to change sign. An application to singular asymptotically $p$-linear boundary problems is given.

## 1. Introduction

Consider the problems

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega, \\ -\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0, 1)$, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$, and $f, g\colon \Omega \to \mathbb{R}$.

In this paper, we shall establish a strong comparison principle

$$u > v \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial\Omega,$$

without requiring that $f \geq g$ a.e. in $\Omega$. Here $\nu$ denotes the outer unit normal vector on $\partial\Omega$. It should be noted that the assumptions $f \geq g$ and $f \not\equiv g$ in $\Omega$ are needed in previous literature (see e.g. [9] and the references therein). We also provide an application to the existence of positive solutions for a class of singular $p$-Laplacian boundary value problems with asymptotically $p$-linear nonlinearity.

Let $d(x) = d(x, \partial\Omega)$ be the distance from $x$ to $\partial\Omega$, we prove the following result:

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2010 Mathematics Subject Classification. 35J95, 35J70.

**Theorem 1.1.** *Let $f, g, g_0 \in L^1(\Omega)$ with $g \geq g_0 \geq 0$, and $g_0 \not\equiv 0$. Suppose there exist constants $C > 0$ and $\gamma \in (0, 1)$ such that*

$$|f(x)|, g(x) \leq \frac{C}{d^\gamma(x)}$$

*for a.e. $x \in \Omega$, and there exist a function $h \in C(\Omega)$, $h > 0$, and constants $\varepsilon \geq 0$, $m, M > 0$ with $m \leq M$ such that*

$$f - g \geq m\left(h - \frac{\varepsilon}{d^\gamma}\right) \quad in \quad \Omega.$$

*Let $u, v \in W_0^{1,p}(\Omega)$ be solutions of (1.1). Then there exists a positive constant $\varepsilon_0$ depending on $n$, $\Omega$, $p$, $\gamma$, $C$, $M$, $h$, $g_0$ (but not on $m$), such that*

$$u > v \quad in \quad \Omega \quad and \quad \frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu} \quad on \quad \partial\Omega$$

*for $\varepsilon < \varepsilon_0$. If $\varepsilon = 0$, the result holds under the weaker condition that $h$ is a nonnegative nontrivial measurable function in $\Omega$.*

REMARK 1.1. When $g \equiv 0$, the conclusion of Theorem 1.1 holds under the weaker assumption that $h$ is a nonnegative nontrivial measurable function in $\Omega$. In this case, $\varepsilon_0$ is independent of $M$. Indeed, let $\bar{u}$, $\bar{v}$ be the solutions of

$$-\Delta_p \bar{u} = \tilde{h} - \frac{\varepsilon}{d^\gamma} \text{ in } \Omega, \quad \bar{u} = 0 \text{ on } \partial\Omega,$$
$$-\Delta_p \bar{v} = \tilde{h} \text{ in } \Omega, \quad \bar{v} = 0 \text{ on } \partial\Omega,$$

respectively, where $\tilde{h} = \min(h, 1/d^\gamma)$. By the strong maximum principle [12, 14], there exists a constant $\delta > 0$ such that $\bar{v} \geq \delta d$ in $\Omega$. Using Lemma 2.3 in Section 2, we deduce that

$$\bar{u} \geq \bar{v} - \frac{\delta}{2} d \geq \frac{\delta}{2} d$$

if $\varepsilon$ is sufficiently small. This implies

$$u \geq m^{1/(p-1)} \bar{u} > m^{1/(p-1)} \frac{\delta}{2} d > 0 \quad \text{in} \quad \Omega$$

and $\partial u / \partial \nu < 0$ on $\partial\Omega$.

As an application of Theorem 1.1, consider the boundary value problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{u^\beta} + \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $\beta \in (0,1)$, $q$, $f$ satisfy the following assumptions:
(A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and there exists a constant $k > 0$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = k.$$

(A2) There exists a constant $\delta \in (0,1)$ such that

$$\limsup_{u\to 0^+} u^{\delta}|f(u)| < \infty.$$

(A3) There exist constants $A, \varepsilon_0 > 0$ such that

$$f(u) \geq ku^{p-1} + \varepsilon_0 \quad \text{for} \quad u > A.$$

(A4) $q : \Omega \to \mathbb{R}$ is measurable and there exist constants $\eta$, $L > 0$ with $\beta + \eta < 1$, such that

$$|q(x)| \leq \frac{L}{d^{\eta}(x)}$$

for a.e. $x \in \Omega$.

Let $\lambda_1$ be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition, and let $\phi_1$ be the corresponding positive eigenfunction with $\|\phi_1\|_\infty = 1$. Note that, since $\partial\phi_1/\partial\nu < 0$ on $\partial\Omega$, Theorem 1.1 holds if $d$ is replaced by $\phi_1$. Let $\lambda_\infty = \lambda_1/k$. Then we have

**Theorem 1.2.** *Let* (A1)–(A4) *hold. Then there exists a constant* $\tilde{\varepsilon} > 0$ *such that for* $\lambda_\infty - \tilde{\varepsilon} < \lambda < \lambda_\infty$, *problem* (1.2) *has a positive solution* $u_\lambda \in C^{1,\kappa}(\bar{\Omega})$ *for some* $\kappa \in (0,1)$ *with*

$$(*) \qquad u_\lambda \geq \left(\frac{\lambda_\infty \varepsilon_0}{4k(\lambda_\infty - \lambda)}\right)^{1/(p-1)} \phi_1 \quad \textit{in} \quad \Omega.$$

**Theorem 1.3.** *Let* $q \geq 0$, $q \not\equiv 0$. *Suppose* $f \geq 0$, (A2), (A4) *hold, and*

$$\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} = k$$

*for some* $k \in (0,\infty)$. *Then problem* (1.2) *has a positive solution* $u_\lambda$ *for* $\lambda < \lambda_\infty$. *If, in addition,*

$$f(u) \geq ku^{p-1} \quad \textit{for all} \quad u > 0,$$

*then* (1.2) *has no positive solution for* $\lambda \geq \lambda_\infty$, *and*

$$\|u_\lambda\|_\infty \to \infty \quad \textit{as} \quad \lambda \to \lambda_\infty^-.$$

EXAMPLE 1.1. (i) Let $f(u) = -1/u^\delta + u^{p-1} + u^q$, where $\delta \in (0, 1)$ and $0 \le q < p-1$. Then $f$ satisfies (A1)–(A3) with $k = 1$, and so (1.2) has a positive solution when $\lambda$ is sufficiently close to $\lambda_1$ and $\lambda < \lambda_1$, by Theorem 1.2.
(ii) Let $f(u) = 1/u^\delta + u^{p-1}(m|\sin u| + e^{1/(1+u)})$, where $\delta \in (0,1)$, $m \ge 0$. Then it follows from Theorem 1.3 that, if $m > 0$, (1.2) has a positive solution for $\lambda < \lambda_1/(m+1)$, and, if $m = 0$, (1.2) has a positive solution if and only if $\lambda < \lambda_1$.

REMARK 1.2. It should be noted that Theorem 1.2 may not be true when $\varepsilon_0 = 0$. Indeed, by multiplying the equation in (1.2) by $u$ and integrating, we see that (1.2) has no positive solution for $\lambda < \lambda_\infty$ when $q \le 0$ and $f(u) = ku^{p-1}$.

REMARK 1.3. In [15], assuming that $f$ is continuous and nonnegative on $[0, \infty)$, $\lim_{u\to\infty} f(u)/u = k \in (0, \infty)$, and $f$ satisfies some additional conditions at 0, Zhang showed via variational method that (1.2) with $p = 2$ has a positive solution for $\lambda \in (0, \lambda_1/k)$, provided that $q \ge 0$, $q \not\equiv 0$, $q\phi_1^{-\beta} \in L^r(\Omega)$, where $n/2 < r$. The result in [15] was improved by Hai in [4], using sub- and super solutions approach. The proof in [4] depends on the linearity of the Laplacian and can not be applied to the general case where $p > 1$, except for radial solutions in a ball [6]. Related results on the case where $f$ is nonsingular can be found in Ambrosetti, Arcoya, and Buffoni [1], Ambrosetti and Hess [2], and Ambrosetti, Garcia Azorero, and Peral [3]. The approach in [1, 2, 3] was via bifurcation theory for $p = 2$ in [1, 2] and $p > 1$ in [3]. Thus, Theorems 1.2 and 1.3 provide extensions of corresponding results in [1, 2, 3, 4, 6, 15] to the singular $p$-Laplacian case. Note that the precise lower bound estimate $(*)$ has not been obtained in previous literature.

## 2. Preliminary results

Let $D$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial D$.

We shall denote the norm in $C^{k,\alpha}(\bar{D})$ and $L^k(D)$ by $|\cdot|_{k,\alpha}$ and $\|\cdot\|_k$ respectively. The distance from $x$ to $\partial D$ is denoted by $d(x, \partial D)$.

We first recall the following regularity result in [5, Lemma 3.1], which plays an important role in the proofs of our main results.

**Lemma A.** *Let $h \in L^\infty_{\text{loc}}(\Omega)$ and suppose there exist numbers $\gamma \in (0,1)$ and $C > 0$ such that*

$$|h(x)| \le \frac{C}{d^\gamma(x)} \tag{3.1}$$

*for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the solution of*

$$\begin{cases} -\Delta_p u = h & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

*Then there exist constants* $\alpha \in (0,1)$ *and* $\tilde{M} > 0$ *depending only on* $C$, $\gamma$, $\Omega$ *such that* $u \in C^{1,\alpha}(\bar{\Omega})$ *and* $|u|_{1,\alpha} < \tilde{M}$.

Let

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where $a_{ij} \in C^{0,\alpha}(\bar{D})$, $1 \le i, j \le n$, for some $\alpha \in (0,1)$, and suppose there exist constants $m_0, m_1 > 0$ such that

$$|a_{ij}|_{0,\alpha} \le m_1 \tag{2.1}$$

for $1 \le i, j \le n$, and

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge m_0|\xi|^2 \tag{2.2}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

**Lemma 2.1.** *Let* $h \in L^1(D)$ *and suppose there exist constants* $C > 0$ *and* $\gamma \in (0,1)$ *such that*

$$|h(x)| \le \frac{C}{d^{\gamma}(x, \partial D)} \tag{2.3}$$

*for a.e.* $x \in D$. *Let* $w \in H_0^1(D)$ *be the solution of*

$$\begin{cases} Lw = h & \text{in} \quad D, \\ w = 0 & \text{on} \quad \partial D. \end{cases} \tag{2.4}$$

*Then there exist constants* $\beta \in (0,1)$ *and* $\tilde{M} > 0$ *depending only on* $m_0$, $m_1$, $C$, $\gamma$, $D$, $n$, *such that* $w \in C^{1,\beta}(\bar{D})$ *and*

$$|w|_{1,\beta} \le \tilde{M}.$$

Proof. Let $\phi \in C^1(\bar{D})$ be the solution of

$$L\phi = 1 \text{ in } D, \quad \phi = 0 \text{ on } \partial D.$$

Then there exists a constant $C_0 > 0$ independent of $a_{ij}$ such that $\phi(x) \le C_0 d(x, \partial D)$ for all $x \in D$. Let $a = 2^{1/(1-\gamma)}\|\phi\|_\infty$ and $h_0 \colon [0, a] \to \mathbb{R}$ satisfy

$$\begin{cases} -h_0'' = \dfrac{1}{t^{\gamma}}, & 0 < t < a, \\ h_0(0) = 0, & h_0'(a) = 0. \end{cases}$$

Note that $h_0(t) = (t/(1-\gamma))(a^{1-\gamma} - t^{1-\gamma}/(2-\gamma))$. A calculation shows that

$$L(h_0(\phi)) = -h_0''(\phi)\sum_{i,j=1}^{n} a_{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} + h_0'(\phi)$$
$$\geq \frac{m_0}{\phi^\gamma}|\nabla\phi|^2 + \frac{a^{1-\gamma}-\phi^{1-\gamma}}{1-\gamma} \geq \frac{m_0}{\phi^\gamma}|\nabla\phi|^2 + \frac{a^{1-\gamma}}{2(1-\gamma)} \geq \frac{m_2}{d^\gamma(x,\partial D)},$$

where $m_2$ is independent of $a_{ij}$. By the weak comparison principle ([11, Lemma A.2], [13, Lemma 3.1]),

$$|w| \leq \frac{C}{m_2}h_0(\phi) \quad \text{in} \quad D,$$

i.e. $w$ is bounded in $D$. By Lemma A, the problem

$$\begin{cases} -\Delta z = h & \text{in} \quad D, \\ z = 0 & \text{on} \quad \partial D, \end{cases}$$

has a solution $z \in C^{1,\alpha}(\bar{D})$ for some $\alpha \in (0,1)$. Since $w$ satisfies

$$-\operatorname{div}(A(x,\nabla w) - \nabla z) = 0 \quad \text{in} \quad D,$$

where $A = (A_1, \ldots, A_n)$, $A_i(x,\eta) = \sum_{j=1}^n a_{ij}(x)\eta_j$, $\eta = (\eta_1, \ldots, \eta_n)$, the result now follows from Lieberman [8, Theorem 1]. □

**Lemma 2.2.** *Let $h$ satisfy* (2.3), $h \geq 0$, $h \not\equiv 0$, *and let $w \in H_0^1(D)$ be the solution of* (2.4). *Then there exists a constant $k_0 > 0$ depending only on $h$, $m_0$, $m_1$, $C$, $\gamma$, $D$, $n$ such that*

$$w(x) \geq k_0 d(x,\partial D)$$

*for all $x \in D$.*

Proof. Let $\Lambda$ be the set of all solutions $w$ of (2.4) among the coefficients $a_{ij}$ that satisfy (2.1) and (2.2). By the strong maximum principle, $w > 0$ in $\Omega$ and $\partial w/\partial\nu < 0$ on $\partial D$. By Lemma 2.1, $w \in C^{1,\beta}(\bar{D})$ and there exists a constant $\tilde{M} > 0$ such that $|w|_{1,\beta} \leq \tilde{M}$ for all $w \in \Lambda$. Since $\Lambda$ is closed in $C^1(\bar{D})$, $\Lambda$ is compact in $C^1(\bar{D})$. Define $G\colon \Lambda \to \mathbb{R}$ by

$$Gw = \inf_{x\in D}\frac{w(x)}{d(x,\partial D)}.$$

Then $G$ is continuous and positive on $\Lambda$, and therefore has a positive minimum, which completes the proof. □

**Lemma 2.3.** *Let* $f, g \in L^1(D)$ *satisfy*

$$|f(x)|, |g(x)| \leq \frac{C}{d^\gamma(x, \partial D)}$$

*for a.e.* $x \in \Omega$ *for some constant* $C > 0$*. Let* $u$*,* $v$ *be the solutions of* (1.1)*. Then* $|u - v|_{0,1} \to 0$ *as* $\|f - g\|_1 \to 0$.

Proof. Note that $f, g \in L^1(\Omega)$ (see [7, p. 6]). By Lemma A, $u, v \in C^{1,\alpha}(\bar{D})$ for some $\alpha \in (0,1)$, and there exists a constant $\tilde{M} > 0$ independent of $u, v$, such that $|u|_{1,\alpha}$, $|v|_{1,\alpha} \leq \tilde{M}$. Multiplying the equation

$$-(\Delta_p u - \Delta_p v) = f - g \quad \text{in} \quad \Omega$$

by $u - v$ and integrating, we obtain

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - v)\, dx = \int_\Omega (f - g)(u - v)\, dx.$$

Using the inequality [10, Lemma 30.1],

$$(|x| + |y|)^{2-\min(p,2)}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq c|x - y|^{\max(p,2)}$$

for $x, y \in \mathbb{R}^n$, where $c$ is a positive constant depending only on $p$, we obtain

$$\int_\Omega |\nabla(u - v)|^r\, dx \leq c_1 \|f - g\|_{L^1} \|u - v\|_\infty \leq c_2 \|f - g\|_{L^1},$$

where $r = \max(p, 2)$ and $c_1$, $c_2$ are constants depending only on $p$, $\tilde{M}$.

Hence

$$\|u - v\|_2 \to 0$$

as $\|f - g\|_1 \to 0$, and since $C^{1,\alpha}(\bar{D})$ is compactly imbedded in $C^1(\bar{D})$, Lemma 2.3 follows. □

## 3. Proofs of the main results

Proof of Theorem 1.1. By the strong maximum principle, there exists a constant $\delta > 0$ such that $v \geq \delta d$ in $\Omega$. Let $\varepsilon \in [0, 1)$, $m_\varepsilon = \min(m, \varepsilon)$ and $\tilde{h} = \min(h, 1/d^\gamma)$. Then

$$f \geq g + m_\varepsilon \tilde{h} - \frac{M\varepsilon}{d^\gamma} \equiv \tilde{f} \quad \text{in} \quad \Omega.$$

Let $\tilde{u}$ satisfy

$$-\Delta_p \tilde{u} = \tilde{f} \text{ in } \Omega, \quad \tilde{u} = 0 \text{ on } \partial\Omega.$$

Then $u \geq \tilde{u}$ in $\Omega$, by the weak comparison principle. Since

$$|\tilde{f}| \leq \frac{C+1+M}{d^\gamma} \equiv \frac{\tilde{C}}{d^\gamma}$$

and

$$\int_\Omega |\tilde{f} - g|\, dx \leq \varepsilon(1+M) \int_\Omega \frac{1}{d^\gamma}\, dx,$$

it follows that $\tilde{f} \to g$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Here we have used the fact that $1/d^\gamma \in L^1(\Omega)$ (see e.g. [7, p. 6]). Using Lemma 2.3, we see that $\tilde{u} \to v$ in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$. Hence

$$u \geq \tilde{u} \geq \frac{\delta d}{2} \quad \text{in} \quad \Omega$$

if $\varepsilon$ is sufficiently small, which we shall assume.

By Lemma A, there exist constants $\tilde{M} > 0$ and $\alpha \in (0,1)$ independent of $u$, $v$ such that $|u|_{1,\alpha}, |v|_{1,\alpha} \leq \tilde{M}$. Thus

$$\frac{\delta d}{2} \leq u, \quad v \leq \tilde{M} d \quad \text{in} \quad \Omega, \tag{3.1}$$

and therefore

$$u \geq c_0 v \quad \text{in} \quad \Omega,$$

where $c_0 = \delta/(2\tilde{M})$.

Let $c$ be the largest number such that $u \geq cv$ in $\Omega$, and suppose that $c \leq 1$. From (3.1), it follows that

$$\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu} \leq -\frac{\delta}{2} \quad \text{on} \quad \partial\Omega.$$

For $t \in [0, 1]$, let $w_t = t\nabla u + (1-t)c\nabla v$. Then

$$\begin{aligned} w_t \cdot \nu &= t\frac{\partial u}{\partial \nu} + (1-t)c\frac{\partial v}{\partial \nu} \leq -\frac{tc\delta}{2} - \frac{(1-t)c\delta}{2} \\ &= -\frac{c\delta}{2} \leq -\frac{c_0\delta}{2} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

which implies

$$|w_t| \geq \frac{c_0\delta}{2} \equiv c_1 \quad \text{on} \quad \partial\Omega \tag{3.2}$$

for all $t \in [0, 1]$.

Let $x \in \Omega$ and $x_0 \in \partial\Omega$ be such that $d(x) = |x - x_0|$. Since $|w_t|_{0,\alpha} \leq \tilde{M}$, it follows that

$$|w_t(x) - w_t(x_0)| \leq \tilde{M} d^\alpha(x),$$

which, together with (3.2), implies

$$|w_t(x)| \geq c_1 - \tilde{M} d^\alpha(x) \geq \frac{c_1}{2} \equiv c_2 \tag{3.3}$$

for $x \in \Omega_\eta \equiv \{x \in \Omega\colon d(x) < \eta\}$, where $\eta = (c_1/2\tilde{M})^{1/\alpha}$.

Next, we have

$$-(\Delta_p u - \Delta_p(cv)) = f - c^{p-1} g \quad \text{in} \quad \Omega, \tag{3.4}$$

and the left hand side of (3.4) can be linearized as $L(u - cv)$, where

$$Lw = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right)$$

and $a_{ij}(x) = \int_0^1 (\partial a^i/\partial z_j)(t\nabla u + (1-t)c\nabla v)\,dt$, $a^i(z) = |z|^{p-2} z$.

Note that, in view of (3.3) and the fact that $|w_t|_{0,\alpha} \leq \tilde{M}$ for $t \in [0, 1]$, the coefficients $a_{ij}$ satisfy (2.1) and (2.2) in $\Omega_\eta$ with $m_0 = (p-1)\min(\tilde{M}^{p-2}, c_2^{p-2})$.

Let $w_0$, $w_1$ be the solutions of

$$Lw_0 = \tilde{h} \text{ in } \Omega_\eta, \quad w_0 = 0 \text{ on } \partial\Omega_\eta,$$

and

$$Lw_1 = \frac{1}{d^\gamma(x, \partial\Omega_\eta)} \text{ in } \Omega_\eta, \quad w_1 = 0 \text{ on } \partial\Omega_\eta$$

respectively. By Lemmas 2.1 and 2.2, there exist positive constants $M_0$ and $k_0$ such that

$$w_0 \geq k_0 d(x, \partial\Omega_\eta), \quad w_1 \leq M_0 d(x, \partial\Omega_\eta) \quad \text{in} \quad \Omega_\eta. \tag{3.5}$$

Since $c \leq 1$ and $d(x) \geq d(x, \partial\Omega_\eta)$ for $x \in \Omega_\eta$,

$$L(u - cv) \geq f - g \geq m\left(\tilde{h} - \frac{\varepsilon}{d^\gamma(x, \partial\Omega_\eta)}\right) \quad \text{in} \quad \Omega_\eta,$$

and since $u \geq cv$ on $\partial\Omega_\eta$, it follows from the weak comparison principle and (3.5) that, for $x \in \Omega_{\eta/2}$,

$$\begin{aligned} u - cv &\geq m(w_0 - \varepsilon w_1) \geq m(k_0 - \varepsilon M_0) d(x, \partial\Omega_\eta) = m(k_0 - \varepsilon M_0) d(x) \\ &\geq m\left(\frac{k_0}{2}\right) d(x) \end{aligned} \tag{3.6}$$

if $\varepsilon < k_0/2M_0$. In particular,

$$u - cv \geq m\left(\frac{k_0}{2}\right)\frac{\eta}{2} \equiv mk_1 \quad \text{when} \quad d(x) = \frac{\eta}{2}.$$

If $\varepsilon = 0$ then it follows from

$$-\Delta_p u = f \geq c^{p-1} g = -\Delta_p(cv + mk_1) \quad \text{in} \quad \Omega$$

and $u \geq cv + mk_1$ on $\partial(\Omega \setminus \Omega_{\eta/2})$ that

$$u \geq cv + mk_1 \quad \text{in} \quad \Omega \setminus \Omega_{\eta/2}. \tag{3.7}$$

Suppose $\varepsilon > 0$ and $h \geq a > 0$ in $\Omega \setminus \Omega_{\eta/2}$. Then we have

$$-\Delta_p u = f \geq g + m\left(a - \frac{\varepsilon}{d^\gamma(x)}\right) \geq g + m\left(a - \frac{\varepsilon}{(\eta/2)^\gamma}\right) \geq g \quad \text{in} \quad \Omega \setminus \Omega_{\eta/2},$$

if $\varepsilon$ is sufficiently small. Hence (3.7) holds by the weak comparison principle. This, together with (3.6), gives the existence of a constant $\tilde{c} > c$ such that $u \geq \tilde{c}v$ in $\Omega$, a contradiction. Hence $c > 1$ and therefore $u > v$ in $\Omega$ and

$$\frac{\partial(u - v)}{\partial \nu} \leq (c - 1)\frac{\partial v}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega,$$

which completes the proof. □

Proof of Theorem 1.2. Let $\lambda > 0$ be such that $\lambda_\infty/2 < \lambda < \lambda_\infty$. Let $c = (\lambda_\infty \varepsilon_0/(4k(\lambda_\infty - \lambda)))^{1/(p-1)}$ and $M$ be a constant such that $M > c$. Define

$$\mathbf{K} = v \in C(\bar{\Omega})\colon c\phi_1 \leq v \leq M\phi_1 \text{ in } \Omega\}.$$

For each $v \in \mathbf{K}$, it follows from Lemma A that the problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{v^\beta} + \lambda f(v) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \equiv Tv \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ such that $|u|_{1,\alpha} < \tilde{M}$, where $\alpha$, $\tilde{M}$ are independent of $v \in \mathbf{K}$. We shall show that $T\colon \mathbf{K} \to C(\bar{\Omega})$ is a compact operator. In view of the compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, we need only to show that $T$ is continuous. Let $(v_n)$ be a sequence in $\mathbf{K}$ such that $v_n \to v$ in $C(\bar{\Omega})$, and let $u_n = Tv_n$, $u = Tv$. Let $G(w) = q(x)/w^\beta + \lambda f(w)$ for $w \in \mathbf{K}$. Then

$$G(v_n) \to G(v) \quad \text{pointwise in} \quad \Omega,$$

and it follows from (A2) and (A4) that there exist constants $K, C_0 > 0$ such that

$$|G(v_n)| \leq \frac{q(x)}{(c\phi_1)^\beta} + \frac{\lambda_\infty K}{(c\phi_1)^\delta} \leq \frac{C_0}{d^\gamma}$$

for all $n$, where $\gamma = \max(\beta + \eta, \delta)$. Hence $G(v_n) \to G(v)$ in $L^1(\Omega)$, and Lemma 2.3 implies $u_n \to u$ in $C^1(\bar{\Omega})$, i.e., $T$ is continuous on $\mathbf{K}$.

Next, we shall show that if $\lambda$ is sufficiently close to $\lambda_\infty$ and $M$ is large enough then $T$ maps $\mathbf{K}$ into $\mathbf{K}$. By (A2) and (A3), there exists a constant $M_0 > 0$ such that

$$f(z) \geq kz^{p-1} + \varepsilon_0 - \frac{M_0}{z^\delta} \tag{3.8}$$

for all $z > 0$. Let $v \in \mathbf{K}$ and $u = Tv$. Then (3.8) and (A4) imply

$$-\Delta_p u \geq -\frac{L}{(c\phi_1)^\beta d^\eta} + \lambda\left(k(c\phi_1)^{p-1} + \varepsilon_0 - \frac{M_0}{(c\phi_1)^\delta}\right) \quad \text{in} \quad \Omega.$$

Consequently,

$$-\Delta_p\left(\frac{u}{c}\right) \geq -\frac{L_1}{c^{p-1+\beta}\phi_1^\gamma} - \frac{\lambda_\infty M_0}{c^{p-1+\delta}\phi_1^\gamma} + \lambda\left(k\phi_1^{p-1} + \frac{\varepsilon_0}{c^{p-1}}\right) \equiv f_{c,\lambda} \quad \text{in} \quad \Omega,$$

where $L_1$ is a positive constant such that $d/\phi_1 \geq (L/L_1)^{1/\eta}$.

Let $\bar{u}_c$, $\bar{z}_c$ be the solutions of

$$-\Delta_p \bar{u}_c = f_{c,\lambda} \text{ in } \Omega, \quad \bar{u}_c = 0 \text{ on } \partial\Omega,$$

and

$$-\Delta_p \bar{z}_c = \lambda\left(k\phi_1^{p-1} + \frac{\varepsilon_0}{2c^{p-1}}\right) \equiv g_{c,\lambda} \text{ in } \Omega, \quad \bar{z}_c = 0 \text{ on } \partial\Omega,$$

respectively. Then $u \geq c\bar{u}_c$ in $\Omega$. Note that

$$|f_{c,\lambda}|, |g_{c,\lambda}| \leq \frac{\tilde{C}}{\phi_1^\gamma},$$

where $\tilde{C} > 0$ depends only on $\varepsilon_0$, $k$, $p$, $L_1$, $\lambda_\infty$, $M_0$. Since

$$f_{c,\lambda} - g_{c,\lambda} \geq \frac{1}{c^{p-1}}\left[\frac{\lambda_\infty \varepsilon_0}{4} - \left(\frac{L_1}{c^\beta} + \frac{\lambda_\infty M_0}{c^\delta}\right)\frac{1}{\phi_1^\gamma}\right] \quad \text{in} \quad \Omega,$$

and

$$c^{1-p} \leq \frac{2k}{\varepsilon_0},$$

it follows from Theorem 1.1 with $m = c^{1-p}$, $M = 2k/\varepsilon_0$, $h = \lambda_\infty\varepsilon_0/4$, $g_0 = (\lambda_\infty/2)k\phi_1^{p-1}$, that $\bar{u}_c > \bar{z}_c$ in $\Omega$ for $c \gg 1$, which implies

$$u \geq c\bar{z}_c \equiv \tilde{z}_c \quad \text{in} \quad \Omega. \tag{3.9}$$

By the choice of $c$,

$$(\lambda_1 - \lambda k)c^{p-1} = \frac{\lambda_\infty \varepsilon_0}{4} \leq \frac{\lambda \varepsilon_0}{2}.$$

Hence

$$-\Delta_p \tilde{z}_c = \lambda k\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{2k}\right) \geq \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

and since

$$-\Delta_p(c\phi_1) = \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

it follows that

$$\tilde{z}_c \geq c\phi_1 \quad \text{in} \quad \Omega. \tag{3.10}$$

Hence, if $\lambda$ is sufficiently close to $\lambda_\infty$, it follows from (3.9) and (3.10) that $u \geq c\phi_1$ in $\Omega$.

Next, let $\tilde{\lambda}_\infty > 0$ and $b > 1$ be such that $\lambda b < \tilde{\lambda}_\infty < \lambda_\infty$. In view of (A1) and (A2), there exists a constant $D > 0$ such that

$$f(z) \leq kbz^{p-1} + \frac{D}{z^\delta}$$

for all $z > 0$. Hence

$$-\Delta_p u \leq \lambda kb(M\phi_1)^{p-1} + \frac{\lambda_\infty D + L_1}{\phi_1^\gamma} \quad \text{in} \quad \Omega,$$

for $c > 1$, which implies

$$-\Delta_p\left(\frac{u}{M}\right) \leq \lambda kb\phi_1^{p-1} + \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma} \equiv f_M \quad \text{in} \quad \Omega.$$

Let $\bar{u}_M$ be the solution of

$$-\Delta_p(\bar{u}_M) = f_M \text{ in } \Omega, \quad \bar{u}_M = 0 \text{ on } \partial\Omega.$$

Then $u \leq M\bar{u}_M$ in $\Omega$. Since

$$-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} \quad \text{in} \quad \Omega,$$

and

$$\begin{aligned} \lambda_1\phi_1^{p-1} - f_M &= (\lambda_1 - \lambda kb)\phi_1^{p-1} - \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma} \\ &\geq k(\lambda_\infty - \tilde{\lambda}_\infty)\phi_1^{p-1} - \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma}, \end{aligned}$$

it follows from Theorem 1.1 with $u = \phi_1$, $v = \bar{u}_M$, $m = 1$, $h = k(\lambda_\infty - \tilde{\lambda}_\infty)\phi_1^{p-1}$ that $\bar{u}_M \leq \phi_1$ in $\Omega$ for $M \gg 1$. Hence $u \leq M\phi_1$ in $\Omega$ for $M \gg 1$. Thus $T: \mathbf{K} \to \mathbf{K}$ and the result now follows from the Schauder fixed point theorem. $\square$

Proof of Theorem 1.3. Let $z \in C^1(\bar{\Omega})$ be the solution of

$$-\Delta_p z = \frac{cq(x)}{\phi_1^\beta} \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega,$$

where $c \in (0, 1)$. By Lemma 2.3, $z \leq \phi_1$ in $\Omega$ if $c$ is sufficiently small, which we assume. Let $M > 1$ be a large constant to be determined later and define

$$\mathbf{C} = \{v \in C(\bar{\Omega}) \colon v \leq M\phi_1 \text{ in } \Omega\}.$$

Fix $\lambda \in (0, \lambda_\infty)$ and choose $b > 1$ so that $\lambda b < \lambda_\infty$. For each $v \in \mathbf{C}$, the problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{\max^\beta(v, z)} + \lambda f(\max(v, z)) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \equiv Sv \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ such that $|u|_{1,\alpha} < \tilde{M}$, where $\alpha$, $\tilde{M}$ are independent of $v \in \mathbf{C}$. Since $z \geq \varepsilon_1 d$ in $\Omega$ for some $\varepsilon_1 > 0$, it follows as in the proof of Theorem 1.2 that $S \colon \mathbf{C} \to C(\bar{\Omega})$ is a compact operator. We shall show that $S \colon \mathbf{C} \to \mathbf{C}$ if $M$ is large enough. Note that any fixed point of $S$ is positive in $\Omega$, by the strong maximum principle. Let $v \in \mathbf{C}$ and $u = Sv$. Since there exists a constant $D > 0$ such that

$$f(t) \leq kbt^{p-1} + \frac{D}{t^\delta}$$

for $t > 0$, it follows that

$$-\Delta_p u \leq \frac{L_1}{z^{\beta+\eta}} + \lambda\left(kb(M\phi_1)^{p-1} + \frac{D}{z^\delta}\right) \quad \text{in} \quad \Omega,$$

where $L_1$ is defined in the proof of Theorem 1.2. This implies

$$-\Delta_p\left(\frac{u}{M}\right) \leq \lambda kb\phi_1^{p-1} + \left(\frac{L_1 + \lambda_\infty D}{M^{p-1}}\right)\frac{1}{z^\gamma} \equiv g_M,$$

where $\gamma = \max(\beta + \eta, \delta)$. Let $u_M$ be the solution of

$$-\Delta_p(u_M) = g_M \text{ in } \Omega, \quad u_M = 0 \text{ on } \partial\Omega.$$

Then $u \leq Mu_M$ in $\Omega$. Since

$$\lambda_1\phi_1^{p-1} - g_M \geq k(\lambda_\infty - \lambda b)\phi_1^{p-1} - \left(\frac{L_1 + \lambda_\infty D}{M^{p-1}}\right)\frac{1}{z^\gamma},$$

it follows from Theorem 1.1 that $u_M \le \phi_1$ in $\Omega$ for $M \gg 1$, which implies

$$u \le M u_M \le M\phi_1 \quad \text{in} \quad \Omega$$

i.e. $u \in \mathbf{C}$ for $M \gg 1$. By the Schauder fixed point theorem, $S$ has a fixed point $u_\lambda$ in $\mathbf{C}$. We claim that $u_\lambda \ge z$ in $\Omega$. Let $D = \{x \in \Omega\colon u_\lambda(x) < z(x)\}$ and suppose that $D \ne \varnothing$. Then, since $f \ge 0$,

$$-\Delta_p u_\lambda \ge \frac{q(x)}{u_\lambda^\beta} \ge \frac{q(x)}{z^\beta} \ge \frac{q(x)}{\phi_1^\beta} \ge -\Delta_p z \quad \text{in} \quad D.$$

Since $u_\lambda = z$ on $\partial D$, this implies $u_\lambda \ge z$ in $D$, a contradiction. Hence $D = \varnothing$ and therefore $u_\lambda \ge z$ in $\Omega$ as claimed. Thus $u_\lambda$ is a positive solution of (1.2).

Next, suppose $f(u) \ge ku^{p-1}$ for $u > 0$. Let $\lambda \ge \lambda_\infty$ and let $u$ be a positive solution of (1.2). Then $u > 0$ in $\Omega$ and $\partial u/\partial\nu < 0$ on $\partial\Omega$ by the strong maximum principle. Let $c > 0$ be the largest number so that $u \ge c\phi_1$ in $\Omega$. Then

$$-\Delta_p u \ge \frac{q(x)}{\|u\|_\infty^\beta} + \lambda k(c\phi_1)^{p-1} \ge \frac{q(x)}{\|u\|_\infty^\beta} + \lambda_1(c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

and since

$$-\Delta_p(c\phi_1) = \lambda_1(c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

it follows from Theorem 1.1 with $\varepsilon = 0$ that $u > c\phi_1$ in $\Omega$ and

$$\frac{\partial u}{\partial \nu} < \frac{\partial(c\phi_1)}{\partial\nu} < 0 \quad \text{on} \quad \partial\Omega.$$

Hence there exists a constant $\tilde{c} > c$ such that $u \ge \tilde{c}\phi_1$ in $\Omega$, a contradiction. Thus (1.2) has no positive solution for $\lambda \ge \lambda_\infty$. We shall verify next that $\lim_{\lambda\to\lambda_\infty^-} \|u_\lambda\|_\infty = \infty$. Suppose otherwise, then there exist a sequence $(\lambda_n) \subset (0, \lambda_\infty)$ and a constant $C > 0$ such that $\lambda_n \to \lambda_\infty^-$ and $\|u_n\|_\infty < C$ for all $n$, where $u_n \equiv u_{\lambda_n}$. Since

$$-\Delta_p u_n \ge \frac{q(x)}{u_n^\beta} \ge \frac{q(x)}{C^\beta} \quad \text{in} \quad \Omega,$$

it follows that there exists a constant $\tilde{k} > 0$ such that $u_n \ge \tilde{k}\phi_1$ in $\Omega$ for all $n$. Hence there exists a constant $\tilde{C} > 0$ such that

$$\frac{q(x)}{u_n^\beta} + \lambda f(u_n) \le \frac{\tilde{C}}{\phi_1^\gamma} \quad \text{in} \quad \Omega$$

for all $n$. By Lemma A, there exist constants $\alpha \in (0, 1)$ and $\tilde{M} > 0$ such that $u_n \in C^{1,\alpha}(\bar{\Omega})$ and $|u_n|_{1,\alpha} < \tilde{M}$ for all $n$. By going to a subsequence, we assume that there exists $u \in C^1(\bar{\Omega})$ such that $u_n \to u$ in $C^1(\bar{\Omega})$. Let $\psi \in W_0^{1,p}(\Omega)$. Then

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi \, dx = \int_\Omega \left(\frac{q(x)}{u_n^\beta} + \lambda_n f(u_n)\right)\psi \, dx \tag{3.11}$$

for all $n$. Let $n \to \infty$ in (3.11) and using the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx = \int_\Omega \left(\frac{q(x)}{u^\beta} + \lambda_\infty f(u)\right)\psi \, dx$$

i.e. $u$ is a positive solution of

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{u^\beta} + \lambda_\infty f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

a contradiction. This completes the proof of Theorem 1.3. □

ACKNOWLEDGEMENT. The author thanks the referee for pointing out some errors in the original manuscript along with suggestions for improvement.

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