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ON HOMOLOGY 3-SPHERES DEFINED BY TWO KNOTS

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Abstract

We show that if each of K_1 and K_2 is a trefoil knot or a figure eight knot, the homology 3-sphere defined by the Kirby diagram which is a simple link of K_1 and K_2 with framing $(0, n)$ is represented by an n-twisted Whitehead double of K_2.

1. Introduction

We define $W_n(K_1, K_2)$ to be the 4-dimensional handlebody represented by Fig. 1.1's Kirby daigram, the following Kirby diagram, and define $M_n(K_1, K_2)$ to be $\partial(W_n)$, where K_1 and K_2 are knots. Note that $M_n(K_1, K_2)$ is a homology 3-sphere.

When K_1 and K_2 are right handed trefoil knots $T_{2.3}$, Y. Matsumoto asked in [5] whether $M_0(T_{2.3}, T_{2.3})$ bounds a contractible 4-manifold or not. By Gordon's result [3], if n is odd, $M_n(T_{2.3}, T_{2.3})$ does not bound any contractible 4-manifold. If n is 6, N. Maruyama [6] proved that $M_6(T_{2.3}, T_{2.3})$ bounds a contractible 4-manifold. If n is 0, S. Akbulut [1] proved that $M_0(T_{2.3}, T_{2.3})$ does not bound any contractible 4-manifold.

In this note, we show that if each of K_1 and K_2 is a trefoil knot or a figure eight knot, the homology 3-sphere defined by Fig. 1.1 is represented by an n-twisted Whitehead double of K_2.

NOTATIONS. (i) Let K be a knot, we define $D_+(K, n)$ (or $D_-(K, n)$) to be the n-twisted Whitehead double of K with a positive hook (or a negative hook). For example, when K is a right handed trefoil knot $T_{2.3}$, $D_+(T_{2.3}, n)$ is the knot represented by Fig. 1.2, and $D_-(T_{2.3}, n)$ is the knot represented by Fig. 1.3.
(ii) We define $S^3_{\pm 1}(K)$ to be the ± 1-surgery along a knot K. For example, when K is a figure eight knot, $S^3_{+1}(D_+(K, n))$ is represented by Fig. 1.4.

Theorem 1.1. *If each of K_1 and K_2 is a trefoil knot or a figure eight knot, $M_n(K_1, K_2)$ is represented by the second column on the following table. $\lambda(S^3_{\pm 1}(D_\pm(K, n)))$ is the Casson invariant of $S^3_{\pm 1}(D_\pm(K, n))$.*

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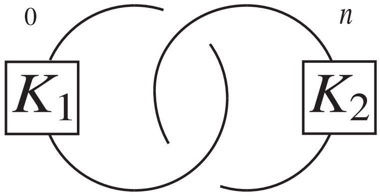


Fig. 1.1. $W_n(K_1, K_2)$.

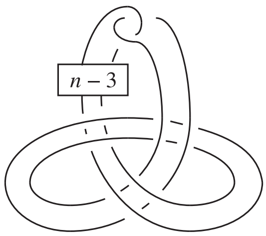


Fig. 1.2. $D_+(T_{2.3}, n)$.

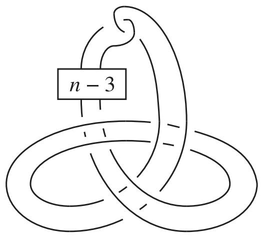


Fig. 1.3. $D_-(T_{2.3}, n)$.

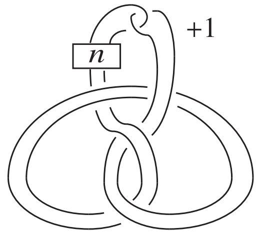


Fig. 1.4. $S^3_{+1}(D_+(K, n))$.

Table 1. Theorem 1.1's table.

$M_n(K_1, K_2)$	$S^3_{\pm 1}(D_\pm(K, n))$	$\lambda(S^3_{\pm 1}(D_\pm(K, n)))$
K_1: right handed trefoil, K_2: right handed trefoil	$S^3_{+1}(D_+(K_2, n))$	$-n$
K_1: left handed trefoil, K_2: right handed trefoil	$S^3_{-1}(D_-(K_2, n))$	$-n$
K_1: figure eight knot, K_2: right handed trefoil	$S^3_{-1}(D_+(K_2, n))$ $\cong$ $S^3_{+1}(D_-(K_2, n))$	n
K_1: right handed trefoil, K_2: figure eight knot	$S^3_{+1}(D_+(K_2, n))$	$-n$
K_1: figure eight knot, K_2: figure eight knot	$S^3_{+1}(D_-(K_2, n))$	n

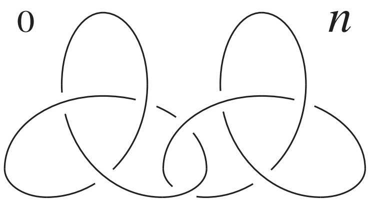
We will prove Theorem 1.1 in Section 2.

REMARK. When n is 0, S. Akbulut [1] shows essentially the same result of the first row on the table by a different method.

Corollary 1.2 (Gordon [3], cf. Y. Matsumoto [7] §3.1)**.** *Let $M_n(K_1, K_2)$ be one of the manifolds in the above table. If n is odd, $M_n(K_1, K_2)$ does not bound any contractible* 4*-manifold.*

Proof. A short proof of this result goes as follows:
The Casson invariant, when reduced modulo 2, is the Rohlin invariant:

$$\lambda(M_n(K_1, K_2)) \equiv \mu(M_n(K_1, K_2)) \mod 2.$$



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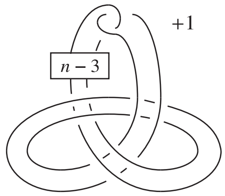
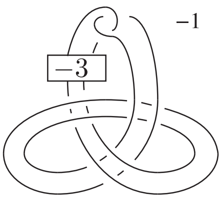
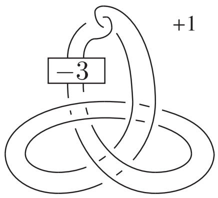


Fig. 1.5. $M_n(T_{2.3},\ T_{2.3})$. Fig. 1.6. $S^3_{+1}(D_+(T_{2.3},\ n))$.





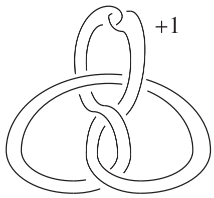


Fig. 1.7. $S^3_{-1}(D_+(T_{2.3},\ 0))$. Fig. 1.8. $S^3_{+1}(D_-(T_{2.3},\ 0))$. Fig. 1.9. $S^3_{+1}(D_+(4_1,\ 0))$.

By Theorem 1.1, $\lambda(M_n(K_1,\ K_2))$ is n or $-n$. Therefore if n is odd, we have $\mu(M_n(K_1,\ K_2)) \equiv 1 \bmod 2$, and so $M_n(K_1,\ K_2)$ does not bound any contractible 4-manifold. □

Corollary 1.3 (N. Maruyama [6]). *If K_1 and K_2 are right handed trefoil knots $T_{2.3}$, $M_6(T_{2.3},\ T_{2.3})$ bounds a contractible 4-manifold.*

Proof. By the first row on Theorem 1.1's table, $M_n(T_{2.3},\ T_{2.3})$ is represented by $S^3_{+1}(D_+(T_{2.3}, n))$. If n is 6, $D_+(T_{2.3}, 6)$ is known to be a slice knot ([8], p. 226). Therefore by [3], $M_6(T_{2.3},\ T_{2.3})$ bounds a contractible 4-manifold. □

Corollary 1.4. *If n is 0, $D_+(T_{2.3},\ 0)$ is not a slice knot.*

Proof. By [1], $M_0(T_{2.3},\ T_{2.3})$ does not bound any contractible 4-manifold. Therefore $D_+(T_{2.3},\ 0)$ is not a slice knot. □

REMARK. M. Hedden [4] showed that if n is smaller than 2, $D_+(T_{2.3},\ n)$ is not a slice knot.

Corollary 1.5. *Let $T_{2.3}$ be a right handed trefoil knot and 4_1 be a figure eight knot. The homology 3-spheres $S^3_{-1}(D_+(T_{2.3},\ 0))$, $S^3_{+1}(D_-(T_{2.3},\ 0))$ and $S^3_{+1}(D_+(4_1,\ 0))$ are pairwise diffeomorphic.*

Proof. By the third row and the fourth row on Theorem 1.1's table, if $n = 0$, the 4-dimensional handlebodies defined by Figs. 1.7, 1.8 and 1.9 have the same boundaries.

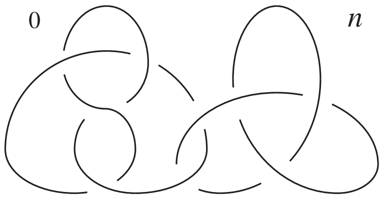


Fig. 1.10. $M_n(K_1, K_2)$.

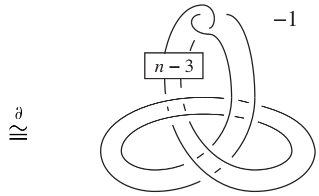


Fig. 1.11. $S^3_{-1}(D_+(K_2, n))$.

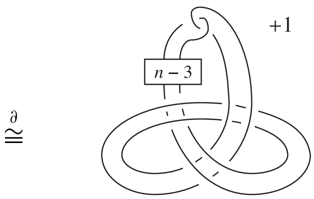


Fig. 1.12. $S^3_{+1}(D_-(K_2, n))$.

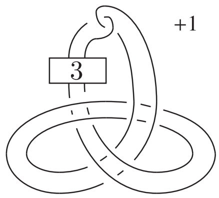


Fig. 1.13. $S^3_{+1}(D_-(T_{2,3}, 6))$.

Therefore the homology 3-spheres $S^3_{-1}(D_+(T_{2.3},0))$, $S^3_{+1}(D_-(T_{2.3},0))$ and $S^3_{+1}(D_+(4_1,0))$ are pairwise diffeomorphic. □

Corollary 1.6. *If K_1 is a figure eight knot and K_2 is a right handed trefoil knot* (*see* Fig. 1.10), *then $M_6(K_1, K_2)$ bounds a contractible* 4*-manifold.*

Proof. By the third row on Theorem 1.1's table, $M_n(K_1, K_2)$ is represented by $S^3_{-1}(D_+(K_2, n))$ and also by $S^3_{+1}(D_-(K_2, n))$. If n is 6, $D_+(K_2, 6)$ is known to be a slice knot ([8], p. 226). Therefore by [3], $M_6(K_1, K_2)$ bounds a contractible 4-manifold. □

REMARK. By Corollary 1.6, the homology 3-sphere $S^3_{+1}(D_-(T_{2,3}, 6))$ bounds a contractible 4-manifold. The author does not know whether the knot $D_-(T_{2,3}, 6)$ is a slice knot or not.

QUESTION. Let V_n^1 be the 4-dimensional handlebody defined by Fig. 1.12, and V_n^2 be the 4-dimensional handlebody defined by Fig. 1.11. Since $\partial(V_n^1)$ is diffeomorphic to $\partial(V_n^2)$ by Theorem 1.1, we have a closed 4-manifold $V_n^1 \cup_\partial (-V_n^2)$. Because $D_+(T_{2,3}, 6)$ is a slice knot, we have a smooth S^2 with self intersection -1 in V_6^2 representing a generator of $H_2(V_6^2)$. Blow down this smooth S^2 from the $V_6^1 \cup_\partial (-V_6^2)$. Then we are left with a closed smooth 4-manifold homotopy equivalent to $\mathbb{C}P^2$. Is this 4-manifold diffeomorphic to $\mathbb{C}P^2$?

Proposition 1.7. *$V_n^1 \cup_\partial (-V_n^2)$ is diffeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$.*

We show this fact in Section 3.

It seems that Theorem 1.1 is related to [6] Corollary 8 (3), but the author could not understand the relationship clearly.

The author does not know whether there is an even number $n \neq 0, 6$, such that $M_n(T_{2,3}, T_{2,3})$ bounds a contractible 4-manifold or not. M. Tange [10] proved that if n is smaller than 2, $M_n(T_{2,3}, T_{2,3})$ does not bound any contractible 4-manifold by computing the Heegaard Floer homology $HF^+(M_n(T_{2,3}, T_{2,3}))$ and the correction term $d(M_n(T_{2,3}, T_{2,3}))$.

2. Proof of Theorem 1.1

In this section, first we show that $M_n(K_1, K_2)$ is represented by $S^3_{\pm 1}(D_\pm(K_2, n))$. Next we compute the Casson invariant $\lambda(S^3_{\pm 1}(D_\pm(K_2, n)))$.

2.1. Proof of the first row on Theorem 1.1's table. K_1 and K_2 are right handed trefoil knots.

Proof. We show that the 4-manifolds represented by Figs. 2.1 and 2.17 have the same boundaries by following Kirby calculus:

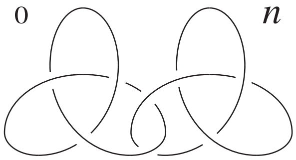


Fig. 2.1. $M_n(K_1, K_2)$.

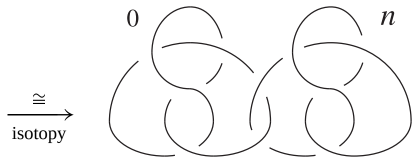


Fig. 2.2.

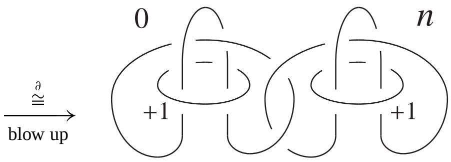


Fig. 2.3.

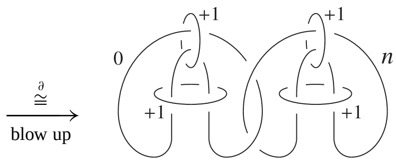


Fig. 2.4.

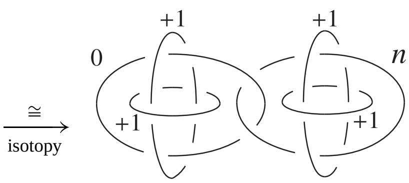


Fig. 2.5.

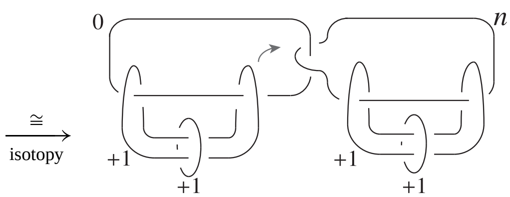


Fig. 2.6.

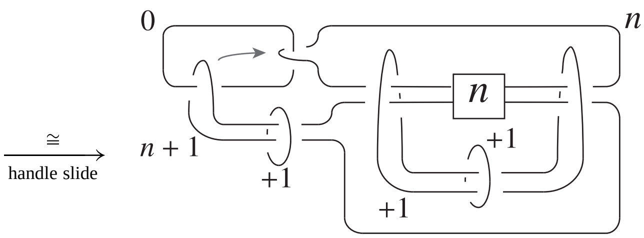


Fig. 2.7.

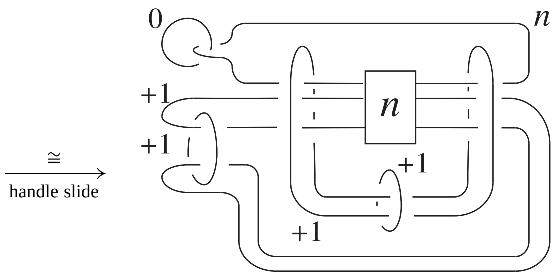


Fig. 2.8.

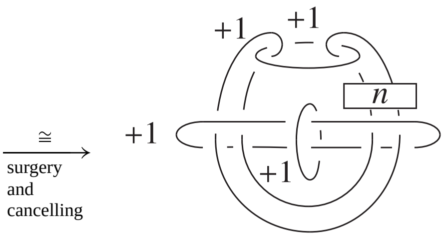


Fig. 2.9.

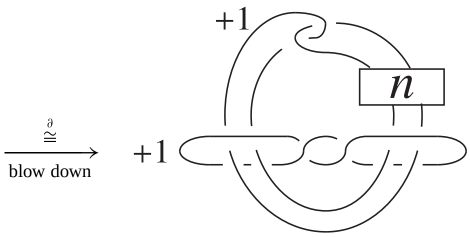


Fig. 2.10.

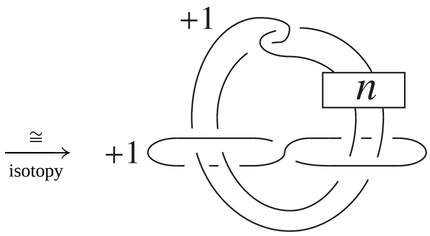


Fig. 2.11.

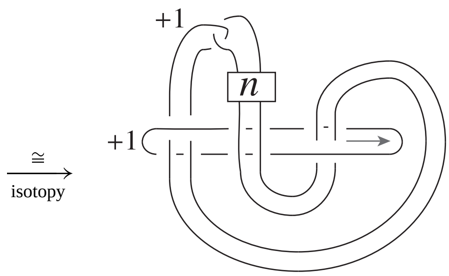


Fig. 2.12.

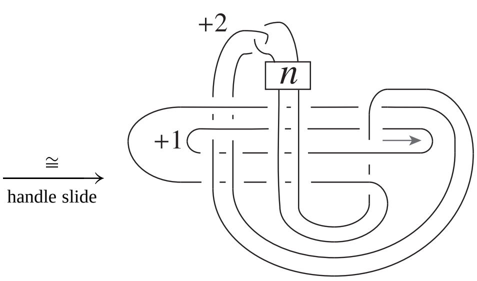


Fig. 2.13.

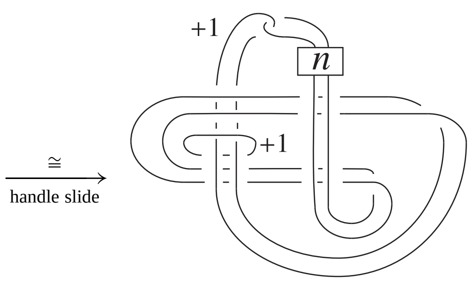


Fig. 2.14.

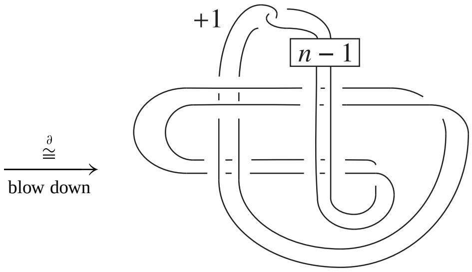


Fig. 2.15.

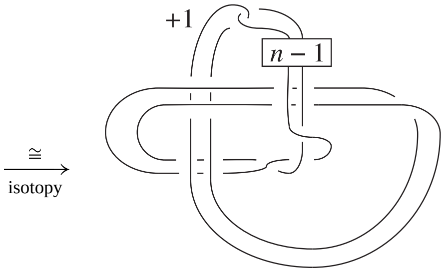


Fig. 2.16.

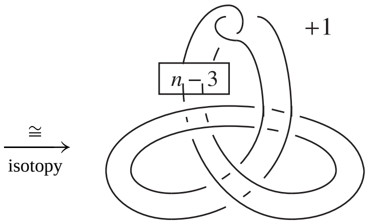


Fig. 2.17. $S^3_{+1}(D_+(K_2, n))$.

□

2.2. Proof of the second row on Theorem 1.1's table. K_1 is a left handed trefoil knot and K_2 is a right handed trefoil knot.

Proof. We show that the 4-manifolds represented by Figs. 2.18 and 2.23 have the same boundaries. □

2.3. Proof of the third row on Theorem 1.1's table. K_1 is a figure eight knot and K_2 is a right handed trefoil knot.

Proof. We show that the 4-manifolds represented by Figs. 2.24, 2.29 and 2.35 have the same boundaries.

Fig. 2.30 is the same diagram of Fig. 2.24, but by using the invertibility of the figure eight knot, we can show that they can be represented by a different doubled knot. □

2.4. Proof of the fourth row on Theorem 1.1's table. K_1 is a right handed trefoil knot and K_2 is a figure eight knot.

Proof. We show that the 4-manifolds represented by Figs. 2.36 and 2.47 have the same boundaries.

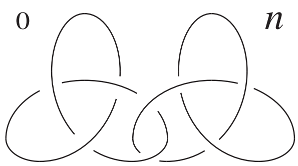


Fig. 2.18. $M_n(K_1, K_2)$.

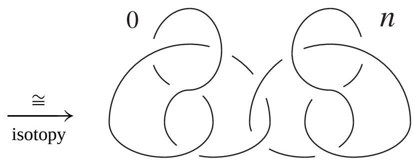


Fig. 2.19.

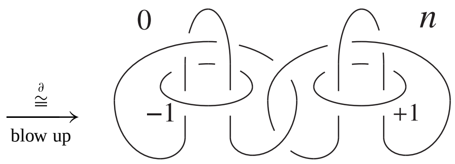


Fig. 2.20.

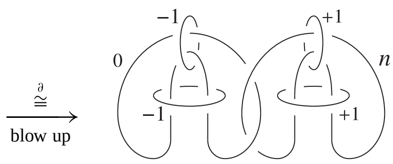


Fig. 2.21.

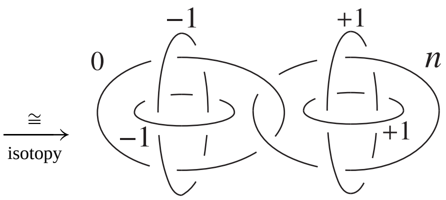


Fig. 2.22.

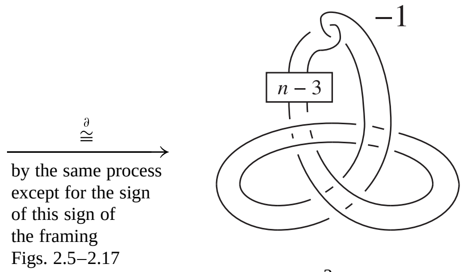


Fig. 2.23. $S^3_{-1}(D_-(K_2, n))$.

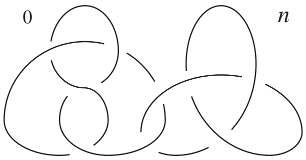


Fig. 2.24. $M_n(K_1, K_2)$.

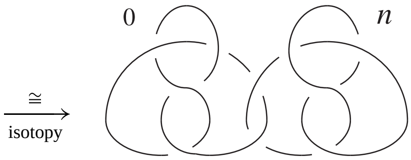


Fig. 2.25.

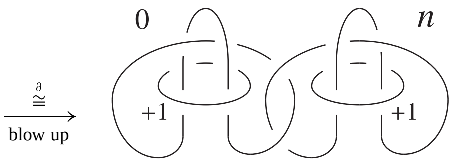


Fig. 2.26.

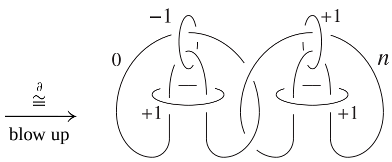
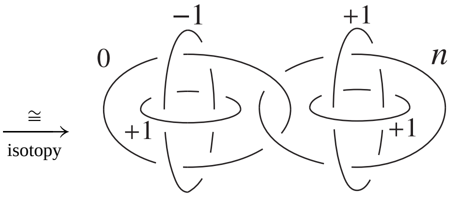


Fig. 2.27.



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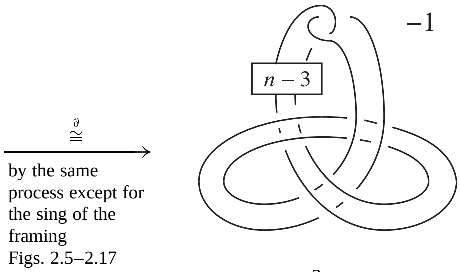


Fig. 2.29. $S^3_{-1}(D_+(K_2, n))$.

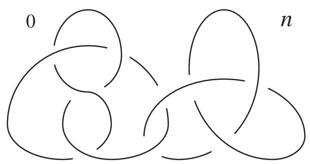


Fig. 2.30. $M_n(K_1, K_2)$.

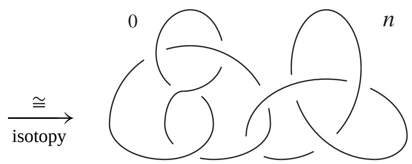


Fig. 2.31.

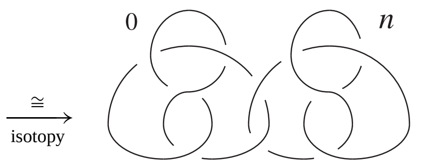


Fig. 2.32.

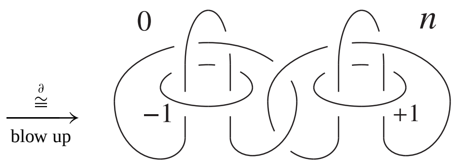


Fig. 2.33.

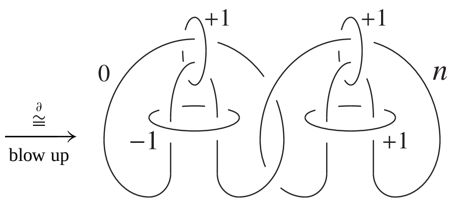


Fig. 2.34.

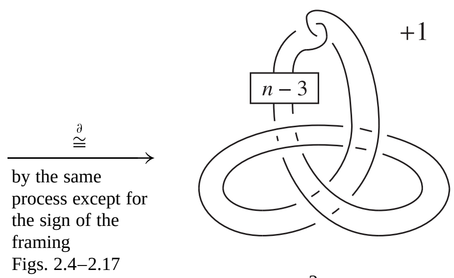
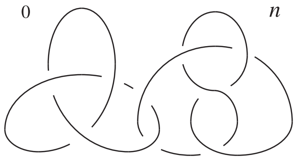


Fig. 2.35. $S^3_{+1}(D_-(K_2, n))$.



≅
isotopy

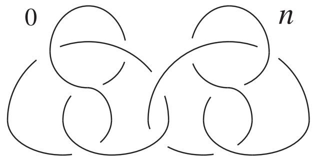
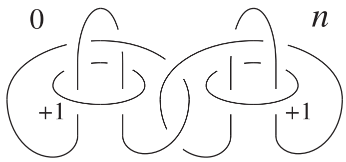


Fig. 2.36. $M_n(K_1, K_2)$).

Fig. 2.37.

$\overset{\partial}{\cong}$
blow up



$\overset{\partial}{\cong}$
blow up

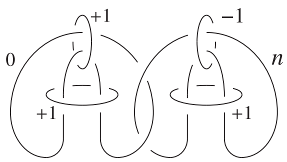
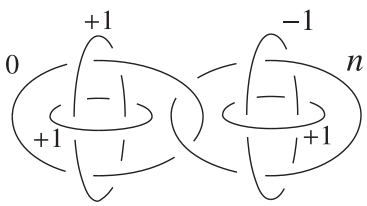


Fig. 2.38.

Fig. 2.39.

≅
isotopy



$\overset{\partial}{\cong}$
by the same process except for the sign of the framing Figs. 2.5–2.11

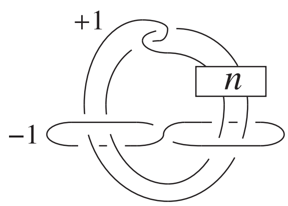
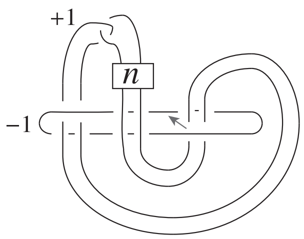


Fig. 2.40.

Fig. 2.41.

≅
isotopy



≅
handle slide

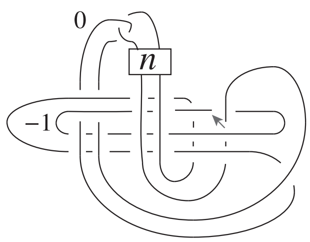
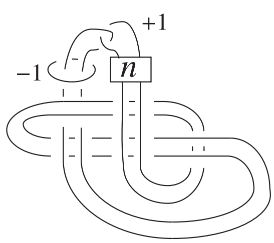


Fig. 2.42.

Fig. 2.43.

≅
handle slide



$\overset{\partial}{\cong}$
blow down

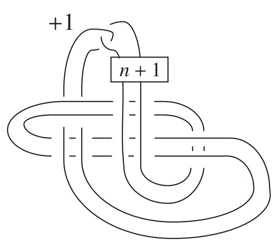


Fig. 2.44.

Fig. 2.45.

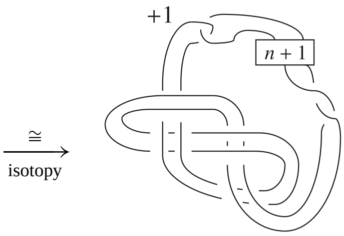


Fig. 2.46.

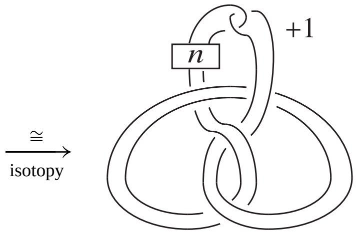


Fig. 2.47. $S^3_{+1}(D_+(K_2, n))$.

□

2.5. Proof of the fifth row on Theorem 1.1's table. K_1 and K_2 are figure eight knots.

Proof. We show that the 4-manifolds represented by Figs. 2.48 and 2.55 have the same boundaries.

Since Fig. 2.53 is the same diagram of Fig. 2.51, we can show that they can be represented by a different double knot. □

Next we compute the Casson invariant $\lambda(M_n(K_1, K_2))$. Now suppose that K_-, K_+ and K_0 are links in S^3 which have projections which differ at a single crossing of K_- as depicted Fig. 2.56.

REMARK. Our convention in Fig. 2.56 is different from that in [2]. In fact, their K_+ (resp.K_-) is our K_- (resp.K_+). We adopt our convention as Fig. 2.56 because by our convention $\lambda'(T_{2,3})$ is computed to be 1, where $T_{2,3}$ is a right handed trefoil knot. While by their convention $\lambda'(T_{2,3})$ is computed to be -1, contradicting the normalization $\lambda'(T_{2,3}) = 1$ ([2] p148, [9] p. 52).

Lemma 2.1 (see [2], p. 143)**.** *Let K_- be a knot in S^3. Let K_+ and K_0 be as above. Then K_0 is a two component link and*:

$$\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$$

where $\lambda'(K)$ is the Casson invariant of a knot K.

Lemma 2.2 (Surgery formula, see [9], p. 52)**.** *Let K be a knot in S^3. The Casson invariant $\lambda(S^3_{+1}(K))$ is equal to $\lambda'(K)$.*

By Lemmas 2.1, 2.2 and the second column on Theorem 1.1's table, we can compute the Casson invariant $\lambda(M_n(K_1, K_2))$.

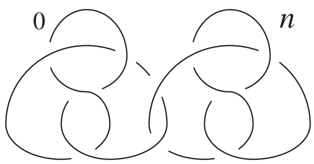


Fig. 2.48. $M_n(K_1, K_2)$.

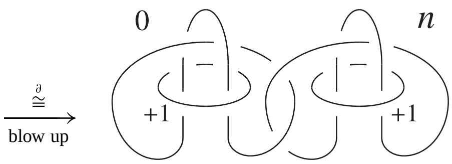


Fig. 2.49.

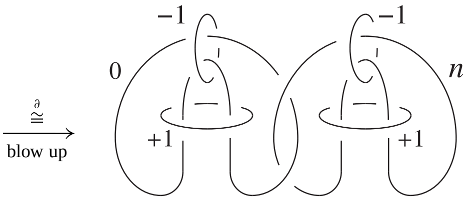


Fig. 2.50.

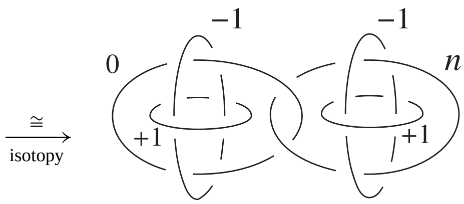


Fig. 2.51.

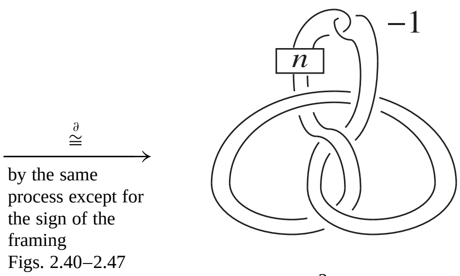


Fig. 2.52. $S^3_{-1}(D_+(K_2, n))$.

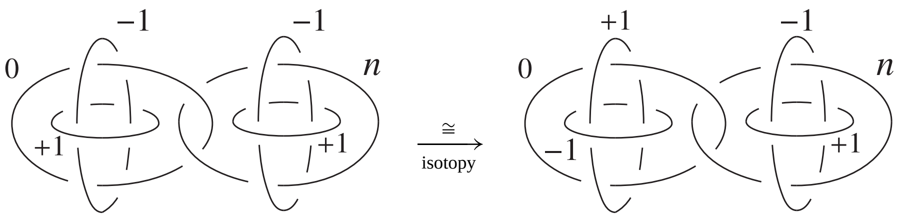


Fig. 2.53.

Fig. 2.54.

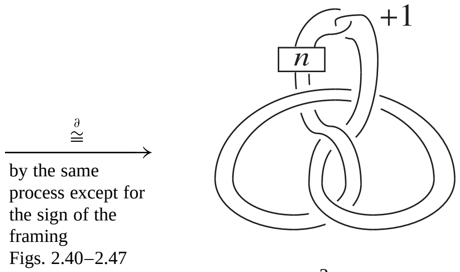


Fig. 2.55. $S^3_{+1}(D_-(K_2, n))$.

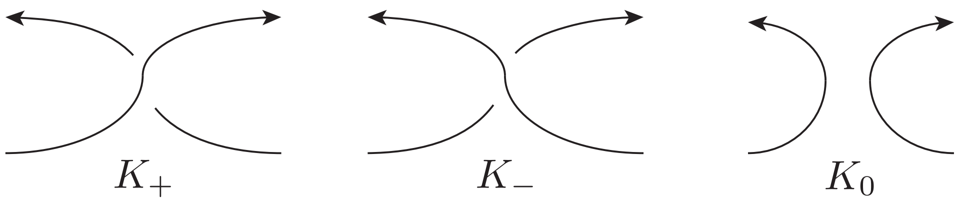


Fig. 2.56.

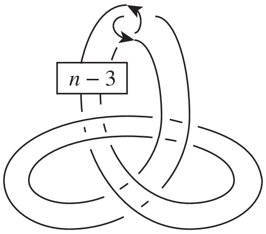


Fig. 2.57. K_+.

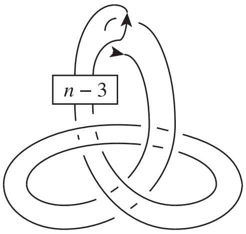


Fig. 2.58. K_-.

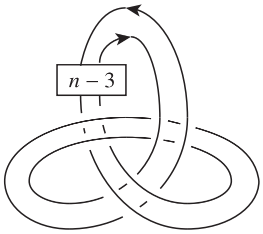


Fig. 2.59. K_0.

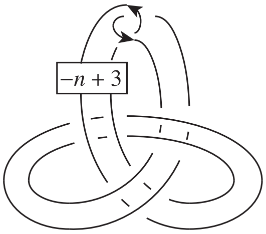


Fig. 2.60. K_+.

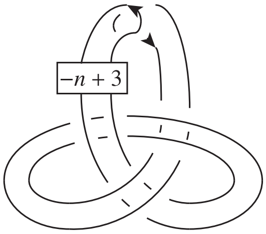


Fig. 2.61. K_-.

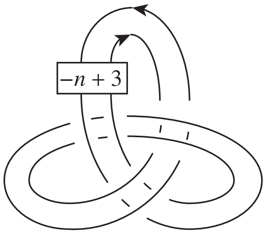


Fig. 2.62. K_0.

2.6. The Casson invariant of the first row on Theorem 1.1's table.

Proof. K_1 and K_2 are right handed trefoil knots. By Section 2.1, $M_n(K_1, K_2)$ is diffeomorphic to $S^3_{+1}(D_+(K_2, n))$. Therefore $\lambda(M_n(K_1, K_2))$ is equal to $\lambda(S^3_{+1}(D_+(K_2, n)))$. By Lemmas 2.1 and 2.2, we can compute the Casson invariant $\lambda(S^3_{+1}(D_+(K_2, n)))$ as Figs. 2.57–2.59.

By Lemma 2.1, $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$. Since K_- is a trivial knot, $\lambda'(K_-) = 0$. $lk(K_0)$ is $-n$. Therefore, $\lambda'(K_+) = -n$. Then $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{+1}(D_+(K_2, n))) = \lambda'(K_+) = -n$. □

2.7. The Casson invariant of the second row on Theorem 1.1's table.

Proof. K_1 is a left handed trefoil knot and K_2 is a right handed trefoil knot. By Section 2.2, $M_n(K_1, K_2)$ is diffeomorphic to $S^3_{-1}(D_-(K_2, n))$. Therefore $\lambda(M_n(K_1, K_2))$ is equal to $\lambda(S^3_{-1}(D_-(K_2, n)))$. Since $\lambda(S^3_{-1}(D_-(K_2, n))) = -\lambda(S^3_{+1}(D_+(K_1, -n)))$ (see [9], p. 52, Theorem 3.1.), we compute the Casson invariant $\lambda(S^3_{+1}(D_+(K_1, -n)))$ as Figs. 2.60–2.62.

By Lemma 2.1, $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$. Since K_- is a trivial knot, $\lambda'(K_-) = 0$. $lk(K_0)$ is n. Therefore, $\lambda'(K_+) = n$. Then $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{-1}(D_-(K_2, n))) = -\lambda(S^3_{+1}(D_+(K_1, -n))) = -\lambda'(K_+) = -n$. □

2.8. The Casson invariant of the third row on Theorem 1.1's table.

Proof. K_1 is a figure eight knot and K_2 is a right handed trefoil knot. By Section 2.3, $M_n(K_1, K_2)$ is diffeomorphic to $S^3_{+1}(D_-(K_2, n))$. Therefore $\lambda(M_n(K_1, K_2))$

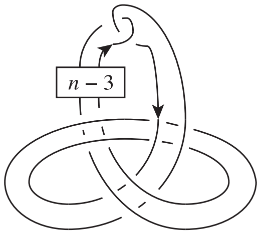


Fig. 2.63. K_-.

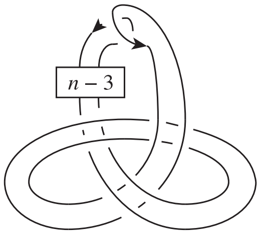


Fig. 2.64. K_+.

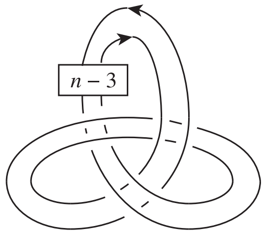


Fig. 2.65. K_0.

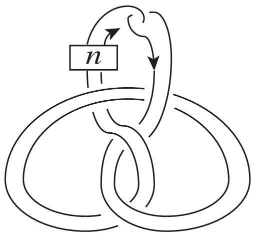


Fig. 2.66. K_+.

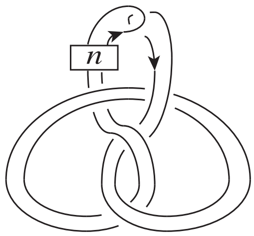


Fig. 2.67. K_-.

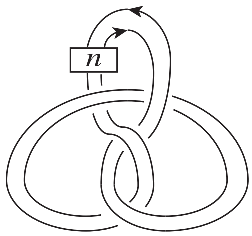


Fig. 2.68. K_0.

is equal to $\lambda(S^3_{+1}(D_-(K_2, n)))$. By Lemmas 2.1 and 2.2, we can compute the Casson invariant $\lambda(S^3_{+1}(D_-(K_2, n)))$ as Figs. 2.63–2.65.

By Lemma 2.1, $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$. Since K_+ is a trivial knot, $\lambda'(K_+) = 0$. $lk(K_0)$ is $-n$. Therefore, $\lambda'(K_-) = n$. Then $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{+1}(D_-(K_2, n))) = \lambda'(K_-) = n$. □

2.9. The Casson invariant of the fourth row on Theorem 1.1's table.

Proof. K_1 is a right handed trefoil knot and K_2 is a figure eight knot. By 2(iv), $M_n(K_1, K_2)$ is diffeomorphic to $S^3_{+1}(D_+(K_2, n))$. Therefore $\lambda(M_n(K_1, K_2))$ is equal to $\lambda(S^3_{+1}(D_+(K_2, n)))$. By Lemmas 2.1 and 2.2, we can compute the Casson invariant $\lambda(S^3_{+1}(D_+(K_2, n)))$ as Figs. 2.66–2.68.

By Lemma 2.1, $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$. Since K_- is a trivial knot, $\lambda'(K_-) = 0$. $lk(K_0)$ is $-n$. Therefore, $\lambda'(K_+) = -n$. Then $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{+1}(D_+(K_2, n))) = \lambda'(K_+) = -n$. □

2.10. The Casson invariant of the fifth row on Theorem 1.1's table.

Proof. K_1 and K_2 are figure eight knots. By Section 2.5, $M_n(K_1, K_2)$ is diffeomorphic to $S^3_{+1}(D_-(K_2, n))$. Therefore $\lambda(M_n(K_1, K_2))$ is equal to $\lambda(S^3_{+1}(D_-(K_2, n)))$. By Lemmas 2.1 and 2.2, we can compute the Casson invariant $\lambda(S^3_{+1}(D_-(K_2, n)))$ as Figs. 2.69–2.71.

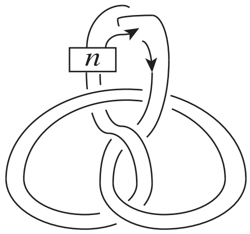


Fig. 2.69. K_-.

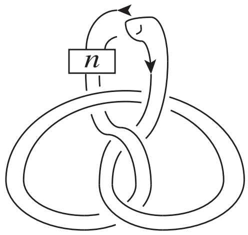


Fig. 2.70. K_+.

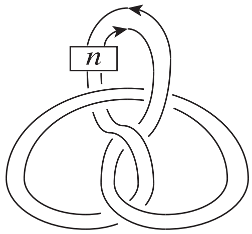


Fig. 2.71. K_0.

By Lemma 2.1, $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$. Since K_+ is a trivial knot, $\lambda'(K_+) = 0$. $lk(K_0)$ is $-n$. Therefore, $\lambda'(K_-) = n$. Then $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{+1}(D_-(K_2, n))) = \lambda'(K_-) = n$. $\square$

3. Proof of Proposition 1.7

We show that $V^1_n \cup_\partial (-V^2_n)$ is diffeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$.

Proof. By Kirby calculus from Figs. 3.1–3.6, we will show that the Kirby diagram of $V^1_n \cup_\partial (-V^2_n)$ is represented by Fig. 3.6. $\square$

4. Appendix

An alternative proof of Corollary 1.2. By [3], if n is odd, $M_n(T_{2,3}, T_{2,3})$ does not bound any contractible 4-manifold. In this Section we will give an alternative proof of this fact. For this purpose, we will prove the following proposition;

Proposition 4.1. *The* 4*-dimensional handlebodies represented by* Figs. 4.1, 4.8 *and* 4.9 *have the same boundaries.*

Proof. See Figs. 4.1–4.9. $\square$

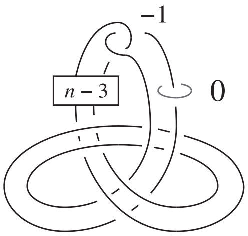


Fig. 3.1.

$\overset{\partial}{\cong}$

by the same process except for the sign of the framing Fig. 2.5–2.17

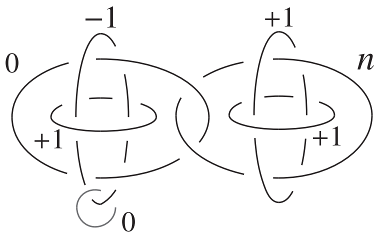


Fig. 3.2.

$\cong$

isotopy

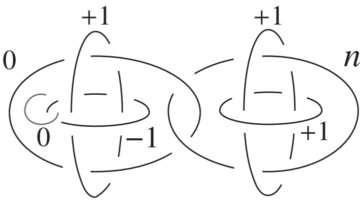


Fig. 3.3.

$\overset{\partial}{\cong}$

by the same process except for the sign of the framing Fig. 2.5–2.9

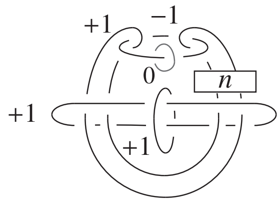


Fig. 3.4.

$\overset{\partial}{\cong}$

blow down

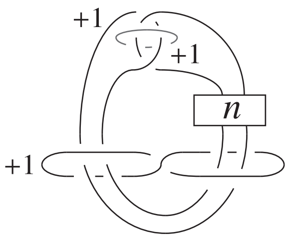


Fig. 3.5.

$\overset{\partial}{\cong}$

blow down

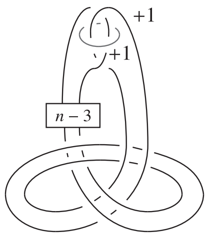


Fig. 3.6. $V_n^1 \cup_\partial (-V_n^2)$.

$\cong$

handle slide

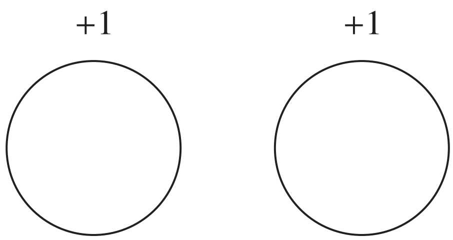


Fig. 3.7. $\mathbb{C}P^2 \# \mathbb{C}P^2$.

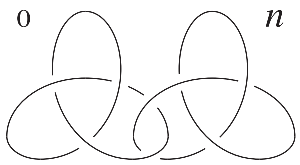


Fig. 4.1. $M_n(T_{2,3}, T_{2,3})$.

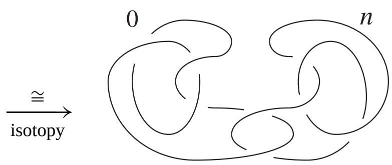


Fig. 4.2.

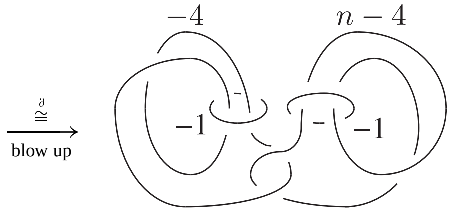


Fig. 4.3.

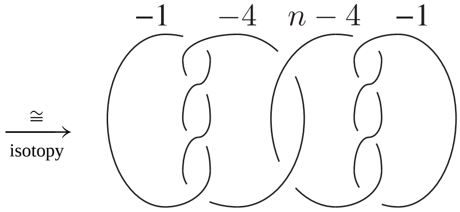


Fig. 4.4.

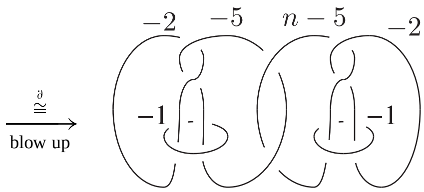


Fig. 4.5.

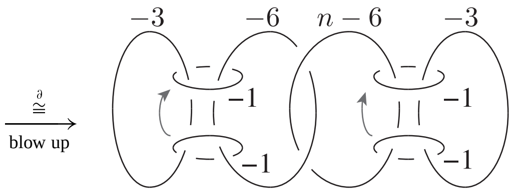


Fig. 4.6.

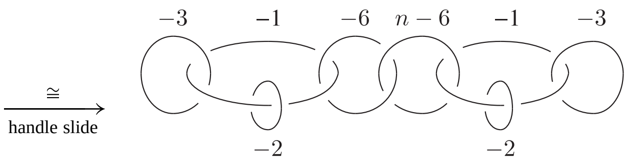


Fig. 4.7.

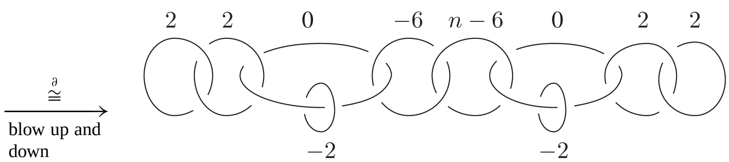


Fig. 4.8.

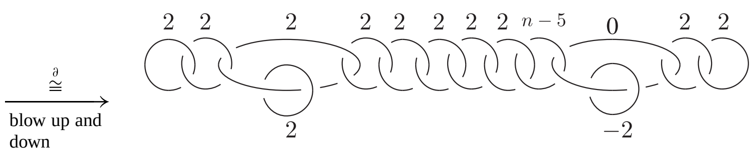


Fig. 4.9.

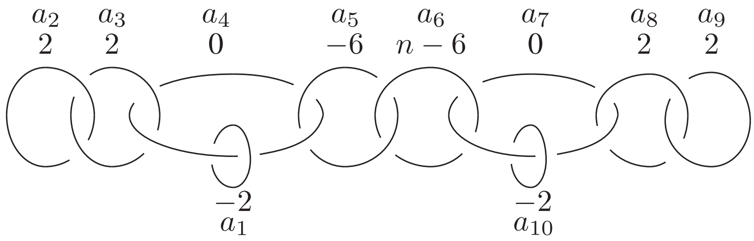


Fig. 4.10. Q_1.

We give an alternative proof of Corollary 1.2.

Proof of Corollary 1.2. Fig. 4.8 gives a smooth 4-manifold Q_1 with intersection form A.

$$A = (\alpha_{ij}), \quad \alpha_{ij} = a_i \cdot a_j, \quad 1 \leq i,\ j \leq 10,$$

$$A = \begin{pmatrix} -2 & & & 1 & & & & & & \\ & 2 & 1 & & & & & & & \\ & 1 & 2 & 1 & & & & & & \\ 1 & & 1 & 0 & 1 & & & & & \\ & & & 1 & -6 & 1 & & & & \\ & & & & 1 & n-6 & 1 & & & \\ & & & & & 1 & 0 & 1 & & 1 \\ & & & & & & 1 & 2 & 1 & \\ & & & & & & & 1 & 2 & \\ & & & & & & 1 & & & -2 \end{pmatrix}.$$

We have $\mathrm{Index}(A) = 0$. Note that A is an even type matrix if n is even.

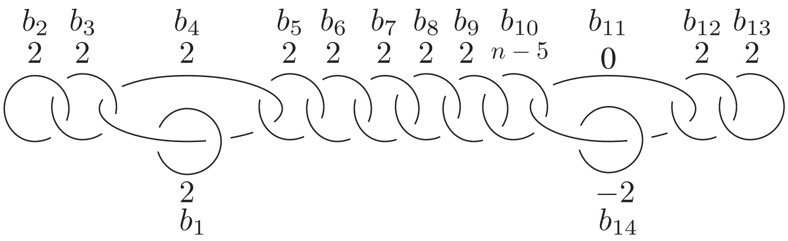


Fig. 4.11. Q_2

Fig. 4.9 gives a smooth 4-manifold Q_2 with intersection form B.

$$B = (\beta_{ij}), \quad \beta_{ij} = b_i \cdot b_j, \quad 1 \le i,\, j \le 14,$$

$$B = \left(\begin{array}{cccccccccccccc}
2 & & & 1 & & & & & & & & & & \\
 & 2 & 1 & & & & & & & & & & & \\
 & 1 & 2 & 1 & & & & & & & & & & \\
1 & & 1 & 2 & 1 & & & & & & & & & \\
 & & & 1 & 2 & 1 & & & & & & & & \\
 & & & & 1 & 2 & 1 & & & & & & & \\
 & & & & & 1 & 2 & 1 & & & & & & \\
 & & & & & & 1 & 2 & 1 & & & & & \\
 & & & & & & & 1 & 2 & 1 & & & & \\
 & & & & & & & & 1 & n-5 & 1 & & & \\
 & & & & & & & & & 1 & 0 & 1 & & 1 \\
 & & & & & & & & & & 1 & 2 & 1 & \\
 & & & & & & & & & & & 1 & 2 & \\
 & & & & & & & & & & 1 & & & -2
\end{array}\right).$$

We have $\mathrm{Index}(B) = 8$. Note that B is an even type matrix if n is odd.

By Proposition 4.1, we have the Rohlin invariant $\mu(M_n(T_{2,3}, T_{2,3}))$ as follows:

$$\mu(M_n(T_{2,3}, T_{2,3})) \equiv \begin{cases} \mathrm{Index}(B) \equiv 1 & (n \text{ is odd}) \\ \mathrm{Index}(A) \equiv 0 & (n \text{ is even}) \end{cases} \mod 2.$$

Therefore if n is odd, $M_n(T_{2,3}, T_{2,3})$ does not bound any contractible 4-manifold. □

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