A COMPARISON PRINCIPLE AND APPLICATIONS TO ASYMPTOTICALLY $p$-LINEAR BOUNDARY VALUE PROBLEMS

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Abstract

Consider the problems
\[
\begin{cases}
-\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial \Omega, \\
-\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. $\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$. We prove a strong comparison principle that allows $f - g$ to change sign. An application to singular asymptotically $p$-linear boundary problems is given.

1. Introduction

Consider the problems
\[
\begin{cases}
-\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial \Omega, \\
-\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial \Omega \in C^{2, \alpha}$ for some $\alpha \in (0, 1)$, $\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$, and $f, g : \Omega \to \mathbb{R}$.

In this paper, we shall establish a strong comparison principle
\[
\begin{align*}
u > v \text{ in } \Omega \quad &\text{and} \quad \frac{\partial u}{\partial v} < \frac{\partial v}{\partial v} \text{ on } \partial \Omega, 
\end{align*}
\]
without requiring that $f \geq g$ a.e. in $\Omega$. Here $v$ denotes the outer unit normal vector on $\partial \Omega$. It should be noted that the assumptions $f \geq g$ and $f \neq g$ in $\Omega$ are needed in previous literature (see e.g. [9] and the references therein). We also provide an application to the existence of positive solutions for a class of singular $p$-Laplacian boundary value problems with asymptotically $p$-linear nonlinearity.

Let $d(x) = d(x, \partial \Omega)$ be the distance from $x$ to $\partial \Omega$, we prove the following result:
Theorem 1.1. Let \( f, g, g_0 \in L^1(\Omega) \) with \( g \geq g_0 \geq 0 \), and \( g_0 \neq 0 \). Suppose there exist constants \( C > 0 \) and \( \gamma \in (0, 1) \) such that

\[
|f(x)|, g(x) \leq \frac{C}{d^\gamma(x)}
\]

for a.e. \( x \in \Omega \), and there exist a function \( h \in C(\Omega) \), \( h > 0 \), and constants \( \varepsilon \geq 0 \), \( m, M > 0 \) with \( m \leq M \) such that

\[
f - g \geq m \left( h - \frac{\varepsilon}{d^\gamma} \right) \text{ in } \Omega.
\]

Let \( u, v \in W_0^{1,p}(\Omega) \) be solutions of (1.1). Then there exists a positive constant \( \varepsilon_0 \) depending on \( n, \Omega, p, \gamma, C, M, h, g_0 \) (but not on \( m \)), such that

\[
u > v \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} \leq \frac{\partial v}{\partial \nu} \text{ on } \partial \Omega
\]

for \( \varepsilon < \varepsilon_0 \). If \( \varepsilon = 0 \), the result holds under the weaker condition that \( h \) is a nonnegative nontrivial measurable function in \( \Omega \).

Remark 1.1. When \( g \equiv 0 \), the conclusion of Theorem 1.1 holds under the weaker assumption that \( h \) is a nonnegative nontrivial measurable function in \( \Omega \). In this case, \( \varepsilon_0 \) is independent of \( M \). Indeed, let \( \tilde{u}, \tilde{v} \) be the solutions of

\[
-\Delta_p \tilde{u} = \tilde{h} - \frac{\varepsilon}{d^\gamma} \text{ in } \Omega, \quad \tilde{u} = 0 \text{ on } \partial \Omega,
\]

\[
-\Delta_p \tilde{v} = \tilde{h} \text{ in } \Omega, \quad \tilde{v} = 0 \text{ on } \partial \Omega,
\]

respectively, where \( \tilde{h} = \min(h, 1/d^\gamma) \). By the strong maximum principle [12, 14], there exists a constant \( \delta > 0 \) such that \( \tilde{v} \geq \delta d \) in \( \Omega \). Using Lemma 2.3 in Section 2, we deduce that

\[
\tilde{u} \geq \tilde{v} - \frac{\delta}{2} \geq \frac{\delta}{2} d
\]

if \( \varepsilon \) is sufficiently small. This implies

\[
u \geq m^{1/(p-1)} \tilde{u} > m^{1/(p-1)} \frac{\delta}{2} d > 0 \text{ in } \Omega
\]

and \( \partial u/\partial \nu < 0 \) on \( \partial \Omega \).

As an application of Theorem 1.1, consider the boundary value problem

\[
(1.2)
\begin{align*}
-\Delta_p u &= \frac{q(x)}{u^\beta} + \lambda f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where $\beta \in (0, 1)$, $q$, $f$ satisfy the following assumptions:

(A1) $f : (0, \infty) \to \mathbb{R}$ is continuous and there exists a constant $k > 0$ such that

$$\lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = k.$$ 

(A2) There exists a constant $\delta \in (0, 1)$ such that

$$\limsup_{u \to 0^+} u^\delta |f(u)| < \infty.$$ 

(A3) There exist constants $A, \varepsilon_0 > 0$ such that

$$f(u) \geq ku^{p-1} + \varepsilon_0 \quad \text{for} \quad u > A.$$ 

(A4) $q : \Omega \to \mathbb{R}$ is measurable and there exist constants $\eta, L > 0$ with $\beta + \eta < 1$, such that

$$|q(x)| \leq \frac{L}{d^n(x)}$$

for a.e. $x \in \Omega$.

Let $\lambda_1$ be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition, and let $\phi_1$ be the corresponding positive eigenfunction with $\|\phi_1\|_\infty = 1$. Note that, since $\partial \phi_1/\partial \nu < 0$ on $\partial \Omega$, Theorem 1.1 holds if $d$ is replaced by $\phi_1$. Let $\lambda_\infty = \lambda_1/k$. Then we have

**Theorem 1.2.** Let (A1)-(A4) hold. Then there exists a constant $\bar{\varepsilon} > 0$ such that for $\lambda_\infty - \bar{\varepsilon} < \lambda < \lambda_\infty$, problem (1.2) has a positive solution $u_\lambda \in C^{1,\kappa}(\bar{\Omega})$ for some $\kappa \in (0, 1)$ with

$$u_\lambda \geq \left(\frac{\lambda_\infty \varepsilon_0}{4k(\lambda_\infty - \lambda)}\right)^{1/(p-1)} \phi_1 \quad \text{in} \quad \Omega.$$ 

**Theorem 1.3.** Let $q \geq 0$, $q \neq 0$. Suppose $f \geq 0$, (A2), (A4) hold, and

$$\limsup_{u \to \infty} \frac{f(u)}{u^{p-1}} = k$$

for some $k \in (0, \infty)$. Then problem (1.2) has a positive solution $u_\lambda$ for $\lambda < \lambda_\infty$. If, in addition,

$$f(u) \geq ku^{p-1} \quad \text{for all} \quad u > 0,$$

then (1.2) has no positive solution for $\lambda \geq \lambda_\infty$, and

$$\|u_\lambda\|_\infty \to \infty \quad \text{as} \quad \lambda \to \lambda_\infty^-.$$
EXAMPLE 1.1. (i) Let \( f(u) = -1/u^\delta + u^{p-1} + u^q \), where \( \delta \in (0, 1) \) and \( 0 \leq q < p - 1 \). Then \( f \) satisfies (A1)–(A3) with \( k = 1 \), and so (1.2) has a positive solution when \( \lambda \) is sufficiently close to \( \lambda_1 \) and \( \lambda < \lambda_1 \), by Theorem 1.2.

(ii) Let \( f(u) = 1/u^\delta + u^{p-1}(m|\sin u| + e^{1/(1+u)}) \), where \( \delta \in (0, 1) \), \( m \geq 0 \). Then it follows from Theorem 1.3 that, if \( m > 0 \), (1.2) has a positive solution for \( \lambda < \lambda_1/(m + 1) \), and, if \( m = 0 \), (1.2) has a positive solution if and only if \( \lambda < \lambda_1 \).

REMARK 1.2. It should be noted that Theorem 1.2 may not be true when \( \varepsilon_0 = 0 \). Indeed, by multiplying the equation in (1.2) by \( u \) and integrating, we see that (1.2) has no positive solution for \( q \leq 0 \) and \( f(u) = ku^{p-1} \).

REMARK 1.3. In [15], assuming that \( f \) is continuous and nonnegative on \([0, \infty)\), \( \lim_{u \to \infty} f(u)/u = k \in (0, \infty) \), and \( f \) satisfies some additional conditions at 0, Zhang showed via variational method that (1.2) with \( p = 2 \) has a positive solution for \( \lambda \in (0, \lambda_1/k) \), provided that \( q \geq 0 \), \( q \neq 0 \), \( q\phi_1^\beta \in L'(\Omega) \), where \( n/2 < r \). The result in [15] was improved by Hai in [4], using sub- and super solutions approach. The proof in [4] depends on the linearity of the Laplacian and can not be applied to the general case where \( p > 1 \), except for radial solutions in a ball [6]. Related results on the case where \( f \) is nonsingular can be found in Ambrosetti, Arcaya, and Buffoni [1], Ambrosetti and Hess [2], and Ambrosetti, Garcia Azorero, and Peral [3]. The approach in [1, 2, 3] was via bifurcation theory for \( p = 2 \) in [1, 2] and \( p > 1 \) in [3]. Thus, Theorems 1.2 and 1.3 provide extensions of corresponding results in [1, 2, 3, 4, 6, 15] to the singular \( p \)-Laplacian case. Note that the precise lower bound estimate (*) has not been obtained in previous literature.

2. Preliminary results

Let \( D \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial D \).

We shall denote the norm in \( C^{k,\alpha}(\bar{D}) \) and \( L^k(D) \) by \( |\cdot|_{k,\alpha} \) and \( \|\cdot\|_k \) respectively. The distance from \( x \) to \( \partial D \) is denoted by \( d(x, \partial D) \).

We first recall the following regularity result in [5, Lemma 3.1], which plays an important role in the proofs of our main results.

**Lemma A.** Let \( h \in L^\infty_{\text{loc}}(\Omega) \) and suppose there exist numbers \( \gamma \in (0, 1) \) and \( C > 0 \) such that

\[
|h(x)| \leq \frac{C}{d^\gamma(x)}
\]

for a.e. \( x \in \Omega \). Let \( u \in W^{1,p}_0(\Omega) \) be the solution of

\[
\begin{aligned}
-\Delta_p u &= h & &\text{in} & &\Omega, \\
u &= 0 & &\text{on} & &\partial\Omega.
\end{aligned}
\]
Then there exist constants $\alpha \in (0, 1)$ and $\tilde{M} > 0$ depending only on $C, \gamma, \Omega$ such that $u \in C^{1,\alpha}(\tilde{\Omega})$ and $|u|_{1,\alpha} < \tilde{M}$.

Let

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right),$$

where $a_{ij} \in C^{0,\alpha}(\tilde{D}), 1 \leq i, j \leq n$, for some $\alpha \in (0, 1)$, and suppose there exist constants $m_0, m_1 > 0$ such that

$$|a_{ij}|_{0,\alpha} \leq m_1$$

for $1 \leq i, j \leq n$, and

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq m_0 |\xi|^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

**Lemma 2.1.** Let $h \in L^1(D)$ and suppose there exist constants $C > 0$ and $\gamma \in (0, 1)$ such that

$$|h(x)| \leq \frac{C}{d^\gamma(x, \partial D)}$$

for a.e. $x \in D$. Let $w \in H_0^1(D)$ be the solution of

$$\begin{cases}
Lw = h & \text{in } D, \\
w = 0 & \text{on } \partial D.
\end{cases}$$

Then there exist constants $\beta \in (0, 1)$ and $\tilde{M} > 0$ depending only on $m_0, m_1, C, \gamma, D, n$, such that $w \in C^{1,\beta}(\tilde{D})$ and $|w|_{1,\beta} \leq \tilde{M}$.

**Proof.** Let $\phi \in C^1(\tilde{D})$ be the solution of

$$L\phi = 1 \text{ in } D, \quad \phi = 0 \text{ on } \partial D.$$ 

Then there exists a constant $C_0 > 0$ independent of $a_{ij}$ such that $\phi(x) \leq C_0 d(x, \partial D)$ for all $x \in D$. Let $a = 2^{1/(1-\gamma)} \|\phi\|_\infty$ and $h_0: [0, a] \to \mathbb{R}$ satisfy

$$\begin{cases}
-h_0'' = \frac{1}{t^\gamma}, & 0 < t < a, \\
h_0(0) = 0, & h_0'(a) = 0.
\end{cases}$$
Note that \( h_0(t) = (t/(1 - \gamma))(a^{1-\gamma} - t^{1-\gamma}/(2 - \gamma)) \). A calculation shows that

\[
L(h_0(\phi)) = -h_0^{\gamma}(\phi) \sum_{i,j=1}^{n} a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + h_0(\phi)
\geq \frac{m_0}{\phi^\gamma} |\nabla \phi|^2 + \frac{a^{1-\gamma} - \phi^{1-\gamma}}{1 - \gamma} \geq \frac{m_0}{\phi^\gamma} |\nabla \phi|^2 + \frac{a^{1-\gamma}}{2(1 - \gamma)} \geq \frac{m_2}{d^\gamma(x, \partial D)},
\]

where \( m_2 \) is independent of \( a_{ij} \). By the weak comparison principle ([11, Lemma A.2], [13, Lemma 3.1]),

\[
|w| \leq \frac{C}{m_2} h_0(\phi) \quad \text{in} \quad D,
\]
i.e. \( w \) is bounded in \( D \). By Lemma A, the problem

\[
\begin{aligned}
-\Delta z &= h \quad \text{in} \quad D, \\
    z &= 0 \quad \text{on} \quad \partial D,
\end{aligned}
\]

has a solution \( z \in C^{1,\alpha}(\bar{D}) \) for some \( \alpha \in (0, 1) \). Since \( w \) satisfies

\[
-\text{div}(A(x, \nabla w) - \nabla z) = 0 \quad \text{in} \quad D,
\]

where \( A = (A_1, \ldots, A_n) \), \( A_i(x, \eta) = \sum_{j=1}^{n} a_{ij}(x) \eta_j \), \( \eta = (\eta_1, \ldots, \eta_n) \), the result now follows from Lieberman [8, Theorem 1].

**Lemma 2.2.** Let \( h \) satisfy (2.3), \( h \geq 0 \), \( h \not\equiv 0 \), and let \( w \in H_0^1(D) \) be the solution of (2.4). Then there exists a constant \( k_0 > 0 \) depending only on \( h \), \( m_0 \), \( m_1 \), \( C \), \( \gamma \), \( D \), \( n \) such that

\[
w(x) \geq k_0 d(x, \partial D)
\]

for all \( x \in D \).

Proof. Let \( \Lambda \) be the set of all solutions \( w \) of (2.4) among the coefficients \( a_{ij} \) that satisfy (2.1) and (2.2). By the strong maximum principle, \( w > 0 \) in \( \Omega \) and \( \partial w/\partial n < 0 \) on \( \partial D \). By Lemma 2.1, \( w \in C^{1,\beta}(\bar{D}) \) and there exists a constant \( \bar{M} > 0 \) such that \( |w|_{1,\beta} \leq \bar{M} \) for all \( w \in \Lambda \). Since \( \Lambda \) is closed in \( C^1(\bar{D}) \), \( \Lambda \) is compact in \( C^1(\bar{D}) \). Define \( G: \Lambda \to \mathbb{R} \) by

\[
G_w = \inf_{x \in \partial D} \frac{w(x)}{d(x, \partial D)}.
\]

Then \( G \) is continuous and positive on \( \Lambda \), and therefore has a positive minimum, which completes the proof.
Lemma 2.3. Let \( f, g \in L^1(D) \) satisfy
\[
|f(x)|, |g(x)| \leq \frac{C}{d^\nu(x, \partial D)}
\]
for a.e. \( x \in \Omega \) for some constant \( C > 0 \). Let \( u, v \) be the solutions of (1.1). Then \( |u - v|_{0,1} \to 0 \) as \( \|f - g\|_1 \to 0 \).

Proof. Note that \( f, g \in L^1(D) \) (see [7, p. 6]). By Lemma A, \( u, v \in C^{1,\alpha}(\tilde{D}) \) for some \( \alpha \in (0, 1) \), and there exists a constant \( \tilde{M} > 0 \) independent of \( u, v \), such that \( |u|_{1,\alpha}, |v|_{1,\alpha} \leq \tilde{M} \). Multiplying the equation
\[
-(\Delta u - \Delta v) = f - g  \quad \text{in} \quad \Omega
\]
by \( u - v \) and integrating, we obtain
\[
\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - v) \, dx = \int_{\Omega} (f - g)(u - v) \, dx.
\]
Using the inequality [10, Lemma 30.1],
\[
(|x| + |y|)^{2-\min(p,2)}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq c |x - y|^{\max(p,2)}
\]
for \( x, y \in \mathbb{R}^n \), where \( c \) is a positive constant depending only on \( p \), we obtain
\[
\int_{\Omega} |\nabla(u - v)|^r \, dx \leq c_1 \|f - g\|_{L^1} \|u - v\|_{\infty} \leq c_2 \|f - g\|_{L^1},
\]
where \( r = \max(p, 2) \) and \( c_1, c_2 \) are constants depending only on \( p, \tilde{M} \).

Hence
\[
\|u - v\|_2 \to 0
\]
as \( \|f - g\|_1 \to 0 \), and since \( C^{1,\alpha}(\tilde{D}) \) is compactly imbedded in \( C^1(\tilde{D}) \), Lemma 2.3 follows.

3. Proofs of the main results

Proof of Theorem 1.1. By the strong maximum principle, there exists a constant \( \delta > 0 \) such that \( v \geq \delta d \) in \( \Omega \). Let \( \varepsilon \in (0, 1) \), \( m_\varepsilon = \min(m, \varepsilon) \) and \( \tilde{h} = \min(h, 1/d^\nu) \). Then
\[
f \geq g + m_\varepsilon \tilde{h} - \frac{M\varepsilon}{d^\nu} \equiv \tilde{f} \quad \text{in} \quad \Omega.
\]
Let \( \tilde{u} \) satisfy
\[
-(\Delta \tilde{u}) = \tilde{f} \quad \text{in} \quad \Omega, \quad \tilde{u} = 0 \quad \text{on} \quad \partial \Omega.
\]
Then \( u \geq \tilde{u} \) in \( \Omega \), by the weak comparison principle. Since

\[
|\tilde{f}| \leq \frac{C + 1 + M}{d^\gamma} = \frac{\tilde{C}}{d^\gamma}
\]

and

\[
\int_{\Omega} |\tilde{f} - g| \, dx \leq \varepsilon(1 + M) \int_{\Omega} \frac{1}{d^\gamma} \, dx,
\]

it follows that \( \tilde{f} \rightarrow g \) in \( L^1(\Omega) \) as \( \varepsilon \rightarrow 0 \). Here we have used the fact that \( 1/d^\gamma \in L^1(\Omega) \) (see e.g. [7, p. 6]). Using Lemma 2.3, we see that \( \tilde{u} \rightarrow v \) in \( C^1(\tilde{\Omega}) \) as \( \varepsilon \rightarrow 0 \).

Hence

\[
u \geq \tilde{u} \geq \frac{\delta d}{2} \quad \text{in} \quad \Omega
\]

if \( \varepsilon \) is sufficiently small, which we shall assume.

By Lemma A, there exist constants \( \tilde{M} > 0 \) and \( \alpha \in (0, 1) \) independent of \( u, v \) such that \( |u|_{1,\alpha}, |v|_{1,\alpha} \leq \tilde{M} \). Thus

\[
(3.1) \quad \frac{\delta d}{2} \leq u, \quad v \leq \tilde{M}d \quad \text{in} \quad \Omega,
\]

and therefore

\[
u \geq c_0 v \quad \text{in} \quad \Omega,
\]

where \( c_0 = \delta/(2\tilde{M}) \).

Let \( c \) be the largest number such that \( u \geq cv \) in \( \Omega \), and suppose that \( c \leq 1 \). From (3.1), it follows that

\[
\frac{\partial u}{\partial v}, \frac{\partial v}{\partial v} \leq -\frac{\delta}{2} \quad \text{on} \quad \partial \Omega.
\]

For \( t \in [0, 1] \), let \( w_t = t\nabla u + (1 - t)c\nabla v \). Then

\[
w_t \cdot v = t \frac{\partial u}{\partial v} + (1 - t)c \frac{\partial v}{\partial v} \leq \frac{tc\delta}{2} - \frac{(1 - t)c\delta}{2} = -\frac{c\delta}{2} \leq -\frac{c_0\delta}{2} \quad \text{on} \quad \partial \Omega,
\]

which implies

\[
(3.2) \quad |w_t| \geq \frac{c_0\delta}{2} = c_1 \quad \text{on} \quad \partial \Omega
\]

for all \( t \in [0, 1] \).

Let \( x \in \Omega \) and \( x_0 \in \partial \Omega \) be such that \( d(x) = |x - x_0| \). Since \( |w_t|_{0,\alpha} \leq \tilde{M} \), it follows that

\[
|w_t(x) - w_t(x_0)| \leq \tilde{M}d^{\alpha}(x),
\]
which, together with (3.2), implies

\[(3.3) \quad |w_t(x)| \geq c_1 - \tilde{M}d^\alpha(x) \geq \frac{c_1}{2} = c_2\]

for \(x \in \Omega_\eta \equiv \{ x \in \Omega : d(x) < \eta \} \), where \(\eta = (c_1/2\tilde{M})^{1/\alpha}\).

Next, we have

\[(3.4) \quad -(\Delta_p u - \Delta_p (c v)) = f - c^{p-1}g \quad \text{in} \quad \Omega,\]

and the left hand side of (3.4) can be linearized as \(L(u - c v)\), where

\[Lw = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial w}{\partial x_j} \right)\]

and \(a_{ij}(x) = \int_0^1 (\partial a'/\partial z_j)(t \nabla u + (1-t)c \nabla v) \, dt, \ a'(z) = |z|^{p-2}z.\)

Note that, in view of (3.3) and the fact that \(\frac{\partial w_t}{\partial \nu} \leq \tilde{M}\) for \(t \in [0, 1]\), the coefficients \(a_{ij}\) satisfy (2.1) and (2.2) in \(\Omega_\eta\) with \(m_0 = (p - 1) \min(\tilde{M}^{p-2}, c^{p-2})\).

Let \(w_0, w_1\) be the solutions of

\[Lw_0 = \tilde{h} \quad \text{in} \quad \Omega_\eta, \quad w_0 = 0 \quad \text{on} \quad \partial \Omega_\eta,\]

and

\[Lw_1 = \frac{1}{d^\gamma(x, \partial \Omega_\eta)} \quad \text{in} \quad \Omega_\eta, \quad w_1 = 0 \quad \text{on} \quad \partial \Omega_\eta\]

respectively. By Lemmas 2.1 and 2.2, there exist positive constants \(M_0\) and \(k_0\) such that

\[(3.5) \quad w_0 \geq k_0 d(x, \partial \Omega_\eta), \quad w_1 \leq M_0 d(x, \partial \Omega_\eta) \quad \text{in} \quad \Omega_\eta.\]

Since \(c \leq 1\) and \(d(x) \geq d(x, \partial \Omega_\eta)\) for \(x \in \Omega_\eta\),

\[L(u - c v) \geq f - g \geq m \left( \tilde{h} - \frac{\epsilon}{d^\gamma(x, \partial \Omega_\eta)} \right) \quad \text{in} \quad \Omega_\eta,\]

and since \(u \geq c v\) on \(\partial \Omega_\eta\), it follows from the weak comparison principle and (3.5) that, for \(x \in \Omega_{\eta/2}\),

\[u - c v \geq m(w_0 - \epsilon w_1) \geq m(k_0 - \epsilon M_0) d(x, \partial \Omega_\eta) = m(k_0 - \epsilon M_0) d(x)\]

\[(3.6) \quad \geq m \left( \frac{k_0}{2} \right) d(x)\]

if \(\epsilon < k_0/2M_0\). In particular,

\[u - c v \geq m \left( \frac{k_0}{2} \right)^2 \leq mk_1 \quad \text{when} \quad d(x) = \frac{\eta}{2}.\]
If \( \varepsilon = 0 \) then it follows from
\[
-\Delta_p u = f \geq c^{p-1} g = -\Delta_p (c v + m k_1) \quad \text{in } \Omega
\]
and \( u \geq c v + m k_1 \) on \( \partial(\Omega \setminus \Omega_{\eta/2}) \) that
\[
(3.7) \quad u \geq c v + m k_1 \quad \text{in } \Omega \setminus \Omega_{\eta/2}.
\]
Suppose \( \varepsilon > 0 \) and \( h \geq a > 0 \) in \( \Omega \setminus \Omega_{\eta/2} \). Then we have
\[
-\Delta_p u = f \geq g + m \left( a - \frac{\varepsilon}{d^\gamma(x)} \right) \geq g + m \left( a - \frac{\varepsilon}{(\eta/2)\gamma} \right) \geq g \quad \text{in } \Omega \setminus \Omega_{\eta/2},
\]
if \( \varepsilon \) is sufficiently small. Hence (3.7) holds by the weak comparison principle. This, together with (3.6), gives the existence of a constant \( \tilde{c} > c \) such that \( u \geq \tilde{c} v \) in \( \Omega \), a contradiction. Hence \( c > 1 \) and therefore \( u > v \) in \( \Omega \) and
\[
\frac{\partial(u - v)}{\partial v} \leq (c - 1)\frac{\partial v}{\partial v} < 0 \quad \text{on } \partial \Omega,
\]
which completes the proof. \( \square \)

Proof of Theorem 1.2. Let \( \lambda > 0 \) be such that \( \lambda_{\infty}/2 < \lambda < \lambda_{\infty} \). Let \( c = (\lambda_{\infty}\varepsilon_0/(4k(\lambda_{\infty} - \lambda)))^{1/(p-1)} \) and \( M \) be a constant such that \( M > c \). Define
\[
K = \{ v \in C(\tilde{\Omega}) : c \phi_1 \leq v \leq M \phi_1 \text{ in } \Omega \}.
\]
For each \( v \in K \), it follows from Lemma A that the problem
\[
\begin{cases}
-\Delta_p u = \frac{q(x)}{w^\beta} + \lambda f(v) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a unique solution \( u \equiv T v \in C^{1,\alpha}(\tilde{\Omega}) \) for some \( \alpha \in (0, 1) \) such that \( |u|_{1,\alpha} < \tilde{M} \), where \( \alpha, \tilde{M} \) are independent of \( v \in K \). We shall show that \( T : K \to C(\tilde{\Omega}) \) is a compact operator. In view of the compact embedding of \( C^{1,\alpha}(\tilde{\Omega}) \) into \( C^1(\tilde{\Omega}) \), we need only to show that \( T \) is continuous. Let \( (v_n) \) be a sequence in \( K \) such that \( v_n \to v \) in \( C(\tilde{\Omega}) \), and let \( u_n = T v_n, u = T v \). Let \( G(w) = q(x)/w^\beta + \lambda f(w) \) for \( w \in K \). Then
\[
G(v_n) \to G(v) \quad \text{pointwise in } \Omega,
\]
and it follows from (A2) and (A4) that there exist constants \( K, C_0 > 0 \) such that
\[
|G(v_n)| \leq \frac{q(x)}{(c \phi_1)^\beta} + \frac{\lambda_{\infty}K}{(c \phi_1)^{\delta}} \leq C_0 d^\gamma,
\]
for all \( n \), where \( \gamma = \max(\beta + \eta, \delta) \). Hence \( G(v_n) \to G(v) \) in \( L^1(\Omega) \), and Lemma 2.3 implies \( u_n \to u \) in \( C^1(\overline{\Omega}) \), i.e., \( T \) is continuous on \( K \).

Next, we shall show that if \( \lambda \) is sufficiently close to \( \lambda_\infty \) and \( M \) is large enough then \( T \) maps \( K \) into \( K \). By (A2) and (A3), there exists a constant \( M_0 > 0 \) such that

\[
(3.8) \quad f(z) \geq k z^{p-1} + \varepsilon_0 - \frac{M_0}{z^{\gamma}}
\]

for all \( z > 0 \). Let \( v \in K \) and \( u = T v \). Then (3.8) and (A4) imply

\[
-\Delta_p u \geq -\frac{L_1}{(c\phi_1)^p d_\eta} + \lambda \left( k(c\phi_1)^{p-1} + \varepsilon_0 - \frac{M_0}{(c\phi_1)^{\gamma}} \right) \quad \text{in} \quad \Omega.
\]

Consequently,

\[
-\Delta_p \left( \frac{u}{c} \right) \geq -\frac{L_1}{c^{p-1} \phi_1^p} - \frac{\lambda_\infty M_0}{c^{p-1+\delta} \phi_1^\gamma} + \lambda \left( k\phi_1^{p-1} + \frac{\varepsilon_0}{2c^{p-1}} \right) = f_{c,\lambda} \quad \text{in} \quad \Omega,
\]

where \( L_1 \) is a positive constant such that \( d/\phi_1 \geq (L/L_1)^{1/\eta} \).

Let \( \tilde{u}_c, \tilde{z}_c \) be the solutions of

\[
-\Delta_p \tilde{u}_c = f_{c,\lambda} \quad \text{in} \quad \Omega, \quad \tilde{u}_c = 0 \quad \text{on} \quad \partial \Omega,
\]

and

\[
-\Delta_p \tilde{z}_c = \lambda \left( k\phi_1^{p-1} + \frac{\varepsilon_0}{2c^{p-1}} \right) = g_{c,\lambda} \quad \text{in} \quad \Omega, \quad \tilde{z}_c = 0 \quad \text{on} \quad \partial \Omega,
\]

respectively. Then \( u \geq c\tilde{u}_c \) in \( \Omega \). Note that

\[
|f_{c,\lambda}|, |g_{c,\lambda}| \leq \frac{\tilde{C}}{\phi_1^\gamma},
\]

where \( \tilde{C} > 0 \) depends only on \( \varepsilon_0, k, p, L_1, \lambda_\infty, M_0 \). Since

\[
f_{c,\lambda} - g_{c,\lambda} \geq \frac{1}{c^{p-1}} \left( \frac{\lambda_\infty \varepsilon_0}{4} - \left( \frac{L_1}{c^p} + \frac{\lambda_\infty M_0}{c^\delta} \right) \frac{1}{\phi_1^\gamma} \right) \quad \text{in} \quad \Omega,
\]

and

\[
c^{1-p} \leq \frac{2k}{\varepsilon_0},
\]

it follows from Theorem 1.1 with \( m = c^{1-p} \), \( M = 2k/\varepsilon_0 \), \( h = \lambda_\infty \varepsilon_0/4 \), \( g_0 = (\lambda_\infty/2)k\phi_1^{p-1} \), that \( \tilde{u}_c > \tilde{z}_c \) in \( \Omega \) for \( c \gg 1 \), which implies

\[
(3.9) \quad u \geq c\tilde{z}_c \equiv \tilde{z}_c \quad \text{in} \quad \Omega.
\]
By the choice of \( c \),
\[
(\lambda_1 - \lambda k) c^{p-1} = \frac{\lambda_{\infty} \varepsilon_0}{4} \leq \frac{\lambda \varepsilon_0}{2}.
\]
Hence
\[
-\Delta_p \tilde{z}_c = \lambda k \left( (c \phi_1)^{p-1} + \frac{\varepsilon_0}{2k} \right) \geq \lambda_1 (c \phi_1)^{p-1} \quad \text{in} \quad \Omega,
\]
and since
\[
-\Delta_p (c \phi_1) = \lambda_1 (c \phi_1)^{p-1} \quad \text{in} \quad \Omega,
\]
it follows that
\[
(3.10) \quad \tilde{z}_c \geq c \phi_1 \quad \text{in} \quad \Omega.
\]
Hence, if \( \lambda \) is sufficiently close to \( \lambda_{\infty} \), it follows from (3.9) and (3.10) that \( u \geq c \phi_1 \) in \( \Omega \).

Next, let \( \tilde{\lambda}_{\infty} > 0 \) and \( b > 1 \) be such that \( \lambda b < \tilde{\lambda}_{\infty} < \lambda_{\infty} \). In view of (A1) and (A2), there exists a constant \( D > 0 \) such that
\[
f(z) \leq k b z^{p-1} + \frac{D}{z^\delta}
\]
for all \( z > 0 \). Hence
\[
-\Delta_p u \leq \lambda b (M \phi_1)^{p-1} + \frac{\lambda_{\infty} D + L_1}{\phi_1^\gamma} \quad \text{in} \quad \Omega,
\]
for \( c > 1 \), which implies
\[
-\Delta_p \left( \frac{u}{M} \right) \leq \lambda b \phi_1^{p-1} + \frac{\lambda_{\infty} D + L_1}{M^{p-1} \phi_1^\gamma} \equiv f_M \quad \text{in} \quad \Omega.
\]
Let \( \tilde{u}_M \) be the solution of
\[
-\Delta_p (\tilde{u}_M) = f_M \quad \text{in} \quad \Omega, \quad \tilde{u}_M = 0 \quad \text{on} \quad \partial \Omega.
\]
Then \( u \leq M \tilde{u}_M \) in \( \Omega \). Since
\[
-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} \quad \text{in} \quad \Omega,
\]
and
\[
\lambda_1 \phi_1^{p-1} - f_M = (\lambda_1 - \lambda k b) \phi_1^{p-1} - \frac{\lambda_{\infty} D + L_1}{M^{p-1} \phi_1^\gamma} \geq k (\lambda_{\infty} - \tilde{\lambda}_{\infty}) \phi_1^{p-1} - \frac{\lambda_{\infty} D + L_1}{M^{p-1} \phi_1^\gamma},
\]
it follows from Theorem 1.1 with \( u = \phi_1, \, v = \tilde{u}_M, \, m = 1, \, h = k(\lambda_\infty - \tilde{\lambda}_\infty) \phi_1^{p-1} \) that \( \tilde{u}_M \leq \phi_1 \) in \( \Omega \) for \( M \gg 1 \). Hence \( u \leq M \phi_1 \) in \( \Omega \) for \( M \gg 1 \). Thus \( T: K \to K \) and the result now follows from the Schauder fixed point theorem.

Proof of Theorem 1.3. Let \( z \in C^1(\tilde{\Omega}) \) be the solution of

\[-\Delta_p z = \frac{cq(x)}{\phi_1^p} \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( c \in (0, 1) \). By Lemma 2.3, \( z \leq \phi_1 \) in \( \Omega \) if \( c \) is sufficiently small, which we assume. Let \( M > 1 \) be a large constant to be determined later and define

\[ C = \{ v \in C(\tilde{\Omega}) : v \leq M \phi_1 \ \text{in} \ \Omega \}.
\]

Fix \( \lambda \in (0, \lambda_\infty) \) and choose \( b > 1 \) so that \( \lambda b < \lambda_\infty \). For each \( v \in C \), the problem

\[
\begin{cases}
-\Delta_p u = \frac{q(x)}{\phi_1^{p-1}} + \lambda f(\max(v, z)) \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

has a unique solution \( u = Sv \in C^{1,\alpha}(\tilde{\Omega}) \) for some \( \alpha \in (0, 1) \) such that \( |u|_{1,\alpha} < \tilde{M} \), where \( \alpha, \tilde{M} \) are independent of \( v \in C \). Since \( z \geq \epsilon_1 d \) in \( \Omega \) for some \( \epsilon_1 > 0 \), it follows as in the proof of Theorem 1.2 that \( S: C \to C(\tilde{\Omega}) \) is a compact operator. We shall show that \( S: C \to C \) if \( M \) is large enough. Note that any fixed point of \( S \) is positive in \( \Omega \), by the strong maximum principle. Let \( v \in C \) and \( u = Sv \). Since there exists a constant \( D > 0 \) such that

\[ f(t) \leq kbt^{p-1} + \frac{D}{t^\delta}
\]

for \( t > 0 \), it follows that

\[-\Delta_p u \leq \frac{L_1}{z^{\beta+\eta}} + \lambda \left( kb(M \phi_1)^{p-1} + \frac{D}{z^\delta} \right) \quad \text{in} \quad \Omega,
\]

where \( L_1 \) is defined in the proof of Theorem 1.2. This implies

\[-\Delta_p \left( \frac{u}{M} \right) \leq \lambda kb \phi_1^{p-1} + \left( \frac{L_1 + \lambda_\infty D}{M^{p-1}} \right) \frac{1}{z^\gamma} = g_M,
\]

where \( \gamma = \max(\beta + \eta, \delta) \). Let \( u_M \) be the solution of

\[-\Delta_p (u_M) = g_M \quad \text{in} \quad \Omega, \quad u_M = 0 \quad \text{on} \quad \partial \Omega.
\]

Then \( u \leq Mu_M \) in \( \Omega \). Since

\[ \lambda_1 \phi_1^{p-1} - g_M \geq k(\lambda_\infty - \lambda b) \phi_1^{p-1} - \left( \frac{L_1 + \lambda_\infty D}{M^{p-1}} \right) \frac{1}{z^\gamma}, \]
it follows from Theorem 1.1 that \( u_M \leq \phi_1 \) in \( \Omega \) for \( M \gg 1 \), which implies

\[
 u \leq Mu_M \leq M\phi_1 \quad \text{in} \quad \Omega
\]

i.e. \( u \in C \) for \( M \gg 1 \). By the Schauder fixed point theorem, \( S \) has a fixed point \( u_\lambda \) in \( C \). We claim that \( u_\lambda \geq z \) in \( \Omega \). Let \( D = \{ x \in \Omega : u_\lambda(x) < z(x) \} \) and suppose that \( D \neq \emptyset \). Then, since \( f \geq 0 \),

\[
 -\Delta_p u_\lambda \leq \frac{q(x)}{|u_\lambda|^{\beta}} \geq \frac{q(x)}{z^{\beta}} \geq \frac{q(x)}{\phi_1^{\beta}} \geq -\Delta_p z \quad \text{in} \quad D.
\]

Since \( u_\lambda = z \) on \( \partial D \), this implies \( u_\lambda \geq z \) in \( D \), a contradiction. Hence \( D = \emptyset \) and therefore \( u_\lambda \geq z \) in \( \Omega \) as claimed. Thus \( u_\lambda \) is a positive solution of (1.2).

Next, suppose \( f(u) \geq ku^{p-1} \) for \( u > 0 \). Let \( \lambda \geq \lambda_\infty \) and let \( u \) be a positive solution of (1.2). Then \( u > 0 \) in \( \Omega \) and \( \partial u/\partial v < 0 \) on \( \partial \Omega \) by the strong maximum principle. Let \( c > 0 \) be the largest number so that \( u \geq c\phi_1 \) in \( \Omega \). Then

\[
 -\Delta_p u \geq \frac{q(x)}{|u|^{\beta}} + \lambda k(c\phi_1)^{p-1} \geq \frac{q(x)}{|u|^{\beta}} + \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,
\]

and since

\[
 -\Delta_p (c\phi_1) = \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,
\]

it follows from Theorem 1.1 with \( \varepsilon = 0 \) that \( u > c\phi_1 \) in \( \Omega \) and

\[
 \frac{\partial u}{\partial v} < \frac{\partial (c\phi_1)}{\partial v} < 0 \quad \text{on} \quad \partial \Omega.
\]

Hence there exists a constant \( \tilde{c} > c \) such that \( u \geq \tilde{c}\phi_1 \) in \( \Omega \), a contradiction. Thus (1.2) has no positive solution for \( \lambda \geq \lambda_\infty \). We shall verify next that \( \lim_{\lambda_\to \lambda_\infty} \|u_\lambda\|_\infty = \infty \).

Suppose otherwise, then there exist a sequence \( (\lambda_n) \subset (0, \lambda_\infty) \) and a constant \( C > 0 \) such that \( \lambda_n \to \lambda_\infty \) and \( \|u_\lambda\|_\infty < C \) for all \( n \), where \( u_n \equiv u_{\lambda_n} \). Since

\[
 -\Delta_p u_n \geq \frac{q(x)}{|u_n|^{\beta}} \geq \frac{q(x)}{C^{\beta}} \quad \text{in} \quad \Omega,
\]

it follows that there exists a constant \( \tilde{k} > 0 \) such that \( u_n \geq \tilde{k}\phi_1 \) in \( \Omega \) for all \( n \). Hence there exists a constant \( \tilde{C} > 0 \) such that

\[
 \frac{q(x)}{|u_n|^{\beta}} + \lambda f(u_n) \leq \frac{\tilde{C}}{\phi_1^{\beta}} \quad \text{in} \quad \Omega
\]

for all \( n \). By Lemma A, there exist constants \( \alpha \in (0, 1) \) and \( \tilde{M} > 0 \) such that \( u_n \in C^{1,\alpha}(\overline{\Omega}) \) and \( |u_n|_{1,\alpha} < \tilde{M} \) for all \( n \). By going to a subsequence, we assume that there
exists \( u \in C^1(\bar{\Omega}) \) such that \( u_n \to u \) in \( C^1(\bar{\Omega}) \). Let \( \psi \in W^{1,\rho}_0(\Omega) \). Then

\[
\int_{\Omega} |\nabla u_n|^{\rho-2} \nabla u_n \cdot \nabla \psi \, dx = \int_{\Omega} \left( \frac{q(x)}{u_n^\rho} + \lambda_n f(u_n) \right) \psi \, dx
\]

for all \( n \). Let \( n \to \infty \) in (3.11) and using the Lebesgue dominated convergence theorem, we obtain

\[
\int_{\Omega} |\nabla u|^{\rho-2} \nabla u \cdot \nabla \psi \, dx = \int_{\Omega} \left( \frac{q(x)}{u^\rho} + \lambda_\infty f(u) \right) \psi \, dx
\]

i.e. \( u \) is a positive solution of

\[
\begin{cases}
-\Delta u = \frac{q(x)}{u^\rho} + \lambda_\infty f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

a contradiction. This completes the proof of Theorem 1.3.

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References


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