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# ON THE CLASSIFICATIONS OF UNITARY MATRICES

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**Abstract**

We classify the dynamical action of matrices in $\mathrm{SU}(p,q)$ using the coefficients of their characteristic polynomial. This generalises earlier work of Goldman for $\mathrm{SU}(2,1)$ and the classical result for $\mathrm{SU}(1,1)$, which is conjugate to $\mathrm{SL}(2,\mathbb{R})$. As geometrical applications, we show how this enables us to classify automorphisms of real and complex hyperbolic space and anti de Sitter space.

## 1. Introduction

In this paper we use the coefficients of the characteristic polynomial to give a dynamical classification of unitary matrices preserving a non-degenerate Hermitian form. The most interesting case is where the Hermitian form has indefinite signature. This includes the case of orthogonal matrices (with respect to a possibly indefinite quadratic form) by restricting to the case where the matrix is real, and so the coefficients of the characteristic polynomial are also real. The application we have in mind is that orthogonal and unitary matrices often act as isometries on metric spaces. The most obvious example of this is when the signature is $(n, 1)$, when orthogonal matrices act on real hyperbolic $n$-space and unitary matrices act on complex hyperbolic $n$-space. There are more exotic examples, however. For example, isometries of quaternionic hyperbolic 1-space and anti de Sitter space may both be embedded in (projectivisations of) $\mathrm{SU}(2, 2)$.

The classification of elements of $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SU}(2,1)$ has been useful in many contexts; see [7], [13] or [18]. Our initial motivation to this work was to provide initial tools for generalisation of these works to $\mathrm{SU}(p,1)$ for $p \geq 3$. As we did so, we realised it is natural to consider Hermitian forms of arbitrary signature. We first give the classification in arbitrary dimensions, and then we go on to consider $\mathrm{SU}(p,q)$ where $p+q=4$.

In order to illustrate and motivate the main results of the paper, let us work through the well known example of $2\times 2$ matrices. In this case, if $A \in \mathrm{SU}(p,q)$ with $p+q=2$

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then the characteristic polynomial of $A$ is

$$\chi_A(X) = X^2 - \tau X + 1$$

where $\tau = \mathrm{tr}(A)$, which is real. There are three possibilities for the eigenvalues $\lambda_1$, $\lambda_2$ of $A$, which are the roots of $\chi_A$ (compare Theorem 4.3.1 of [2] for example). Namely,
(i) $\tau^2 < 4$ and $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$.
(ii) $\tau^2 = 4$ and $\lambda_1 = \lambda_2 = \pm 1$.
(iii) $\tau^2 > 4$ and, reordering if necessary, $\lambda_1 = \pm e^{l}$, $\lambda_2 = \pm e^{-l}$ where $l > 0$.
Based on standard terminology from hyperbolic geometry we refer to these cases as *elliptic*, *parabolic* (provided $A \neq \pm I$) and *loxodromic* respectively. Suppose that $A \in \mathrm{SU}(p, q)$ with $p + q = 2$ satisfies the conditions of case (iii). Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be non-zero eigenvectors with eigenvalues $\lambda_1 = \pm e^{l}$ and $\lambda_2 = \pm e^{-l}$ respectively. It is not hard to show that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be null vectors with respect to the Hermitian form. Therefore $p = q = 1$. A similar argument shows that in case (ii) either $A = \pm I$ or $A$ is not diagonalisable and $p = q = 1$.

We want to reformulate this classification in terms that may be generalised. A key to this classification is the *resultant* $R(\chi_A, \chi_A')$, which determines when $\chi_A$ and $\chi_A'$ have a common root, and hence $\chi_A(X)$ has a repeated root. In the case where $p + q = 2$ the resultant is $4 - \tau^2$. Therefore we have
(i) $A$ is elliptic if and only if $R(\chi_A, \chi_A') = 4 - \tau^2 > 0$.
(ii) $A$ is parabolic (or $\pm I$) if and only if $R(\chi_A, \chi_A') = 4 - \tau^2 = 0$.
(iii) $A$ is loxodromic if and only if $R(\chi_A, \chi_A') = 4 - \tau^2 < 0$.
The case (ii) where $A$ has a repeated eigenvalue is more complicated than the other cases. In what follows we will not discuss the details of this case.

This argument was generalised to the case where $p + q = 3$ by Goldman in [8]; see also Parker [18]. This is the main motivation for our work here. In fact Goldman's work concentrated on the case $p = 2$, $q = 1$, but it is not hard to see how to generalise this to other signatures when $p + q = 3$. We give a summary of Goldman's results in Section 2.3 below, but we generalise his methods to arbitrary signature. In the case when $n = 3$, the locus where $R(\chi_A, \chi_A') = 0$ is a classical curve called a *deltoid*. Goldman's work has been generalised in a different direction by Navarrete [16] who considers elements of $\mathrm{SL}(3, \mathbb{C})$. This is related to the theory of complex Kleinian groups; see the book [3].

Our aim in this paper is to generalise this classification to higher values of $p+q = n$. First, we consider arbitrary $n$ and give a general result, Theorem 3.1. We refer to later sections for the precise definitions contained in this theorem. In particular *regular* means that the eigenvalues of $A$ are distinct. For the definition of $k$-loxodromic see Section 2.2. Roughly speaking, this means that $A$ has $k$ pairs of distinct eigenvalues related by inversion in the unit circle and all other eigenvalues lie on the unit circle, so regular 0-loxodromic maps are elliptic.

**Theorem 3.1.** *Let $A \in \mathrm{SU}(p,q)$. Let $R(\chi_A, \chi'_A)$ denote the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi'_A(X)$. Then for $m \geq 0$, we have the following.*
(i) *$A$ is regular $2m$-loxodromic if and only if $R(\chi_A, \chi'_A) > 0$.*
(ii) *$A$ is regular $(2m+1)$-loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.*
(iii) *$A$ has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$.*

An immediate corollary of Theorem 3.1 is a classification for $\mathrm{SU}(p,1)$. Since $q = 1$, if $A$ is loxodromic it must be 1-loxodromic. This simplifies the classification:

**Corollary 3.2.** *Let $A \in \mathrm{SU}(p,1)$. Let $R(\chi_A, \chi'_A)$ denote the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi'_A(X)$. Then we have the following.*
(i) *$A$ is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$.*
(ii) *$A$ is regular loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.*
(iii) *$A$ has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$.*

Secondly, we give a much more detailed description in the case $p + q = 4$. Here the characteristic polynomial is

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \overline{\tau} X + 1$$

where $\tau = \mathrm{tr}(A)$, which is complex, and $\sigma = (\mathrm{tr}^2(A) - \mathrm{tr}(A^2))/2$, which is real. In this case, the locus where $R(\chi_A, \chi'_A) = 0$ was studied by Poston and Stewart [21] following earlier work by Chillingworth [5]. They named this object the *holy grail*. As a subset of three dimensional space, parametrised by $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$, the holy grail comprises a ruled surface together with four space curves, called *whiskers*. We devote some space to different ways of parametrising the holy grail and the different components of its complement. The parametrisation of the corresponding object (a deltoid) in the case of $p + q = 3$ has been useful when studying complex hyperbolic representation spaces (see [10], [20] or the survey [18]) and we believe that the results in this paper will be foundational to the generalisation of these theorems to higher dimensions. The main theorem of this section is:

**Theorem 4.9.** *Let $A \in \mathrm{SU}(p, q)$ where $p + q = 4$ and let $\tau = \mathrm{tr}(A)$ and $\sigma = (\mathrm{tr}^2(A) - \mathrm{tr}(A^2))/2$. Let $\chi_A(X)$ be the characteristic polynomial of $A$ and let $R(\chi_A, \chi'_A)$ be the resultant of $\chi_A(X)$ and $\chi'_A(X)$. Then*
(i) *$A$ is regular 2-loxodromic if and only if $R(\chi_A, \chi'_A) > 0$ and*

$$\min\{\Re(\tau)^2 - 4\sigma + 8,\ \Im(\tau)^2 + 4\sigma + 8,\ 6 - \sigma,\ 6 + \sigma\} < 0.$$

(ii) *$A$ is regular 1-loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.*
(iii) *$A$ is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$ and*

$$\Re(\tau)^2 - 4\sigma + 8 > 0, \quad \Im(\tau)^2 + 4\sigma + 8 > 0, \quad -6 < \sigma < 6.$$

(iv) *A has a repeated eigenvalue if and only if* $R(\chi_A, \chi_A') = 0$.

In our first geometric application, Section 5.2, we take $p = 3$ and $q = 1$. We express Corollary 3.2 in terms of $\tau$ and $\sigma$ and discuss the geometry of the action of $A$ on complex hyperbolic 3-space $\mathbf{H}^3_{\mathbb{C}}$.

Our second geometric application, Section 5.3, concerns isometries of the quaternionic hyperbolic line $\mathbf{H}^1_{\mathbb{H}}$. These isometries are (projections of) matrices in Sp(1, 1) preserving a quaternionic Hermitian form. Identifying the quaternions with $\mathbb{C}^2$ gives a map of Sp(1, 1) into SU(2, 2). Using this we give the connection between our main results and Gonogopadhyay's classification [11] of elements of $\mathrm{SL}(2, \mathbb{H})$.

Finally in Section 5.4, we consider the automorphisms of anti de Sitter space, which may be canonically identified with $\mathrm{PSL}(2, \mathbb{R})$. This gives an identification between the automorphisms of anti de Sitter space and $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$. By translating such an automorphism to PSO(2, 2) we can use our classification to determine the dynamics. In this case "regular" refers to the map in PSO(2, 2) not having a repeated eigenvalue. Specifically we have

**Theorem 5.5.** *Let* $(A_1, A_2) \in \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ *be an automorphism of anti de Sitter space. Then*

(i) $(A_1, A_2)$ *is regular* 2*-loxodromic if at least one of* $A_1$ *and* $A_2$ *is loxodromic, and also* $\mathrm{tr}^2(A_1)$ *and* $\mathrm{tr}^2(A_2)$ *are distinct and neither of them equals* 4.

(ii) $(A_1, A_2)$ *is regular elliptic if* $A_1$ *and* $A_2$ *are both elliptic and* $\mathrm{tr}^2(A_1)$ *does not equal* $\mathrm{tr}^2(A_2)$.

(iii) $(A_1, A_2)$ *is not regular if* $\mathrm{tr}^2(A_1) = 4$ *or* $\mathrm{tr}^2(A_2) = 4$ *or* $\mathrm{tr}^2(A_1) = \mathrm{tr}^2(A_2)$.

## 2. Preliminaries

**2.1. Hermitian forms.** Consider a complex vector space $V = \mathbb{C}^n$ equipped with the non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$. Suppose the associated matrix $H$ has $p$ positive eigenvalues and $q$ negative eigenvalues. Therefore $p + q = n$ and we say that both $\langle \cdot, \cdot \rangle$ and $H$ have *signature* $(p, q)$.

For example, suppose that $H$ is the $n \times n$ diagonal matrix, $p$ of whose diagonal entries are $+1$ and $q$ are $-1$. Then clearly $H$ is Hermitian with signature $(p, q)$. Such a Hermitian space $(V, H)$ is referred to as a pseudo-Hermitian space often by mathematical physicists, see [1]. It is well-known that Hermitian forms over the complex numbers are classified by their signatures and so, up to equivalence, we can always take a pseudo-Hermitian form to work on a Hermitian space.

Let $\mathbf{v} \in V$. We say that $\mathbf{v}$ is *positive*, *null* or *negative* if $\langle \mathbf{v}, \mathbf{v} \rangle$ is greater than, equal to or less than zero, respectively. Sometimes terminology from special relativity is used and these vectors are called *spacelike*, *lightlike* or *timelike* respectively. Motivated by this, we define

$$V_+ = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{v} \rangle > 0\}, \tag{2.1}$$

$$V_0 = \{\mathbf{v} \in V - \{\mathbf{0}\} : \langle \mathbf{v}, \mathbf{v} \rangle = 0\}, \tag{2.2}$$

$$V_- = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{v} \rangle < 0\}. \tag{2.3}$$

Notice that if $\lambda$ is a non-zero complex scalar then $\langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle$. Thus if $\mathbf{v}$ is positive, null or negative then so is any non-trivial vector in the subspace of $V$ spanned by $\mathbf{v}$. More generally, if $U$ is a vector subspace of $V$ then we say that $U$ is *positive*, *null* or *negative* if every vector in $U - \{\mathbf{0}\}$ is positive, null or negative. Similarly, a vector subspace is *non-negative* or *non-positive* if it contains positive (respectively negative) vectors and non-trivial null vectors. Likewise we say that a vector subspace $U$ is *indefinite* if $U$ contains both positive and negative vectors (and necessarily null vectors as well). We remark that, since $\langle \cdot, \cdot \rangle$ is non-degenerate, all null subspaces are one (complex) dimensional.

**2.2. The group U($p,q$).** Let $V$ denote a vector space of dimension $n$ with a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(p, q)$. An $n \times n$ matrix $A$ is *unitary* with respect to this form if $\langle A\mathbf{v}, A\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$. We let $\mathrm{U}(p, q)$ denote the group of matrices that are unitary with respect to this form. We often wish to consider unitary matrices with determinant equal to 1. Such matrices form the group $\mathrm{SU}(p, q)$.

We remark that if $\langle \cdot, \cdot \rangle$ has signature $(p, q)$ then $-\langle \cdot, \cdot \rangle$ has signature $(q, p)$. Thus any matrix in $\mathrm{U}(p, q)$ is also in $\mathrm{U}(q, p)$. Hence we may suppose that $p \geq q$.

We will be interested in eigenvalues and eigenspaces of unitary matrices. If $A \in \mathrm{U}(p, q)$ has distinct eigenvalues then we call it *regular*. This automatically means that $A$ is diagonalisable. Let $A \in \mathrm{U}(p, q)$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$. First, since $A$ is unitary we must have $\lambda \neq 0$. Let $V_\lambda$ be the eigenspace associated to $\lambda$. Then we say that $\lambda$ is of *positive type*, *null type*, *negative type*, *non-negative type*, *non-positive type* or *indefinite type* if $V_\lambda$ is positive, null, negative, non-negative, non-positive or indefinite respectively.

We will heavily use the following simple lemma.

**Lemma 2.1** (Lemma 6.2.5 of Goldman [8]). *Let $V$ be a Hermitian vector space and $A$ a unitary automorphism of $V$. If $\lambda$ is an eigenvalue of $A$ then $\bar{\lambda}^{-1}$ is also an eigenvalue of $A$ with the same multiplicity as $\lambda$. That is, the collection of eigenvalues of $A$ is invariant under inversion in the unit circle.*

Note that if $|\lambda| = 1$ then $\bar{\lambda}^{-1} = \lambda$ and this statement is vacuous. Clearly if $|\lambda| \neq 1$ then $\lambda$ and $\bar{\lambda}^{-1}$ are distinct.

Furthermore, suppose that $\lambda$ is an eigenvalue of $A$ with $|\lambda| \neq 1$ and multiplicity 1. Then $\bar{\lambda}^{-1}$ is also an eigenvalue of $A$ with multiplicity 1. In this case, the eigenspaces

$V_\lambda$ and $V_{\overline{\lambda}^{-1}}$ are both null one dimensional vector subspaces. Moreover, $V_\lambda \oplus V_{\overline{\lambda}^{-1}}$ is an indefinite subspace of $V$ and the restriction of the Hermitian form to this subspace has signature (1, 1).

More generally, if $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and ordered so that $|\lambda_1| \geq \cdots \geq |\lambda_k| > 1$. Then $\overline{\lambda}_1^{-1}, \ldots, \overline{\lambda}_k^{-1}$ are also distinct eigenvalues. Then the corresponding eigenspaces $V_{\lambda_j}$ and $V_{\overline{\lambda}_j^{-1}}$ are all null and of dimension 1. Moreover $V_{\lambda_i} \oplus V_{\overline{\lambda}_i^{-1}}$ and $V_{\lambda_j} \oplus V_{\overline{\lambda}_j^{-1}}$ are orthogonal and so

$$V_{\lambda_1} \oplus V_{\overline{\lambda}_1^{-1}} \oplus \cdots \oplus V_{\lambda_k} \oplus V_{\overline{\lambda}_k^{-1}}$$

is a vector subspace of signature $(k, k)$. In particular, $k \leq \min\{p, q\}$. In this case, we say that $A \in \mathrm{U}(p, q)$ is *regular k-loxodromic*. If the eigenvalues of $A$ are distinct and all have unit modulus, in other words $A$ is regular 0-loxodromic, then we say $A$ is *regular elliptic*. There are further divisions when $A$ has repeated eigenvalues. These cases depend on the modulus of the eigenvalues, whether $A$ is diagonalisable and the minimum polynomial of $A$. We will not distinguish between these cases in this paper and so we will not discuss them here.

**2.3. Goldman's classification in the case of $p + q = 3$.** Goldman considered the case of $\mathrm{SU}(p, q)$ where $p + q = 3$ in Section 6.2 of [8]. Our treatment is motivated by this account and we now give a brief summary of Goldman's work. Let $A \in \mathrm{SU}(p, q)$ where $p + q = 3$. Then the characteristic polynomial of $A$ is

$$\chi_A(X) = X^3 - \tau X^2 + \overline{\tau} X - 1 \tag{2.4}$$

where $\tau = \mathrm{tr}(A)$. The resultant of $\chi_A$ and $\chi'_A$ is

$$R(\chi_A, \chi'_A) = -|\tau|^2 + 8\Re(\tau^3) - 18|\tau|^2 + 27. \tag{2.5}$$

The locus where $R(\chi_A, \chi'_A) = 0$ is a classical curve called a *deltoid*, see pp. 26–27 of Kirwan [14]. We can extend the definitions of elliptic, parabolic and loxodromic as follows. We say $A$ is *regular elliptic* if the eigenvalues of $A$ are distinct and have modulus 1. We say $A$ is *loxodromic* if $A$ has a pair of eigenvalues $\lambda_1$ and $\lambda_2$ with $|\lambda_1| > 1 > |\lambda_2|$. In fact, using Lemma 2.1, this implies that $\lambda_2 = \overline{\lambda}_1^{-1}$. If $A$ has a repeated eigenvalue then $A$ is said to be *parabolic* if it is not diagonalisable and *boundary elliptic* if it is diagonalisable and not a scalar multiple of the identity. If $A$ is a scalar multiple of the identity then it acts as the identity on the corresponding projective space. Goldman's classification result is:

**Theorem 2.2** (Theorem 6.2.4 of Goldman [8]). *Let $A \in \mathrm{SU}(p,q)$ with $p+q = 3$. The characteristic polynomial $\chi_A$ and resultant $R(\chi_A, \chi'_A)$ are given in* (2.4) *and* (2.5). *Then*

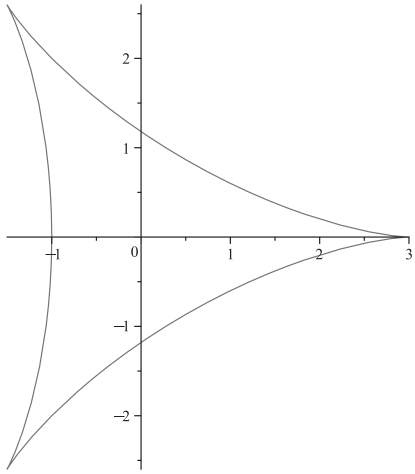


Fig. 1. The deltoid.

(i) *A is regular elliptic if and only if* $R(\chi_A, \chi_A') > 0$.
(ii) *A has a repeated eigenvalue if and only if* $R(\chi_A, \chi_A') = 0$. *In this case A is either parabolic or boundary elliptic.*
(iii) *A is loxodromic if and only if* $R(\chi_A, \chi_A') < 0$.
*Moreover, if A is loxodromic or parabolic then* $(p, q) = (2, 1)$ *or* $(1, 2)$.

Furthermore, in the case of loxodromic maps the matrix $A$ is determined up to conjugation by $\tau$ and hence by $\chi_A$. For regular elliptic maps this is almost true as well. (There is a small error in Goldman's statement at this point.) In order to discuss this further, we need to talk about the signature of eigenspaces. All three eigenspaces will be definite, therefore $p$ of them will be positive (contained in $V_+$) and $q$ will be negative (contained in $V_-$). Clearly, it is not possible to conjugate an element of $\mathrm{SU}(p, q)$ so that a positive eigenvector becomes negative or vice versa. Thus if $p = 0$ or $q = 0$ the eigenvalues determine the group up to conjugacy; if $p = 1$ (or $q = 1$) then there are three possible conjugacy classes depending on the choice of positive eigenspace (respectively negative eigenspace).

The following statement is a combination of the remaining statement of Theorem 6.2.4 of [8] and Proposition 3.6 of Parker [18] (see also Proposition 3.8 of [18]).

**Proposition 2.3.** *Suppose that* $A \in \mathrm{SU}(p, q)$ *with* $p + q = 3$ *and* $\tau = \mathrm{tr}(A)$.
(i) *If A is loxodromic then A is determined up to conjugacy by* $\tau$.
(ii) *If A is regular elliptic and* $(p, q) = (3, 0)$ *or* $(0, 3)$ *then A is determined up to conjugacy by* $\tau$.
(iii) *If A is regular elliptic and* $(p,q) = (2,1)$ *or* $(1,2)$ *each value of* $\tau$ *determines three conjugacy classes, these classes being determined by the signature of the eigenspaces.*

## 3. Classification of elements in SU($p, q$)

**3.1. Introduction.** In this section we consider matrices in SU($p,q$) for arbitrary $n = p+q$. We discuss how to use the resultant to enumerate the different possibilities for such matrices. We will also use the description of the resultant of $p$ and $q$ as a determinant of an $(r+s)\times(r+s)$ matrix; for more details see p. 52 of Kirwan [14].

**3.2. Classification when $p+q=n$.** A matrix $A$ in SU($p$, $q$) is called *$k$-loxodromic* if it has $k$ pairs of eigenvalues $r_j e^{i\theta_j}$ and $r_j^{-1}e^{i\theta_j}$ with $r_j > 1$ for $j = 1, \ldots, k$, and all other eigenvalues are unit modulus complex numbers. We adopt the convention of taking $k \geq 0$ with the understanding that a 0-loxodromic means that all eigenvalues are unit modulus complex numbers. Note that in SU($p$, $q$) we have $k \leq \min\{p, q\}$.

Also, $A$ is said to be *regular* if the eigenvalues are mutually distinct, that is $A$ has no repeated eigenvalues.

**Theorem 3.1.** *Let $A \in$ SU($p,q$). Let $R(\chi_A, \chi_A')$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi_A'(X)$. Then for $m \geq 0$, we have the following.*
(i) *$A$ is regular $2m$-loxodromic if and only if $R(\chi_A, \chi_A') > 0$.*
(ii) *$A$ is regular $(2m+1)$-loxodromic if and only if $R(\chi_A, \chi_A') < 0$.*
(iii) *$A$ has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.*

Proof. Write $p+q=n$.

Suppose $A$ is $r$-loxodromic, including the case where $r=0$ and so $A$ is elliptic. Then $A$ has mutually distinct eigenvalues

$$\lambda_j = e^{l_j + i\phi_j}, \quad \overline{\lambda}_j^{-1} = e^{-l_j + i\phi_j}, \quad \mu_k = e^{i\theta_k},$$

where $l_j$ is a positive real number, $j = 1, \ldots, r$, $k = 1, \ldots, s$ and $2r+s = p+q = n$. Then the squares of the differences of these eigenvalues are

$$\begin{aligned}
&(\lambda_j - \overline{\lambda}_j^{-1})^2 = e^{2i\phi_j} 4\sinh^2(l_j),\\
&(\lambda_j - \lambda_k)^2(\overline{\lambda}_j^{-1} - \overline{\lambda}_k^{-1})^2 = e^{2i\phi_j + 2i\phi_k}(2\cosh(l_j - l_k) - 2\cos(\phi_j - \phi_k))^2,\\
&(\lambda_j - \overline{\lambda}_k^{-1})^2(\overline{\lambda}_j^{-1} - \lambda_k)^2 = e^{2i\phi_j + 2i\phi_k}(2\cosh(l_j + l_k) - 2\cos(\phi_j - \phi_k))^2,\\
&(\lambda_j - \mu_k)^2(\overline{\lambda}_j^{-1} - \mu_k)^2 = e^{2i\phi_j + 2i\theta_k}(2\cosh(l_j) - 2\cos(\phi_j - \theta_k))^2,\\
&(\mu_j - \mu_k)^2 = -e^{i\theta_j + i\theta_k}(2 - 2\cos(\theta_j - \theta_k)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&R(\chi_A, \chi_A') \\
&= (-1)^{n(n-1)/2} \prod_{j} (\lambda_j - \bar{\lambda}_j^{-1})^2 \prod_{j<k} (\lambda_j - \lambda_k)^2 (\bar{\lambda}_j^{-1} - \bar{\lambda}_k^{-1})^2 (\lambda_j - \bar{\lambda}_k^{-1})^2 (\bar{\lambda}_j^{-1} - \lambda_k)^2 \\
&\quad \cdot \prod_{j,k} (\lambda_j - \mu_k)^2 (\bar{\lambda}_j^{-1} - \mu_k)^2 \prod_{j<k} (\mu_j - \mu_k)^2 \\
&= (-1)^{n(n-1)/2} (-1)^{s(s-1)/2} \prod_{j=1}^{r} e^{(n-1)2i\phi_j} \prod_{k=1}^{s} e^{(n-1)i\theta_k} \prod_{j} 4\sinh^2(l_j) \\
&\quad \cdot \prod_{j<k} (2\cosh(l_j - l_k) - 2\cos(\phi_j - \phi_k))^2 (2\cosh(l_j + l_k) - 2\cos(\phi_j - \phi_k))^2 \\
&\quad \cdot \prod_{j,k} (2\cosh(l_j) - 2\cos(\phi_j - \theta_k))^2 \prod_{j<k} (2 - 2\cos(\theta_j - \theta_k)) \\
&= (-1)^{n(n-1)/2 + s(s-1)/2} \prod_{j} 4\sinh^2(l_j) \\
&\quad \cdot \prod_{j<k} (2\cosh(l_j - l_k) - 2\cos(\phi_j - \phi_k))^2 (2\cosh(l_j + l_k) - 2\cos(\phi_j - \phi_k))^2 \\
&\quad \cdot \prod_{j,k} (2\cosh(l_j) - 2\cos(\phi_j - \theta_k))^2 \prod_{j<k} (2 - 2\cos(\theta_j - \theta_k)),
\end{aligned}$$

where we have used

$$\prod_{j=1}^{r} e^{(n-1)2i\phi_j} \prod_{k=1}^{s} e^{(n-1)i\theta_k} = (\det(A))^{n-1} = 1.$$

All the product terms are real and positive provided $l_j > 0$ and $\theta_j \neq \theta_k$. Thus we must find the power of $(-1)$. Since $n = 2r + s$ we have

$$n(n-1) + s(s-1) = 2n(n-1) - 4rn + 4r^2 + 2r.$$

Since $2n(n-1)$ is even, this implies $(-1)^{n(n-1)/2+s(s-1)/2} = (-1)^r$. This proves assertions (i) and (ii). Assertion (iii) follows from the definition of the resultant. □

**Corollary 3.2.** *Let $A \in \mathrm{SU}(p,1)$. Let $R(\chi_A, \chi_A')$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi_A'(X)$. Then we have the following.*
(i) *$A$ is regular elliptic if and only if $R(\chi_A, \chi_A') > 0$.*
(ii) *$A$ is regular loxodromic if and only if $R(\chi_A, \chi_A') < 0$.*
(iii) *$A$ has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.*

## 4. Classification of matrices in SU($p, q$) with $p + q = 4$

**4.1. Introduction.** In this section we consider the case of SU($p, q$) where $p + q = 4$. In fact, up to changing the sign of the Hermitian form, there are three possible groups SU(4, 0) = SU(4), SU(3, 1) and SU(2, 2). Our goal will be to extend Goldman's classification of matrices in SU(2, 1) using the resultant $R(\chi_A, \chi'_A)$ as a polynomial in $\mathrm{tr}(A)$ and $\overline{\mathrm{tr}(A)}$. In this case, the characteristic polynomial is determined by a complex and a real parameter (see [13, Section 4.5]):

**Lemma 4.1.** *Let $A$ be in* SU($p, q$), *where $p + q = 4$, with characteristic polynomial $\chi_A(X)$. Write $\tau = \mathrm{tr}(A)$ and $\sigma = (1/2)(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)) \in \mathbb{R}$. Then*

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \overline{\tau} X + 1. \tag{4.1}$$

If $\lambda_i$ for $i = 1, 2, 3, 4$ are the eigenvalues of $A$, then note that

$$\tau = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \tag{4.2}$$

$$\sigma = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4. \tag{4.3}$$

We want conditions on $\sigma$, $\tau$ characterising when $\chi_A(X) = 0$ has repeated solutions, or equivalently when $\chi_A(X)$ and its derivative $\chi'_A(X)$ have a common root. Note that:

$$\chi'_A(X) = 4X^3 - 3\tau X^2 + 2\sigma X - \overline{\tau}. \tag{4.4}$$

Therefore we need to find the locus of points $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ where the resultant $R(\chi_A, \chi'_A) = 0$. This problem was studied by Poston and Stewart [21]. Based on earlier work of Chillingworth [5], they call the locus of points where this resultant vanishes the *holy grail*; see Fig. 2. This generalises the deltoid, Fig. 1, which is the zero locus of the resultant for SU(2, 1).

In this section we investigate the dynamics of isometries whose parameters $(\tau, \sigma)$ lie on each part of the holy grail and in each component of the complement. In this section no assumption is made about the signature of $H$, but readers should recall that a $k$-loxodromic map can only occur in SU($p, q$) when $k \leq \min\{p, q\}$.

**4.2. Eigenvalues and parameters.** Consider a unitary matrix $A$ in SU($p,q$) with $p + q = 4$, but at this stage we will not specify the signature of the Hermitian form. Suppose that the eigenvalues of $A$ (that is the roots of the characteristic polynomial) are $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$. Recall from Goldman's lemma, Lemma 2.1, the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is closed under the map $\lambda \mapsto \overline{\lambda}^{-1}$. Note that an even number of eigenvalues satisfy $|\lambda| \neq 1$ and so an even number satisfy $|\lambda| = 1$. In what follows, after rearranging them if necessary, suppose that the eigenvalues are paired up as follows.

- if $|\lambda_1| \neq 1$ then $\lambda_2 = \overline{\lambda}_1^{-1}$; if $|\lambda_1| = 1$ then $|\lambda_2| = 1$;

- if $|\lambda_2| \neq 1$ then $\lambda_1 = \overline{\lambda}_2^{-1}$; if $|\lambda_2| = 1$ then $|\lambda_1| = 1$;
- if $|\lambda_3| \neq 1$ then $\lambda_4 = \overline{\lambda}_3^{-1}$; if $|\lambda_3| = 1$ then $|\lambda_4| = 1$;
- if $|\lambda_4| \neq 1$ then $\lambda_3 = \overline{\lambda}_4^{-1}$; if $|\lambda_4| = 1$ then $|\lambda_3| = 1$.

With this ordering of eigenvalues, note that $|\lambda_1\lambda_2| = |\lambda_3\lambda_4| = 1$. Define $\phi \in [0, \pi)$ by $\lambda_1\lambda_2 = e^{2i\phi}$. Moreover, since the product of the eigenvalues is 1, we also have $\lambda_3\lambda_4 = e^{-2i\phi}$. The following parameters will simplify our calculations:

$$x = (\lambda_1 + \lambda_2)e^{-i\phi}, \quad y = (\lambda_3 + \lambda_4)e^{i\phi}, \quad t = 2\cos(2\phi). \tag{4.5}$$

The rest of this section will be devoted to investigating the properties of the change of parameters $(\tau, \sigma) \leftrightarrow (x, y, \phi)$.

**Lemma 4.2.** *The parameters $x$, $y$ and $t$ defined by* (4.5) *are all real.*

*Proof.* Clearly $t$ is real. In order to see that $x$ is real, note that either $|\lambda_1| = |\lambda_2|^{-1} \neq 1$ and $\overline{\lambda}_1 = \lambda_2^{-1}$, $\overline{\lambda}_2 = \lambda_1^{-1}$ or else $|\lambda_1| = |\lambda_2| = 1$ and $\overline{\lambda}_1 = \lambda_1^{-1}$, $\overline{\lambda}_2 = \lambda_2^{-1}$. In the either case

$$\overline{x} = (\overline{\lambda}_1 + \overline{\lambda}_2)e^{i\phi} = (\lambda_1^{-1} + \lambda_2^{-1})e^{i\phi} = (\lambda_1 + \lambda_2)e^{-i\phi} = x$$

where we have used $\lambda_1\lambda_2 = e^{2i\phi}$. Thus $x$ is real. Similarly $y$ is real. □

**Lemma 4.3.** *With $\tau$, $\sigma$ and $x$, $y$, $\phi$ as in* (4.5), *we have*

$$\tau = xe^{i\phi} + ye^{-i\phi}, \tag{4.6}$$

$$\sigma = xy + 2\cos(2\phi). \tag{4.7}$$

*Proof.* From the definition of $x$, $y$ and $\phi$ we have

$$\begin{aligned} \tau &= (\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) = xe^{i\phi} + ye^{-i\phi}, \\ \sigma &= (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_1\lambda_2 + \lambda_3\lambda_4 = xe^{i\phi}ye^{-i\phi} + e^{2i\phi} + e^{-2i\phi}. \end{aligned}$$ □

We now characterise when this change of variables is a local diffeomorphism.

**Proposition 4.4.** *The change of parameters $\mathbb{R}^2 \times S^1 \to \mathbb{C} \times \mathbb{R}$ given by*

$$(x, y, e^{i\phi}) \mapsto (\tau, \sigma) = (xe^{i\phi} + ye^{-i\phi}, xy + e^{2i\phi} + e^{-2i\phi})$$

*is a local diffeomorphism provided*

$$x^2 + y^2 - 4 - 2xy\cos(2\phi) + 4\cos^2(2\phi) \neq 0.$$

Proof. Consider the change of coordinates

$$\Re(\tau) = (x+y)\cos(\phi), \quad \Im(\tau) = (x-y)\sin(\phi), \quad \sigma = xy + e^{2i\phi} + e^{-2i\phi}.$$

Then the Jacobian is

$$\begin{aligned} J &= \det\begin{pmatrix} \cos(\phi) & \cos(\phi) & -(x+y)\sin(\phi) \\ \sin(\phi) & -\sin(\phi) & (x-y)\cos(\phi) \\ y & x & -4\sin(2\phi) \end{pmatrix} \\ &= 4\sin^2(2\phi) - (x+y)^2\sin^2(\phi) - (x-y)^2\cos^2(\phi) \\ &= -x^2 - y^2 + 4 + 2xy\cos(2\phi) - 4\cos^2(2\phi). \end{aligned}$$

□

Now we show the change of variables is surjective (compare Lemma 3.8 of [18]).

**Proposition 4.5.** *Given* $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ *then there exist* $(x, y, e^{i\phi}) \in \mathbb{R}^2 \times S^1$ *so that*

$$\Re(\tau) = (x+y)\cos(\phi), \quad \Im(\tau) = (x-y)\sin(\phi), \quad \sigma = xy + e^{2i\phi} + e^{-2i\phi}. \tag{4.8}$$

Proof. If there exist such $x$, $y$, $e^{i\phi}$ then, writing $t = 2\cos(2\phi)$, we have

$$|\tau|^2 = \Re(\tau)^2 + \Im(\tau)^2 = x^2 + y^2 + xyt, \tag{4.9}$$

$$2\Re(\tau^2) = 2\Re(\tau)^2 - 2\Im(\tau)^2 = (x^2+y^2)t + 4xy, \tag{4.10}$$

$$\sigma = xy + t.$$

Eliminating $x$ and $y$ we see that $t$ must satisfy $q(t) = 0$ where

$$q(X) = X^3 - \sigma X^2 - 4X + \Re(\tau)^2 X + \Im(\tau)^2 X + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2.$$

Evaluating at $X = \pm 2$ we see that

$$\begin{aligned} q(2) &= 8 - 4\sigma - 8 + 2\Re(\tau)^2 + 2\Im(\tau)^2 + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2 = 4\Im(\tau)^2 \geq 0, \\ q(-2) &= -8 - 4\sigma + 8 - 2\Re(\tau)^2 - 2\Im(\tau)^2 + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2 = -4\Re(\tau)^2 \leq 0. \end{aligned}$$

If $\Re(\tau) \neq 0$ and $\Im(\tau) \neq 0$ then, by the intermediate value theorem, we can find $t$ with $-2 < t < 2$ so that $q(t) = 0$. Define $\phi$ by $2\cos(2\phi) = t$. As $\cos(2\phi) \neq \pm 1$ we have $\sin(2\phi) \neq 0$. In this case $x$ and $y$ are given by

$$x = \frac{\Re(\tau)\sin(\phi) + \Im(\tau)\cos(\phi)}{\sin(2\phi)}, \quad y = \frac{\Re(\tau)\sin(\phi) - \Im(\tau)\cos(\phi)}{\sin(2\phi)}.$$

If $\Im(\tau) = 0$ and $\Re(\tau) \neq 0$ then $q(2) = 0$ and

$$q_0(X) = q(X)/(X-2) = X^2 + 2X - \sigma X - 2\sigma + \Re(\tau)^2.$$

We have

$$q_0(2) = 8 - 4\sigma + \Re(\tau)^2, \quad q_0(-2) = \Re(\tau)^2 > 0.$$

If $\Re(\tau)^2 < 4\sigma - 8$ we have $q_0(2) < 0 < q_0(-2)$ and we can find $t$ with $-2 < t < 2$ and $q_0(t) = 0$. In this case define $t = 2\cos(2\phi)$ and proceed as above. If $\Re(\tau)^2 \geq 4\sigma - 8$ then define $\phi = 0$. We must solve $\Re(\tau) = x + y$ and $\sigma = xy + 2$. A solution is

$$x = \frac{\Re(\tau) + \sqrt{\Re(\tau)^2 - 4\sigma + 8}}{2}, \quad y = \frac{\Re(\tau) - \sqrt{\Re(\tau)^2 - 4\sigma + 8}}{2}.$$

If $\Re(\tau) = 0$ and $\Im(\tau) \neq 0$ then $q(-2) = 0$. As above, if $\Im(\tau)^2 < -8 - 4\sigma$ then we can find $t$ with $-2 < t < 2$ and $q(t) = 0$, giving a similar solution as before. If $\Im(\tau)^2 > -8 - 4\sigma$ then $\phi = \pi/2$ and

$$x = \frac{\Im(\tau) + \sqrt{\Im(\tau)^2 + 4\sigma + 8}}{2}, \quad y = \frac{\Im(\tau) - \sqrt{\Im(\tau)^2 + 4\sigma + 8}}{2}.$$

Finally, suppose $\Re(\tau) = \Im(\tau) = 0$. If $\sigma \geq 0$ then define $\phi = \pi/2$ and $x = y = \sqrt{\sigma + 2}$; if $\sigma < 0$ define $\phi = 0$ and $x = -y = \sqrt{-\sigma + 2}$. $\square$

**4.3. The resultant.** Let $\chi_A(x)$ be the characteristic polynomial of $A \in \mathrm{SU}(p, q)$ with $p + q = 4$. We have expressions for $\chi_A(x)$ and $\chi_A'(x)$ in (4.1) and (4.4). We now calculate their resultant $R(\chi_A, \chi_A')$ as a polynomial in $\tau$, $\overline{\tau}$ and $\sigma$:

$$\begin{aligned}
R(\chi_A, \chi_A') &= \det \begin{pmatrix} 1 & -\tau & \sigma & -\overline{\tau} & 1 & 0 & 0 \\ 0 & 1 & -\tau & \sigma & -\overline{\tau} & 1 & 0 \\ 0 & 0 & 1 & -\tau & \sigma & -\overline{\tau} & 1 \\ 4 & -3\tau & 2\sigma & -\overline{\tau} & 0 & 0 & 0 \\ 0 & 4 & -3\tau & 2\sigma & -\overline{\tau} & 0 & 0 \\ 0 & 0 & 4 & -3\tau & 2\sigma & -\overline{\tau} & 0 \\ 0 & 0 & 0 & 4 & -3\tau & 2\sigma & -\overline{\tau} \end{pmatrix} \\
&= 16\sigma^4 - 4\sigma^3(\tau^2 + \overline{\tau}^2) + \sigma^2|\tau|^4 - 80\sigma^2|\tau|^2 - 128\sigma^2 \\
&\quad + 18\sigma(\tau^2 + \overline{\tau}^2)|\tau|^2 + 144\sigma(\tau^2 + \overline{\tau}^2) \\
&\quad - 4|\tau|^6 - 27(\tau^2 + \overline{\tau}^2)^2 + 48|\tau|^4 - 192|\tau|^2 + 256 \\
&= 4(\sigma^2/3 - |\tau|^2 + 4)^3 - 27(2\sigma^3/27 - |\tau|^2\sigma/3 - 8\sigma/3 + (\tau^2 + \overline{\tau}^2))^2.
\end{aligned}$$

In [21] Poston and Stewart considered the locus of points where

$$f(z, \overline{z}) = \Re(\alpha z^4 + \beta z^3\overline{z} + \gamma z^2\overline{z}^2)$$

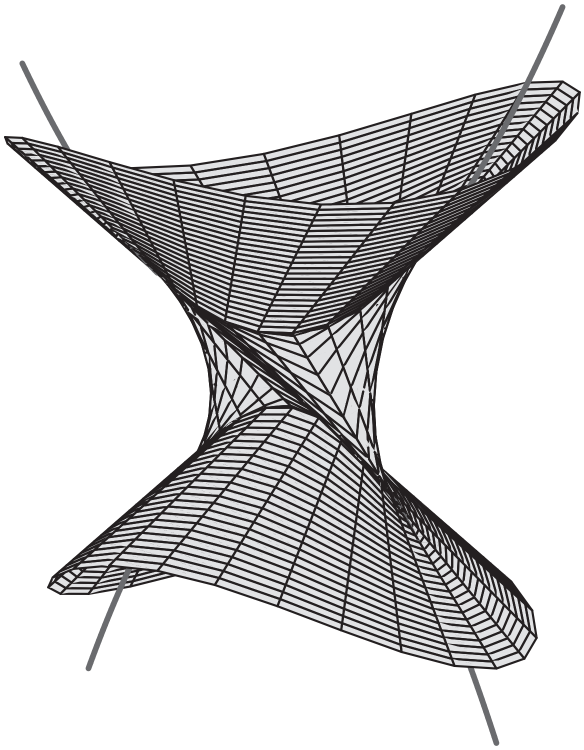

Fig. 2. The holy grail. Here points of $\mathbb{R}^3$ have coordinates $(\Re(\tau), \Im(\tau), \sigma)$.

has repeated roots. Based on earlier work of Chillingworth [5], they call the locus of these points the *holy grail*; see Fig. 2, which should be compared with Figs. 4 and 5 of [21]. In order to see the connection between the two problems, observe that by setting $\alpha = 1$, $\beta = \tau$ and $\gamma = \sigma/2$ we have

$$f(z, \overline{z}) = \overline{z}^4 \chi_A(-z/\overline{z}).$$

When $\alpha = 1$, Poston and Stewart's equation for the holy grail, p. 268 of [21], is

$$\Delta = (4\gamma^2/3 - |\beta|^2 + 4)^3 - 27(8\gamma^3/27 - |\beta|^2\gamma/3 - 8\gamma/3 + (\beta^2 + \overline{\beta}^2)/2)^2.$$

Clearly, the above substitution makes $\Delta$ agree with our expression for $R(\chi_A, \chi_A')$.

We now express $R(\chi_A, \chi_A')$ in terms of $x$, $y$ and $t$. A consequence of this and Proposition 4.4 is that the change of parameters $(\tau, \sigma) \leftrightarrow (x, y, t)$ is a local diffeomorphism when $R(\chi_A, \chi_A') \neq 0$.

**Proposition 4.6.** *In terms of the parameters $x$, $y$ and $t$ given in* (4.5) *the resultant is given by the following expression*:

$$R(\chi_A, \chi'_A) = (x^2 - 4)(y^2 - 4)(x^2 + y^2 - 4 - xyt + t^2)^2.$$

Proof. We use equations (4.9), (4.10) and (4.7) substitute for $\tau$ and $\sigma$ in terms of $x$, $y$ and $t = 2\cos(2\phi)$. Then, expanding and simplifying, we obtain

$$\begin{aligned} R(\chi_A, \chi'_A) &= 16\sigma^4 - 4\sigma^3(\tau^2 + \overline{\tau}^2) + \sigma^2|\tau|^4 - 80\sigma^2|\tau|^2 \\ &\quad - 128\sigma^2 + 18\sigma(\tau^2 + \overline{\tau}^2)|\tau|^2 + 144\sigma(\tau^2 + \overline{\tau}^2) \\ &\quad - 4|\tau|^6 - 27(\tau^2 + \overline{\tau}^2)^2 + 48|\tau|^4 - 192|\tau|^2 + 256 \\ &= (x^2 - 4)(y^2 - 4)(x^2 + y^2 - 4 - xyt + t^2)^2. \end{aligned}$$ □

We remark that there is a symmetry that arises from multiplying $A$ by powers of $i$. In several places below we will use this symmetry to avoid repetition. We note that for our geometrical applications, we will be interested in $\mathrm{PSU}(p,q) = \mathrm{SU}(p,q)/\{\pm I, \pm iI\}$ and so $A$ is only defined up to multiplication by $i$.

**Corollary 4.7.** *Let $x$, $y$ and $t$ be the parameters given in* (4.5). *The resultant $R(\chi_A, \chi'_A)$ is preserved by the changes of variable where $(x, y, t)$ is sent to one of*

$$\begin{gathered} (x, y, t), \quad (x, -y, -t), \quad (-x, y, -t), \quad (-x, -y, t), \\ (y, x, t), \quad (y, -x, -t), \quad (-y, x, -t), \quad (-y, -x, t). \end{gathered}$$

*Moreover, this automorphism group is generated by $(\lambda_1, \lambda_2) \leftrightarrow (\lambda_3, \lambda_4)$. and $A \to iA$.*

Proof. It is easy to see in that all the changes of variable stated above preserve the expression for $R(\chi_A, \chi'_A)$ from Proposition 4.6.

Now consider the effect of multiplying $A$ by $i$. In the following table we give the various changes to our parameters.

| $A$ | $\tau$ | $\sigma$ | $\phi$ | $x$ | $y$ | $t$ |
|---|---|---|---|---|---|---|
| $iA$ | $i\tau$ | $-\sigma$ | $\phi + \pi/2$ | $x$ | $-y$ | $-t$ |
| $-A$ | $-\tau$ | $\sigma$ | $\phi + \pi$ | $x$ | $y$ | $t$ |
| $-iA$ | $-i\tau$ | $-\sigma$ | $\phi + 3\pi/2$ | $x$ | $-y$ | $-t$ |

A further symmetry may be obtained by interchanging the pairs of eigenvalues $(\lambda_1, \lambda_2)$ and $(\lambda_3, \lambda_4)$. It is easy to see from (4.5) that this has the effect of sending $(x, y, t)$ to $(y, x, t)$. Repeated application of the automorphisms $A \to iA$ and $(\lambda_1, \lambda_2) \leftrightarrow (\lambda_3, \lambda_4)$ give all the changes of variable in the statement of the corollary. □

Using Proposition 4.6, the condition $R(\chi_A, \chi'_A) > 0$ implies $(x^2-4)(y^2-4) > 0$. Thus, either $x^2$ and $y^2$ are both greater than 4, or they are both less than 4. In the former case $A$ is 2-loxodromic and in the latter case it is elliptic. Thus it is useful to distinguish when $xy > 4$, $-4 < xy < 4$ and $xy < -4$. In the following lemma, we express these conditions in terms of $\sigma$ and $\tau$.

**Lemma 4.8.** *Let $\tau$ and $\sigma$ be given by* (4.6) *and* (4.7). *Suppose that $R(\chi_A, \chi'_A) > 0$. Then $xy \neq \pm 4$. Furthermore*:
(i) *$xy > 4$ if and only if either $\Re(\tau)^2 - 4\sigma + 8 < 0$ or $\sigma > 6$.*
(ii) *$xy < 4$ if and only if both $\Re(\tau)^2 - 4\sigma + 8 > 0$ and $\sigma < 6$.*
(iii) *$xy > -4$ if and only if both $\Im(\tau)^2 + 4\sigma + 8 > 0$ and $\sigma > -6$.*
(iv) *$xy < -4$ if and only if $\Im(\tau)^2 + 4\sigma + 8 < 0$ or $\sigma < -6$.*

Note that a simple consequence of this lemma is that if $R(\chi_A, \chi'_A) > 0$ then both $\min\{\Re(\tau)^2 - 4\sigma + 8, 6-\sigma\}$ and $\min\{\Im(\tau)^2 + 4\sigma + 8, 6+\sigma\}$ are both non-zero.

Proof of Lemma 4.8. If $R(\chi_A, \chi'_A) > 0$ then we have

$$0 < (x^2-4)(y^2-4) = (xy+4)^2 - 4(x+y)^2 = (xy-4)^2 - 4(x-y)^2.$$

Therefore $xy \neq \pm 4$. The remaining cases exhaust the other possibilities. Therefore, by process of elimination, it suffices to prove only one direction of the implications. We choose to do this from right to left.

If $\sigma > 6$ then

$$6 < \sigma = xy + 2\cos(2\phi) \leq xy + 2.$$

Therefore $xy > 4$. Similarly, if $\sigma < -6$ then $xy < -4$.

If $\Re(\tau)^2 - 4\sigma + 8 < 0$ then

$$0 > \Re(\tau)^2 - 4\sigma + 8 = (x-y)^2\cos^2\phi + (16-4xy)\sin^2\phi \geq (16-4xy)\sin^2\phi$$

and so $xy > 4$. Similarly, if $\Im(\tau)^2 + 4\sigma + 8 > 0$ then $xy < -4$.

Now assume that $\Re(\tau)^2 - 4\sigma + 8 > 0$, $\sigma < 6$ and $R(\chi_A, \chi'_A) > 0$. We note that in terms of $x$, $y$ and $\phi$ these inequalities imply

$$0 < (x-y)^2\cos^2\phi + (16-4xy)\sin^2\phi, \tag{4.11}$$

$$xy - 4 < 4\sin^2\phi, \tag{4.12}$$

$$4(x-y)^2 < (4-xy)^2. \tag{4.13}$$

Using (4.13) to eliminate $(x-y)^2$ from (4.11), we see that

$$0 < 4(x-y)^2\cos^2\phi + 16(4-xy)\sin^2\phi < (4-xy)((4-xy)\cos^2\phi + 16\sin^2\phi).$$

Using (4.12) we see that

$$(4 - xy)\cos^2\phi + 16\sin^2\phi > 4\sin^2\phi(4 - \cos^2\phi) > 0.$$

Therefore $xy < 4$ as claimed.

Similarly, if $\Im(\tau)^2 + 4\sigma + 8 > 0$, $\sigma > -6$ and $R(\chi_A, \chi'_A) > 0$ then $xy > -4$. □

Putting this together, we have the following theorem:

**Theorem 4.9.** *Let $A \in \mathrm{SU}(p, q)$ where $p + q = 4$ and let $\tau = \mathrm{tr}(A)$ and $\sigma = (\mathrm{tr}^2(A) - \mathrm{tr}(A^2))/2$. Let $\chi_A(X)$ be the characteristic polynomial of $A$ and let $R(\chi_A, \chi'_A)$ be the resultant of $\chi_A(X)$ and $\chi'_A(X)$. Then*

(i) *$A$ is regular 2-loxodromic if and only if $R(\chi_A, \chi'_A) > 0$ and*

$$\min\{\Re(\tau)^2 - 4\sigma + 8, \Im(\tau)^2 + 4\sigma + 8, 6 - \sigma, 6 + \sigma\} < 0.$$

(ii) *$A$ is regular 1-loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.*

(iii) *$A$ is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$ and*

$$\Re(\tau)^2 - 4\sigma + 8 > 0, \quad \Im(\tau)^2 + 4\sigma + 8 > 0, \quad -6 < \sigma < 6.$$

(iv) *$A$ has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$.*

**4.4. Parametrising the holy grail.** In this section we consider the points where $R(\chi_A, \chi'_A) = 0$, called the *holy grail*. We claim that, after reordering eigenvalues, we may suppose that either $y = 2$ or else $x^2y^2 > 16$ and $x^2 + y^2 - 4 - xyt + t^2 = 0$. The former condition determines a ruled surface made up of three parts, the *upper bowl*, *central tetrahedron* and *lower bowl*, names introduced by Poston and Stewart. The latter condition determines four space curves called the *whiskers*. This is illustrated in Fig. 2 of this paper or in Fig. 5 of Poston and Stewart [21], where the different parts are labelled.

**Proposition 4.10.** *Let $x$, $y$ and $t$ be the parameters given by* (4.5). *Up to applying one of the automorphisms given in* Corollary 4.7, *the condition $R(\chi_A, \chi'_A) = 0$ is equivalent to one of the following equations*

(i) $y = 2$;

(ii) $(x^2 - 4)(y^2 - 4) > 0$ *and* $x^2 + y^2 - 4 - xyt + t^2 = 0$.

Proof. Using Proposition 4.6 we see that points on the holy grail are given by

$$0 = (x^2 - 4)(y^2 - 4)(x^2 + y^2 - 4 - xyt + t^2)^2.$$

If $(x^4 - 4)(y^2 - 4) = 0$ then either $x = \pm 2$ or $y = \pm 2$. After applying the automorphisms from Corollary 4.7, we see that we may take $y = 2$.

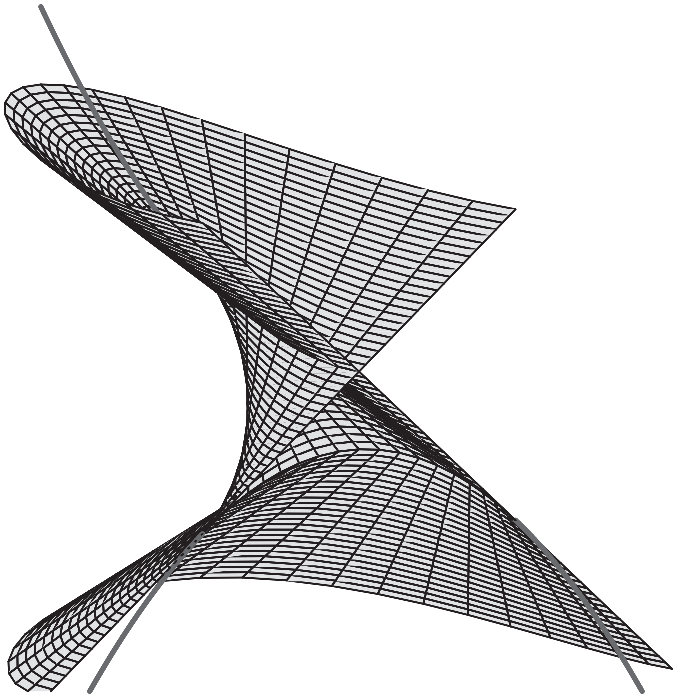

Fig. 3. A cross section through the holy grail.

If $(x^2-4)(y^2-4) \neq 0$ then $x^2+y^2-4-xyt+t^2=0$. Hence

$$t = \frac{xy \pm \sqrt{(x^2-4)(y^2-4)}}{2}.$$

Since $t$ is real, we must have $(x^2-4)(y^2-4) > 0$. $\square$

The following result is stated on page 269 of Poston and Stewart [21]. It is illustrated in the cross-section drawn in Fig. 3.

**Corollary 4.11.** *The points on the holy grail with $y=2$ form a ruled surface in $\mathbb{C} \times \mathbb{R}$.*

Proof. The points in $\mathbb{C} \times \mathbb{R}$ for which $y=2$ are

$$\begin{aligned}(\tau, \sigma) &= (xe^{i\phi} + 2e^{-i\phi}, 2x + 2\cos(2\phi)) \\ &= (2e^{-i\phi}, 2\cos(2\phi)) + x(e^{i\phi}, 2).\end{aligned}$$

This is the equation of a ruled surface (see Section 3.5 of do Carmo [6], for example). $\square$

Suppose that $y = 2$. Then the three main parts of the holy grail are determined by the conditions $x > 2$, $-2 \leq x \leq 2$ and $x < -2$.

**Corollary 4.12.** *Suppose that $y = 2$. Then the parameters $\tau$ and $\sigma$ are given by*
(i) *If $x = 2\cosh(l) > 2$ then*

$$\tau = 2\cosh(l)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = 4\cosh(l) + 2\cos(2\phi).$$

(ii) *If $x = 2\cos(\theta) \in [-2, 2]$ then*

$$\tau = 2\cos(\theta)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = 4\cos(\theta) + 2\cos(2\phi).$$

(iii) *If $x = -2\cosh(l) < -2$ then*

$$\tau = -2\cosh(l)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = -4\cosh(l) + 2\cos(2\phi).$$

The parameter values of Corollary 4.12 exhaust the possibilities when condition (i) of Proposition 4.10 is satisfied. They correspond to the *upper bowl*, *central tetrahedron* and *lower bowl* respectively. We can relate these parameter values to the possible Jordan decompositions that can arise.

**Proposition 4.13.** *Suppose that $A \in \mathrm{SU}(p, q)$ and $y = 2$.*
(i) *If $x = 2\cosh(l) > 2$ or $x = -2\cosh(l) < -2$ then $A$ is either diagonalisable or its Jordan normal form has a $2 \times 2$ Jordan block associated to the eigenvalue $e^{-i\theta}$. The latter can only happen if $p = q = 2$.*
(ii) *If $x = 2\cos(\theta) \in [-2, 2]$ then $A$ can have any Jordan normal form. There can be at most $\min\{p, q\}$ Jordan blocks of size at least 2.*

Proof. The eigenspace associated to each Jordan block of size at least 2 is spanned by a null vector. These null vectors are linearly independent. Therefore there can only be $\min\{p, q\}$ Jordan blocks of size at least 2.

In (i) the eigenvectors corresponding to the eigenvalues $e^{\pm l+i\phi}$ or $-e^{\pm l+i\phi}$ span a subspace where the restriction of $H$ has signature $(1, 1)$. If the other eigenvalues correspond to a Jordan block of size 2, then its eigenvector is linearly independent from the above subspace. Therefore $\min\{p, q\}$ is at least 2. Since $p + q = 4$ we have $p = q = 2$.

In (ii) all eigenvalues have absolute value 1, so there is no further restriction.

In both cases, it is an easy exercise to write down matrices and Hermitian forms to demonstrate that there are no further restrictions. $\square$

We now consider what happens when condition (ii) of Proposition 4.10 is satisfied. Suppose that $(x^2-4)(y^2-4) > 0$ and $-4 \leq xy \leq 4$. Then $-2 < x < 2$ and $-2 < y < 2$.

Write $x = 2\cos(\theta)$ and $y = 2\cos(\psi)$. If we also have $x^2 + y^2 - 4 - xyt + t^2 = 0$ then $t = 2\cos(2\phi) = 2\cos(\theta \pm \psi)$. In other words, $2\phi = \theta \pm \psi$ or $2\phi = -\theta \pm \psi$. There are several cases. We choose the case $2\phi = \theta + \psi$. Eliminating $\psi$, the eigenvalues are

$$\lambda_1 = e^{i\theta+i\phi}, \quad \lambda_2 = e^{-i\theta+i\phi}, \quad \lambda_3 = e^{-i\theta+i\phi}, \quad \lambda_4 = e^{i\theta-3i\phi}.$$

Reorder the eigenvalues by swapping $\lambda_2$ and $\lambda_4$.

$$\lambda_1' = e^{i\theta+i\phi}, \quad \lambda_2' = e^{i\theta-3i\phi}, \quad \lambda_3' = e^{-i\theta+i\phi}, \quad \lambda_4' = e^{-i\theta+i\phi}.$$

With this new parametrisation we get new parameters $e^{2i\phi'} = \lambda_1'\lambda_2' = e^{2i\theta-2i\phi}$ and

$$x' = (\lambda_1' + \lambda_2')e^{-i\phi'} = 2\cos(2\phi), \quad y' = (\lambda_3' + \lambda_4')e^{i\phi'} = 2, \quad t' = 2\cos(2\theta - 2\phi).$$

Therefore, this is a point on the central tetrahedron. The other cases are similar.

We therefore concentrate on the points with $xy > 4$ or $xy < -4$.

**Lemma 4.14.** *Suppose $x^2 + y^2 - 4 - xyt + t^2 = 0$ and $-2 \le t \le 2$.*
(i) *If $xy > 4$ then $x = y$ and $t = 2$.*
(ii) *If $xy < -4$ then $x = -y$ and $t = -2$.*

Proof. We have

$$0 = x^2 + y^2 - 4 - xyt + t^2 = (x - y)^2 + (2 - t)(xy - 4) + (2 - t)^2.$$

Since $-2 \le t \le 2$ we see that if $xy > 4$ we must have $(x-y)^2 = (2-t)^2 = 0$. Similarly

$$0 = x^2 + y^2 - 4 - xyt + t^2 = (x + y)^2 + (2 + t)(-xy - 4) + (2 + t)^2.$$

If $xy < -4$ then $(x + y)^2 = (2 + t)^2 = 0$. □

The locus of points described in Lemma 4.14 are the *whiskers*.

**Corollary 4.15.** *The whiskers are given by*

$$(\tau, \sigma) = (\pm 2\cosh(l), 4\cosh^2(l) + 2),$$
$$(\tau, \sigma) = (\pm 2i\cosh(l), -4\cosh^2(l) - 2)$$

*where $l > 0$ is a real parameter.*

**Proposition 4.16.** *Suppose that $A \in \mathrm{SU}(p, q)$ satisfies the hypotheses of* Lemma 4.14. *Then $p = q = 2$ and $A$ is either diagonalisable or its Jordan normal form has two blocks of size 2.*

Proof. In this case, (up to multiplying $A$ by a power of $i$) the eigenvalues are $e^l$, $e^l$, $e^{-l}$, $e^{-l}$ where $l > 0$. Since there are two eigenvectors that are greater than 1, we see that $\min\{p, q\} \geq 2$. Thus $p = q = 2$.

Since each eigenvalue has multiplicity 2, the possible Jordan blocks have size 1 or 2. Using the same argument as in Lemma 2.1, we see that the eigenspace associated to $e^l$ has the same dimension as the eigenspace associated to $e^{-l}$. Therefore $A$ is either diagonalisable or has two Jordan blocks of size 2. It is easy to write down matrices that show both possibilities can arise (see comment after Theorem 5.5). □

**4.5. When $A$ is 2-loxodromic.** In the next three sections we give a few more details about the components of the complement of the holy grail. In particular, we relate the coordinates $(x, y, t)$ with more geometrical parameters.

Suppose that $|\lambda_1| = |\lambda_2|^{-1} > 1$ and $|\lambda_3| = |\lambda_4|^{-1} > 1$. In this case, (after possibly multiplying $A$ by a power of $i$ if necessary) we can write

$$\lambda_1 = e^{l+i\phi}, \quad \lambda_2 = e^{-l+i\phi}, \quad \lambda_3 = e^{m-i\phi}, \quad \lambda_4 = e^{-m-i\phi}$$

where $l > 0$ and $m > 0$. Hence

$$\tau = 2\cosh(l)e^{i\phi} + 2\cosh(m)e^{-i\phi}, \quad \sigma = 4\cosh(l)\cosh(m) + 2\cos(2\phi). \tag{4.14}$$

and $x = 2\cosh(l)$, $y = 2\cosh(m)$, $t = 2\cos(2\phi)$. In this case

$$\begin{aligned} &R(\chi_A, \chi_A') \\ &= 256\sinh^2(l)\sinh^2(m)(\cosh(l+m) - \cos(2\phi))^2(\cosh(l-m) - \cos(2\phi))^2. \end{aligned}$$

When $l = m$ and $\phi = \pi/2$ then we see that $\tau = 0$ and $\sigma = 4\cosh^2(l) - 2 = 2\cosh(2l)$. Such points lie inside the top bowl of the holy grail. Therefore, by continuity, this region comprises points where $R(\chi_A, \chi_A') > 0$. The presence of the whiskers in this bowl mean these two components of the set where $R(\chi_A, \chi_A') > 0$ are not simply connected. This leads to subtleties when it comes to giving parameters. The whiskers comprise points with $l = m$ and $\phi = 0$ or $\phi = \pi$. We now give a characterisation in terms of $\sigma$ and $\tau$ of the points where exactly one of these conditions is satisfied.

**Lemma 4.17.** *Suppose that $\tau$ and $\sigma$ satisfy* (4.14).
(i) *If $\phi = 0$ and $l \neq m$ then $\Im(\tau) = 0$, $\Re(\tau) > 0$ and $\Re(\tau)^2 - 4\sigma + 8 > 0$.*
(ii) *If $\phi = \pi$ and $l \neq m$ then $\Im(\tau) = 0$, $\Re(\tau) < 0$ and $\Re(\tau)^2 - 4\sigma + 8 > 0$.*
(iii) *If $\phi \neq 0, \pi$ and $l = m$ then $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 < 0$.*

Proof. If $\phi = 0$ and $l \neq m$ then

$$\tau = 2\cosh(l) + 2\cosh(m), \quad \sigma = 4\cosh(l)\cosh(m) + 2.$$

Clearly $\Im(\tau) = 0$ and $\Re(\tau) > 0$. Also

$$\Re(\tau)^2 - 4\sigma + 8 = (2\cosh(l) - 2\cosh(m))^2 > 0.$$

The case where $\phi = \pi$ and $l \neq m$ is similar.

If $\phi \neq 0, \pi$ and $l = m$ then

$$\tau = 4\cosh(l)\cos(\phi), \quad \sigma = 4\cosh^2(l) + 2\cos(2\phi).$$

Clearly $\Im(\tau) = 0$. Also,

$$\Re(\tau)^2 - 4\sigma + 8 = -16\sinh^2(l)\sin^2(\phi) < 0. \qquad \square$$

Define $\mathcal{C}$ to be the set of all $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ satisfying

(i) $R(\chi_A, \chi'_A) > 0$,

(ii) $\min\{\Re(\tau)^2 - 4\sigma + 8, 6 - \sigma\} < 0$,

(iii) $\max\{\Re(\tau)^2 - 4\sigma + 8, \Im(\tau)^2\} > 0$.

Geometrically, conditions (i) and (ii) imply that $\mathcal{C}$ is contained "inside" or "above" the upper bowl of the holy grail. Condition (iii) means that the points with both $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 \leq 0$ are not in $\mathcal{C}$. Using Lemma 4.17 (iii) and the description of the whiskers, we see that this excludes those points with $l = m$.

**Proposition 4.18.** *The map*

$$\Phi: \{(l, m, e^{i\phi}) \in \mathbb{R}_+^2 \times S^1 : l > m\} \to \mathcal{C}$$

*given by* (4.14) *is a diffeomorphism.*

Proof. We have seen above that if $\tau$ and $\sigma$ are given by (4.14) then $R(\chi_A, \chi'_A) > 0$. Moreover since $xy = 4\cosh(l)\cosh(m) > 4$, using Lemma 4.8 we see that

$$\min\{\Re(\tau)^2 - 4\sigma + 8, 6 - \sigma\} < 0.$$

In addition,

$$\Re(\tau)^2 - 4\sigma + 8 = 4(\cosh(l) - \cosh(m))^2 - 16((\cosh(l) + \cosh(m))^2 - 1)\sin^2\phi,$$
$$\Im(\tau)^2 = 4(\cosh(l) - \cosh(m))^2 \sin^2\phi.$$

Since $l \neq m$ either $\Im(\tau)^2 > 0$ or $\sin^2\phi = 0$. In the latter case, $\Re(\tau)^2 - 4\sigma + 8 > 0$. Therefore

$$\max\{\Re(\tau)^2 - 4\sigma + 8, \Im(\tau)^2\} > 0.$$

Hence the image of $\Phi$ is contained $\mathcal{C}$.

Conversely, Proposition 4.5 implies that given any $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ we can find $(x, y, e^{i\phi})$ satisfying (4.8). Using Lemma 4.8 (i) we see that if

$$R(\chi_A, \chi_A') > 0, \quad \min\{\Re(\tau)^2 - 4\sigma + 8, 6 - \sigma\} < 0$$

then $(x^2-4)(y^2-4) > 0$ and $xy > 4$. Thus $x > 2$ and $y > 2$. We can write $x = 2\cosh(l)$ and $y = 2\cosh(m)$. Using Lemma 4.17 (iii) we see that if

$$\max\{\Re(\tau)^2 - 4\sigma + 8, \Im(\tau)^2\} > 0$$

then $l \neq m$. Swapping the roles of $x$ and $y$ if necessary (as in Corollary 4.7) we may assume that $l > m$. Therefore $\Phi$ is onto.

In real coordinates

$$\begin{aligned} \Re(\tau) &= 2(\cosh(l) + \cosh(m))\cos(\phi), \\ \Im(\tau) &= 2(\cosh(l) - \cosh(m))\sin(\phi), \\ \sigma &= 4\cosh(l)\cosh(m) + 2\cos(2\phi). \end{aligned}$$

This change of variables leads to the Jacobian

$$\begin{aligned} J &= 16\sinh(l)\sinh(m)\det\begin{pmatrix} \cos(\phi) & \cos(\phi) & -(\cosh(l)+\cosh(m))\sin(\phi) \\ \sin(\phi) & -\sin(\phi) & (\cosh(l)-\cosh(m))\cos(\phi) \\ \cosh(m) & \cosh(l) & -\sin(2\phi) \end{pmatrix} \\ &= -16\sinh(l)\sinh(m)(\cosh(l+m) - \cos(2\phi))(\cosh(l-m) - \cos(2\phi)). \end{aligned}$$

This is clearly non-zero when $l > m > 0$. Therefore $\Phi$ is a local diffeomorphism.

As $m$ tends to 0 then $(\tau, \sigma)$ tends to the upper bowl of the holy grail; as $l - m$ tends to 0 then $(\tau, \sigma)$ tends to points where $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 \leq 0$; as $l$ tends to $\infty$ then $(\tau, \sigma)$ tends to infinity. Therefore $\Phi$ is proper.

Therefore $\Phi$ is a covering map. For fixed $m$ and very large values of $l$ we have $(\tau, \sigma) \sim (e^l e^{i\phi}, 2e^l \cosh(m))$. Hence $\Phi$ has winding number 1 for such values of $l$ and hence everywhere. Thus $\Phi$ is a global diffeomorphism. $\square$

**4.6. When $A$ is simple loxodromic.** Suppose that $|\lambda_1| = |\lambda_2|^{-1} > 1$ and $|\lambda_3| = |\lambda_4|^{-1} = 1$. In this case, (after possibly multiplying $A$ by a power of $i$ if necessary) we can write

$$\lambda_1 = e^{l+i\phi}, \quad \lambda_2 = e^{-l+i\phi}, \quad \lambda_3 = e^{i\psi-i\phi}, \quad \lambda_4 = e^{-i\psi-i\phi}$$

where $l > 0$. Then

$$\tau = 2\cosh(l)e^{i\phi} + 2\cos(\psi)e^{-i\phi}, \quad \sigma = 4\cosh(l)\cos(\psi) + 2\cos(2\phi) \tag{4.15}$$

and $x = 2\cosh(l)$, $y = 2\cos(\psi)$, $t = 2\cos(2\phi)$. In this case

$$\begin{aligned}&R(\chi_A, \chi_A')\\ &= -256\sinh^2(l)\sin^2(\psi)(\cosh(l) - \cos(\psi + 2\phi))^2(\cosh(l) - \cos(\psi - 2\phi))^2.\end{aligned}$$

When $\psi = \pi/2$ and $\phi = \pi/4$ then $\tau = \sqrt{2}\cosh(l)(1+i)$. Such points are outside the holy grail. Therefore by continuity, $R(\chi_A, \chi_A') < 0$ in this region. The following proposition may be proved in a similar manner to Proposition 4.18 (compare Proposition 3.8 of [18]).

**Proposition 4.19.** *The map*

$$\Phi\colon \{(l, \psi, e^{i\phi}) \in \mathbb{R}_+ \times (0, \pi) \times S^1\} \to \{(\tau, \sigma) \in \mathbb{C} \times \mathbb{R} \colon R(\chi_A, \chi_A') < 0\}$$

*given by* (4.15) *is a diffeomorphism.*

We remark that, depending on the signature of the Hermitian form, Proposition 4.19 may still not mean that $A$ is determined up to conjugacy by $(\tau, \sigma)$. Suppose that the eigenvalue $\lambda_j$ corresponds to the eigenspace $U_j$. Since $|\lambda_1| = |\lambda_2|^{-1} > 1$, the eigenspaces $U_1$ and $U_2$ must both be null and the Hermitian form restricted to $U_1 \oplus U_2$ must have signature $(1, 1)$. If the signature of the form is $(3, 1)$ or $(1, 3)$ then $U_3$ and $U_4$ must both be positive or negative respectively. On the other hand, if the form has signature $(2, 2)$ then one of $U_3$ or $U_4$ is positive and the other is negative. This determines two conjugacy classes in this case. For example, if the form is the standard diagonal form $\mathrm{diag}(1, 1, -1, -1)$ then for $\varepsilon = \pm 1$ consider the following matrices in $\mathrm{SU}(2, 2)$

$$A_\varepsilon = \begin{pmatrix} \cosh(l)e^{i\phi} & 0 & 0 & \sinh(l)e^{i\phi} \\ 0 & e^{i\varepsilon\psi - i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\varepsilon\psi - i\phi} & 0 \\ \sinh(l)e^{i\phi} & 0 & 0 & \cosh(l)e^{i\phi} \end{pmatrix}.$$

Both these matrices have the same values of $\tau$ and $\sigma$ but yet they are not conjugate within $\mathrm{SU}(2, 2)$ (even though they are conjugate in $\mathrm{SL}(4, \mathbb{C})$).

**4.7. When $A$ is regular elliptic.** Suppose that $|\lambda_1| = |\lambda_2|^{-1} = 1$ and $|\lambda_3| = |\lambda_4|^{-1} = 1$. In this case, (after possibly multiplying $A$ by a power of $i$ if necessary) we can write

$$\lambda_1 = e^{i\theta + i\phi}, \quad \lambda_2 = e^{-i\theta + i\phi}, \quad \lambda_3 = e^{i\psi - i\phi}, \quad \lambda_4 = e^{-i\psi - i\phi}.$$

Then

$$\tau = 2\cos(\theta)e^{i\phi} + 2\cos(\psi)e^{-i\phi}, \quad \sigma = 4\cos(\theta)\cos(\psi) + 2\cos(2\phi).$$

and $x = 2\cos(\theta)$, $y = 2\cos(\psi)$, $t = 2\cos(2\phi)$. In this case

$$R(\chi_A, \chi_A') = 256\sin^2(\theta)\sin^2(\psi)\sin^2(\phi + (\theta+\psi)/2)\sin^2(\phi - (\theta+\psi)/2) \cdot \sin^2(\phi + (\theta-\psi)/2)\sin^2(\phi - (\theta-\psi)/2).$$

When $\theta = \psi$ and $\phi = \pi/2$ then we see that $\tau = 0$ and $\sigma = 4\cos^2(\theta) - 2 = 2\cos(2\theta)$. This lies in the central tetrahedron of the holy grail. Therefore, by continuity, this region comprises points where $R(\chi_A, \chi_A') > 0$.

## 5. Geometrical applications

**5.1. Introduction.** Our primary motivation for the classification of elements of $\mathrm{SU}(p, q)$ with $p + q = 4$ was to consider SU(3, 1), a four fold cover of PSU(3, 1), the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$. In order to demonstrate that this classification is also of interest in the case of SU(2, 2), we use our results in two special cases. First we show that we can embed the orientation preserving isometry group of $\mathbf{H}^1_{\mathbb{H}}$, which is isometric to $\mathbf{H}^4_{\mathbb{R}}$, into PSU(2,2). Secondly, we do a similar thing with automorphisms of anti de Sitter space.

**5.2. Isometries of complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$.** Let $\langle\cdot, \cdot\rangle$ be a Hermitian form of signature (3, 1) on $\mathbb{C}^4$. Recall from Section 2.1 the definitions (2.3) and (2.2) of $V_-$, the negative vectors, and $V_0$, the null vectors. Let $\mathbb{P}$ be the canonical projection map from $\mathbb{C}^4 - \{\mathbf{0}\}$ to $\mathbb{CP}^3$ then Recall that if $\mathbf{v}$ is in $V_-$ or $V_0$ then so is $\lambda\mathbf{v}$ for any non-zero complex scalar $\lambda$. Thus it makes sense to speak of $\mathbb{P}V_-$ and $\mathbb{P}V_0$ as subsets of $\mathbb{CP}^3$. *Complex hyperbolic* 3*-space* $\mathbf{H}^3_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and its boundary is defined to be $\mathbb{P}V_0$; see [8] for many more details.

Let $v$ and $w$ be points in $\mathbf{H}^3_{\mathbb{C}} = \mathbb{P}V_-$ corresponding to vectors $\mathbf{v}$ and $\mathbf{w}$ in $V_-$. Then the *Bergman distance* $\rho(v, w)$ between then is defined in terms of the Hermitian form as follows (see Section 3.1.7 of [8] for example):

$$\cosh^2\left(\frac{\rho(v, w)}{2}\right) = \frac{\langle\mathbf{v}, \mathbf{w}\rangle\langle\mathbf{w}, \mathbf{v}\rangle}{\langle\mathbf{v}, \mathbf{v}\rangle\langle\mathbf{w}, \mathbf{w}\rangle}.$$

The holomorphic isometry group of complex hyperbolic 3-space $\mathbf{H}^3_{\mathbb{C}}$ is the projective unitary group $\mathrm{PSU}(3, 1) = \mathrm{SU}(3, 1)/\{\pm I, \pm iI\}$. In this group all loxodromic maps are simple, that is they have a single pair of eigenvalues $\lambda_1$ and $\lambda_2 = \overline{\lambda}_1^{-1}$ with absolute value different from 1, as described in Section 4.6. The classification of elements of SU(3, 1) via their resultant is simply the case $p = 3$ of Corollary 3.2:

**Proposition 5.1.** *Let $A \in \mathrm{SU}(3,1)$. Let $R(\chi_A, \chi_A')$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi_A'(X)$. Then we have the following.*
(i) *A is regular elliptic if and only if $R(\chi_A, \chi_A') > 0$.*

(ii) *A is regular loxodromic if and only if* $R(\chi_A, \chi_A') < 0$.
(iii) *A has a repeated eigenvalue if and only if* $R(\chi_A, \chi_A') = 0$.

Furthermore, using Proposition 4.13 we can say slightly more about the case when $A$ has a repeated eigenvalue.

**Proposition 5.2.** *Suppose that* $A \in \mathrm{SU}(3,1)$ *has a repeated eigenvalue. If* $A$ *is diagonalisable, then it is either elliptic or loxodromic (and both possibilities arise). Otherwise it is parabolic, and the possible minimal polynomials of* $A$ *are*:
(i) $m(x) = (x - e^{-i\phi})^2(x - e^{i\theta+i\phi})(x - e^{-i\theta+i\phi})$ *where* $\theta \neq 0, \pi, \pm 2\phi \pmod{2\pi}$;
(ii) $m(x) = (x - e^{-i\phi})^2(x - e^{i\phi})$ *where* $\phi \neq 0, \pi \pmod{2\pi}$;
(iii) $m(x) = (x - e^{-i\phi})^2(x - e^{3i\phi})$ *where* $\phi \neq 0, \pi/2, \pi, 3\pi/2 \pmod{2\pi}$;
(iv) $m(x) = (x - e^{-i\phi})^3(x - e^{3i\phi})$ *where* $\phi \neq 0, \pi/2, \pi, 3\pi/2 \pmod{2\pi}$;
(v) $m(x) = (x - e^{-ik\pi/2})^2$ *for* $k = 0, 1, 2, 3$;
(vi) $m(x) = (x - e^{-ik\pi/2})^3$ *for* $k = 0, 1, 2, 3$.

For a detailed classification of elements of SU(3, 1) with repeated eigenvalues see [12]. With respect to the Hermitian form

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

we can find representatives of cases (i) to (vi) with one of the following two forms:

$$A_1 = \begin{pmatrix} e^{-i\phi} & 0 & 0 & ie^{-i\phi} \\ 0 & e^{i\theta+i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\theta+i\phi} & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix}, \quad A_2 = \begin{pmatrix} e^{-i\phi} & 0 & -2e^{-i\phi} & -2e^{-i\phi} \\ 0 & e^{3i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\phi} & 2e^{-i\phi} \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix}.$$

In (i) we have $A_1$; in (ii) we have $A_1$ with $\theta = 0$; in (iii) we have $A_1$ with $\theta = 2\phi$; in (iv) we have $A_2$; in (v) we have $A_1$ with $\theta = 0$ and $\phi = k\pi/2$; in (vi) we have $A_2$ with $\phi = k\pi/2$.

Our goal in remainder of this section is to relate our parameters for loxodromic maps in SU(3, 1) with the geometry of their action on $\mathbf{H}^3_{\mathbb{C}}$. This generalises the work in Parker [18] where the geometry of loxodromic maps in SU(2, 1) was considered.

We now recall the notation of Section 4.6. Suppose that $A \in \mathrm{SU}(3,1)$ has eigenvalues

$$\lambda_1 = e^{l+i\phi}, \quad \lambda_2 = e^{-l+i\phi}, \quad \lambda_3 = e^{i\psi-i\phi}, \quad \lambda_4 = e^{-i\psi-i\phi}. \tag{5.2}$$

The eigenspaces $V_1$ and $V_2$ in $\mathbb{C}^{3,1}$ corresponding to $\lambda_1$ and $\lambda_2$ are both null. After projectivisation, they correspond to fixed points $q_1$ and $q_2$ of $A$ on $\partial\mathbf{H}^3_{\mathbb{C}}$. Also, $V_1 \oplus V_2$

is indefinite. Its projectivisation is a complex line, whose intersection $L$ with $\mathbf{H}^3_{\mathbb{C}}$ is a copy of the Poincaré disc model of the hyperbolic plane, called the *complex axis* of $A$. The (Poincaré) geodesic in $L$ with endpoints $q_1$ and $q_2$ is called the *axis* of $A$ and is denoted $\alpha(A)$. The eigenspaces $V_3$ and $V_4$ in $\mathbb{C}^{3,1}$ corresponding to $\lambda_3$ and $\lambda_4$ are each positive. They are orthogonal to $V_1 \oplus V_2$, whose projectivisation intersects $\mathbf{H}^3_{\mathbb{C}}$ in $L$.

**Proposition 5.3.** *Let $A$ in* SU(3,1) *be a loxodromic map with axis $\alpha$ and complex axis $L$. Let $l$, $\phi$ and $\psi$ be the parameters associated to $A$ given by* (5.2)*. Then $A$ translates a Bergman distance $2l$ along $\alpha$ and rotates the complex lines orthogonal to $L$ by angles $-2\phi+\psi$ and $-2\phi-\psi$.*

Proof. We use the diagonal Hermitian form $\langle\ ,\ \rangle$ given by the matrix $H$ from (5.1) and we follow the ideas of Parker [18, Proposition 3.10]. In this case we may represent points $z$ in $\mathbf{H}^3_{\mathbb{C}}$ by $(z_1, z_2, z_3) \in \mathbb{C}^3$ with $2\Re(z_1) + |z_2|^2 + |z_3|^2 < 0$. If the eigenvalues of $A$ are given by (5.2) then, up to conjugacy, we may suppose

$$A = \operatorname{diag}(e^{l+i\phi}, e^{i\psi-i\phi}, e^{-i\psi-i\phi}, e^{-l+i\phi}).$$

Thus $A$ fixes $o = (0, 0, 0, 1)$ and $\infty = (1, 0, 0, 0)$. The action of $A$ on $\mathbf{H}^3_{\mathbb{C}}$ is given by

$$A\colon (z_1, z_2, z_3) \mapsto (e^{2l} z_1, e^{l+i\psi-2i\phi} z_1, e^{l-i\psi-2i\phi} z_2).$$

The axis of $A$ is the geodesic $\alpha$ joining the fixed points and the complex axis of $A$ is the unique complex line containing $\alpha$. They are given by

$$\alpha = \{(-x, 0, 0) \in \mathbf{H}^3_{\mathbb{C}} : x > 0\}, \quad L = \{(-x+iy, 0, 0) \in \mathbf{H}^3_{\mathbb{C}} : x > 0\}.$$

Suppose that $p = (-x, 0, 0)$ is a point of the axis $\alpha$ of $A$. Let $\mathbf{p}$ denote the lift of $p$ to $\mathbb{C}^4$ given by $\mathbf{p} = (-x, 0, 0, 1)^t$. Then the translation length of $A$ along $\alpha$ is $\rho(A(p), p)$. We have

$$\cosh(\rho(A(p), p)/2) = \left|\frac{\langle A\mathbf{p}, \mathbf{p}\rangle}{\langle \mathbf{p}, \mathbf{p}\rangle}\right| = \left|\frac{-xe^{l+i\phi} - xe^{-l+i\phi}}{-2x}\right| = \cosh(l).$$

This implies $\rho(A(p), p) = 2l$ as claimed.

The tangent vectors to $\mathbf{H}^3_{\mathbb{C}}$ spanning the complex lines orthogonal to $L$ are given by $\xi = (0, 1, 0)^t$ and $\eta = (0, 0, 1)^t$. Clearly the (projective) action of $A$ sends $\xi$ in $T_p(\mathbf{H}^2_{\mathbb{C}})$ to $e^{l+i\psi-2i\phi}\xi$ in $T_{A(p)}(\mathbf{H}^2_{\mathbb{C}})$ and $\eta$ to $e^{l-i\psi-2i\phi}\eta$. The rest of the result follows. □

**5.3. Isometries of $\mathbf{H}^1_{\mathbb{H}} = \mathbf{H}^4_{\mathbb{R}}$.** Quaternionic hyperbolic 1-space $\mathbf{H}^1_{\mathbb{H}}$ may be identified with hyperbolic 4-space $\mathbf{H}^4_{\mathbb{R}}$. The isometries of quaternionic hyperbolic 1-space are contained in the projective symplectic group $\mathrm{PSp}(1, 1) = \mathrm{Sp}(1, 1)/(\pm I)$. The group

Sp(1, 1) is the group of $2 \times 2$ quaternionic matrices preserving a quaternionic Hermitian form of signature (1, 1); see Parker [17] for example. There is a canonical way to identify a quaternion with a $2 \times 2$ complex matrix and therefore to identify a $2 \times 2$ quaternionic matrix with a $4 \times 4$ complex matrix; see Gongopadhyay [11] for example. When we do this, the quaternionic Hermitian form of signature (1, 1) becomes a complex Hermitian form of signature (2, 2). The upshot of this construction is that it is possible to embed (the double cover of) the group of orientation preserving isometries of hyperbolic 4-space into SU(2, 2). In this section we show how the classification given in the previous sections relate to the well known classification of four dimensional hyperbolic isometries. Our construction follows Gongopadhyay [11], where arbitrary invertible $2 \times 2$ quaternionic matrices were considered. See also Parker and Short [19] for an alternative method of classifying quaternionic Möbius transformations.

Let $A_{\mathbb{H}}$ be a $2 \times 2$ matrix of quaternions acting on a column vector $\mathbf{z}_{\mathbb{H}}$ of quaternions as

$$A_{\mathbb{H}}\mathbf{z}_{\mathbb{H}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az + bw \\ cz + dw \end{pmatrix}.$$

If $A$ is in Sp(1, 1) then $|a| = |d|$, $|b| = |c|$, $|a|^2 - |c|^2 = 1$, $\overline{a}b = \overline{c}d$ and $a\overline{c} = b\overline{d}$; see Lemma 1.1 of [4] or Proposition 6.3.1 of [17] for example. If $a$ is a quaternion we can write it as $a = a_1 + ja_2$ where $a_1, a_2 \in \mathbb{C}$. Then $a$ corresponds to the following matrix:

$$\begin{pmatrix} a_1 & -\overline{a}_2 \\ a_2 & \overline{a}_1 \end{pmatrix}.$$

It is not hard to show that this identification is a group homomorphism from $\mathbb{H}$ with quaternionic multiplication to M(2, $\mathbb{C}$) with matrix multiplication. Using this identification, the matrix $A_{\mathbb{H}}$ corresponds to a $4 \times 4$ complex matrix $A$ given by:

$$A = \begin{pmatrix} a_1 & -\overline{a}_2 & b_1 & -\overline{b}_2 \\ a_2 & \overline{a}_1 & b_2 & \overline{b}_1 \\ c_1 & -\overline{c}_2 & d_1 & -\overline{d}_2 \\ c_2 & \overline{c}_1 & d_2 & \overline{d}_1 \end{pmatrix}.$$

Likewise $\mathbf{z}_{\mathbb{H}}$ corresponds to a $4 \times 2$ matrix and we only consider its first column, which is a vector $\mathbf{z}$ in $\mathbb{C}^4$. The action of $A_{\mathbb{H}}$ on $\mathbf{z}_{\mathbb{H}}$ induces the standard action of $A$ on $\mathbf{z} \in \mathbb{C}^4$ by matrix multiplication. Using this identification, we see that if $A_{\mathbb{H}}$ is in Sp(1, 1) then $A \in$ SU(2, 2).

Suppose that $\lambda_{\mathbb{H}} \in \mathbb{H}$ is a right eigenvalue for $A_{\mathbb{H}}$. This means that there is a quaternionic vector $\mathbf{v}$ so that $A_{\mathbb{H}}\mathbf{v} = \mathbf{v}\lambda_{\mathbb{H}}$. It is always possible to find a unit quaternion $\mu$ so that $\lambda = \mu^{-1}\lambda_{\mathbb{H}}\mu$ is in $\mathbb{C}$; see Parker and Short [19] or Gongopadhyay [11] for example. (That is, writing $\lambda = \lambda_1 + j\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ gives $\lambda_2 = 0$.) In this case

$$A_{\mathbb{H}}(\mathbf{v}\mu) = \mathbf{v}\lambda_{\mathbb{H}}\mu = (\mathbf{v}\mu)\lambda.$$

Hence $\lambda \in \mathbb{C}$ is also a right eigenvalue of $A_{\mathbb{H}}$. (In the language of quaternions, right eigenvalues of quaternionic matrices are defined up to similarity.) It is easy to show that $\lambda$ is also an eigenvalue of $A$. Since we can also find $\nu \in \mathbb{H}$ so that $\overline{\lambda} = \nu^{-1}\lambda_{\mathbb{H}}\nu$, a similar argument shows that $\overline{\lambda}$ is also an eigenvalue of $A$. Hence, if $|\lambda| \neq 1$, using Lemma 2.1 the eigenvalues of $A$ are

$$\lambda, \quad \overline{\lambda}, \quad \lambda^{-1}, \quad \overline{\lambda}^{-1}.$$

If $|\lambda| = 1$ then this is true of all eigenvalues and they are

$$e^{i\theta}, \quad e^{-i\theta}, \quad e^{i\psi}, \quad e^{-i\psi}.$$

This implies that $\tau$ is real (which could have been seen by inspection) and so the characteristic polynomial $\chi_A(X)$ of $A$ has real coefficients. Hence the coefficients of $X$ and $X^3$ in $\chi_A(X)$ are the same. This rules out case (i) of [11] Theorem 1.1; see also Corollary 6.2 of Parker and Short [19]. Putting $\tau \in \mathbb{R}$ in the expression for $R(\chi_A, \chi_A')$ in terms of $\sigma$ and $\tau$ in Section 4.3 gives.

$$\begin{aligned} R(\chi_A, \chi_A') &= (\sigma^2 + 4\sigma + 4 - 4\tau^2)(\tau^2 - 4\sigma + 8)^2 \\ &= (\sigma + 2 - 2\tau)(\sigma + 2 + 2\tau)(\tau^2 - 4\sigma + 8)^2. \end{aligned}$$

We can now state our classification theorem, which should be compared to Theorem 1.1 of Gongopadhyay [11].

**Proposition 5.4.** *Let* $A \in \mathrm{SU}(2,2)$ *correspond to a map in* $\mathrm{Sp}(1,1)$. *Then* $A$ *has characteristic polynomial*

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \tau X + 1$$

*where* $\mathrm{tr}(A) = \tau \in \mathbb{R}$ *and* $\sigma \in \mathbb{R}$. *Moreover*

(i) $A$ *is regular* 2*-loxodromic if and only if* $\tau^2 - 4\sigma + 8 < 0$.

(ii) $A$ *is regular elliptic if and only if* $\tau^2 - 4\sigma + 8 > 0$ *and* $(\sigma + 2)^2 \neq 4\tau^2$.

(iii) $A$ *has a repeated eigenvalue if and only if* $\tau^2 - 4\sigma + 8 = 0$ *or* $(\sigma + 2)^2 = 4\tau^2$.

We note that the connection between our notation and that of Gongopadhyay is that $c_1 = c_3 = \tau^2/4$ and $c_2 = \sigma$. The main difference between our result and Theorem 1.1 of Gongopadhyay [11] is that his result does not involve $(\sigma+2)^2 - 4\tau^2$. We now explain this. Using our expression for the eigenvalues of $A$, we see that when $|\lambda| \neq 1$ then

$$(\sigma + 2 - 2\tau)(\sigma + 2 + 2\tau) = |\lambda + \lambda^{-1} - 2|^2 |\lambda + \lambda^{-1} + 2|^2 > 0.$$

Otherwise $\tau = 2\cos(\theta) + 2\cos(\psi)$ and $\sigma = 4\cos(\theta)\cos(\psi) + 2$ and

$$\begin{aligned} (\sigma + 2 - 2\tau)(\sigma + 2 + 2\tau) &= 16(1 - \cos(\theta))(1 - \cos(\psi))(1 + \cos(\theta))(1 + \cos(\psi)) \\ &\geq 0. \end{aligned}$$

Hence $(\sigma + 2 - 2\tau)(\sigma + 2 + 2\tau) = 0$ if and only if $e^{i\theta} = \pm 1$ or $e^{i\psi} = \pm 1$. If both of these are true then $\tau^2 - 4\sigma + 8 = 0$. Otherwise, the eigenvalues of $A$ are

$$e^{i\theta}, \quad e^{-i\theta}, \quad \pm 1, \quad \pm 1,$$

where $e^{i\theta} \neq \pm 1$. In this case $\tau^2 - 4\sigma + 8 = 4(1 \mp \cos\theta)^2 > 0$. Furthermore, the repeated eigenvalue $\lambda = \pm 1$ corresponds to the same quaternionic eigenvector $\lambda_{\mathbb{H}} = \pm 1$. Thus there is a two dimensional complex eigenspace associated to $\lambda$, and so $A$ is elliptic.

**5.4. Automorphisms of anti de Sitter space.** There is a canonical identification between $\mathbb{R}^4$ and $\mathrm{M}(2, \mathbb{R})$, the collection of $2 \times 2$ real matrices. Under this identification, the determinant map $\det\colon \mathrm{M}(2, \mathbb{R}) \to \mathbb{R}$ corresponds to a quadratic form of signature $(2, 2)$ on $\mathbb{R}^4$. *Anti de Sitter space* is the projectivisation of the positive vectors with respect to this quadratic form. It may be canonically identified with $\mathrm{PSL}(2, \mathbb{R})$ by considering the section where this quadratic form takes the value $+1$; see Section 7 of Mess [15] or Section 2 of Goldman [9]. The automorphism group of anti de Sitter space with its Lorentz structure is $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. Using the identification of anti de Sitter space with $\mathbb{R}^4$ gives an isomorphism between $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSO}_0(2, 2) = \mathrm{SO}_0(2, 2)/(\pm I)$, where $\mathrm{SO}_0(2, 2)$ is the identity component of $\mathrm{SO}(2, 2)$; again see Mess [15] or Goldman [9].

Let us make this explicit. Identify $\mathbb{R}^4$ and $\mathrm{M}(2, \mathbb{R})$ by the map:

$$F\colon \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The determinant map $\det(X)$ corresponds to the quadratic form $Q(\mathbf{x}) = x_1 x_4 - x_2 x_3$. This is associated to the symmetric matrix $H$ of signature $(2, 2)$ where

$$H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $A_1, A_2 \in \mathrm{SL}(2, \mathbb{R})$. Then the pair $(A_1, A_2)$ acts on $\mathrm{SL}(2, \mathbb{R})$ and this action corresponds to $A \in \mathrm{SO}(2, 2)$ as follows:

$$F(A\mathbf{x}) = A_1 F(\mathbf{x}) A_2^{-1}.$$

(Note we invert the matrix on the right so that the map from $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ to $\mathrm{SO}(2, 2)$ is a homomorphism.) If

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then it is easy to see that

$$A = \begin{pmatrix} a_1d_2 & -a_1c_2 & b_1d_2 & -b_1c_2 \\ -a_1b_2 & a_1a_2 & -b_1b_2 & b_1a_2 \\ c_1d_2 & -c_1c_2 & d_1d_2 & -d_1c_2 \\ -c_1b_2 & c_1a_2 & -d_1b_2 & d_1a_2 \end{pmatrix}.$$

Clearly $\tau = \operatorname{tr}(A) = (a_1 + d_1)(a_2 + d_2) = \operatorname{tr}(A_1)\operatorname{tr}(A_2)$. It is not hard to see that

$$\begin{aligned} \sigma &= \frac{1}{2}(\operatorname{tr}^2(A) - \operatorname{tr}(A^2)) \\ &= \frac{1}{2}(\operatorname{tr}^2(A_1)\operatorname{tr}^2(A_2) - \operatorname{tr}(A_1^2)\operatorname{tr}(A_2^2)) \\ &= \frac{1}{2}(\operatorname{tr}^2(A_1)\operatorname{tr}^2(A_2) - (\operatorname{tr}^2(A_1) - 2)(\operatorname{tr}^2(A_2) - 2)) \\ &= \operatorname{tr}^2(A_1) + \operatorname{tr}^2(A_2) - 2. \end{aligned}$$

**Theorem 5.5.** *Let* $(A_1, A_2) \in \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ *be an automorphism of anti de Sitter space. Then*
(i) $(A_1, A_2)$ *is regular* 2*-loxodromic if either* $A_1$ *or* $A_2$ *is loxodromic and also* $4 \neq \operatorname{tr}^2(A_1) \neq \operatorname{tr}^2(A_2) \neq 4$.
(ii) $(A_1, A_2)$ *is regular elliptic if* $A_1$ *and* $A_2$ *are both elliptic and* $\operatorname{tr}^2(A_1) \neq \operatorname{tr}^2(A_2)$.
(iii) $(A_1, A_2)$ *is not regular if* $\operatorname{tr}^2(A_1) = 4$ *or* $\operatorname{tr}^2(A_2) = 4$ *or* $\operatorname{tr}^2(A_1) = \operatorname{tr}^2(A_2)$.

Proof. Consider the parameters $x$, $y$ and $t$ defined in (4.5). Since $\operatorname{tr}(A)$ is real, we have $t = 2$, that is $\phi = 0$ or $\phi = \pi$. Moreover

$$\begin{aligned} (x + y)^2 &= |\tau|^2 = \operatorname{tr}^2(A_1)\operatorname{tr}^2(A_2), \\ xy + 2 &= \sigma = \operatorname{tr}^2(A_1) + \operatorname{tr}^2(A_2) - 2. \end{aligned}$$

A consequence of this is that

$$\begin{aligned} (x^2 - 4)(y^2 - 4) &= (xy)^2 - 4(x^2 + y^2) + 16 = (\operatorname{tr}^2(A_1) - \operatorname{tr}^2(A_2))^2, \\ x^2 + y^2 - 4 - xyt + t^2 &= (x + y)^2 - 4xy = (\operatorname{tr}^2(A_1) - 4)(\operatorname{tr}^2(A_2) - 4). \end{aligned}$$

Therefore, using the identity from Proposition 4.6, we have

$$\begin{aligned} R(\chi_A, \chi_A') &= (x^2 - 4)(y^2 - 4)(x^2 + y^2 - 4 - xyt + t^2)^2 \\ &= (\operatorname{tr}^2(A_1) - \operatorname{tr}^2(A_2))^2(\operatorname{tr}^2(A_1) - 4)^2(\operatorname{tr}^2(A_2) - 4)^2. \end{aligned}$$

Then $A$ has a repeated eigenvalue if and only if one of the following conditions hold:

$$\operatorname{tr}(A_2) = \pm\operatorname{tr}(A_1), \quad \operatorname{tr}(A_1) = \pm 2, \quad \operatorname{tr}(A_2) = \pm 2.$$

Otherwise $A$ is 2-loxodromic or elliptic. Furthermore, we have

$$\begin{aligned}\Re(\tau)^2 - 4\sigma + 8 &= (\mathrm{tr}^2(A_1) - 4)(\mathrm{tr}^2(A_2) - 4),\\ \Im(\tau)^2 + 4\sigma + 8 &= 4\,\mathrm{tr}^2(A_1) + 4\,\mathrm{tr}^2(A_2),\\ 6 - \sigma &= 8 - \mathrm{tr}^2(A_1) - \mathrm{tr}^2(A_2).\end{aligned}$$

Then using Theorem 4.9 we see $(A_1, A_2)$ is elliptic if and only if $A_1$ and $A_2$ are both elliptic with $\mathrm{tr}^2(A_1) \neq \mathrm{tr}^2(A_2)$. $\square$

Note that taking $A_1$ to be loxodromic and $A_2$ to be parabolic gives an example of a matrix in $\mathrm{SU}(2,2)$ lying on one of the whiskers and whose Jordan normal form has two blocks of size 2; see Proposition 4.16.

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## References


[1] P. Anglès: Conformal Groups in Geometry and Spin Structures, Progress in Mathematical Physics **50**, Birkhäuser Boston, Boston, MA, 2008.

[2] A.F. Beardon: The Geometry of Discrete Groups, Springer, New York, 1983.

[3] A. Cano, J.P. Navarrete and J. Seade: Complex Kleinian Groups, Birkhäuser/Springer Basel AG, Basel, 2013.

[4] W. Cao, J.R. Parker and X. Wang: *On the classification of quaternionic Möbius transformations*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 349–361.

[5] D. Chillingworth: *The ubiquitous astroid*; in The Physics of Structure Formation (Tübingen, 1986), Springer, Berlin, 1987, 372–386.

[6] M.P. do Carmo: Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.

[7] W.M. Goldman: *Trace coordinates on Fricke spaces of some simple hyperbolic surfaces*; in Handbook of Teichmüller Theory, II, IRMA Lect. Math. Theor. Phys., **13**, Eur. Math. Soc., Zürich, 2009, 611–684.

[8] W.M. Goldman: Complex Hyperbolic Geometry, Oxford Univ. Press, Oxford, 1999.

[9] W.M. Goldman: *Crooked surfaces and anti-de Sitter geometry*, Geom. Dedicata **175** (2015), 159–187.

[10] W.M. Goldman and J.R. Parker: *Complex hyperbolic ideal triangle groups*, J. Reine Angew. Math. **425** (1992), 71–86.

[11] K. Gongopadhyay: *Algebraic characterization of the isometries of the hyperbolic* 5*-space*, Geom. Dedicata **144** (2010), 157–170.

[12] K. Gongopadhyay: *Algebraic characterization of isometries of the complex and the quaternionic hyperbolic* 3*-spaces*, Proc. Amer. Math. Soc. **141** (2013), 1017–1027.

[13] K. Gongopadhyay and J.R. Parker: *Reversible complex hyperbolic isometries*, Linear Algebra Appl. **438** (2013), 2728–2739.

[14] F. Kirwan: Complex Algebraic Curves, London Mathematical Society Student Texts **23**, Cambridge Univ. Press, Cambridge, 1992.

[15] G. Mess: *Lorentz spacetimes of constant curvature*, Geom. Dedicata **126** (2007), 3–45.

[16] J.-P. Navarrete: *The trace function and complex Kleinian groups in* $\mathbb{P}^2_{\mathbb{C}}$, Internat. J. Math. **19** (2008), 865–890.

[17] J.R. Parker: Hyperbolic Spaces, Jyväskylä Lectures in Mathematics **2**, 2008.

[18] J.R. Parker: *Traces in complex hyperbolic geometry*; in Geometry, Topology and Dynamics of Character Varieties, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **23**, World Sci. Publ., Hackensack, NJ, 2012, 191–245.

[19] J.R. Parker and I. Short: *Conjugacy classification of quaternionic Möbius transformations*, Comput. Methods Funct. Theory **9** (2009), 13–25.

[20] J.R. Parker and P. Will: *Complex hyperbolic free groups with many parabolic elements*; in Geometry, Groups and Dynamics, Contemp. Math. **639**, Amer. Math. Soc., Providence RI, 2015, 327–348.

[21] T. Poston and I. Stewart: *The cross-ratio foliation of binary quartic forms*, Geom. Dedicata **27** (1988), 263–280.

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