

Rito, C. and Sánchez, M.M.
Osaka J. Math.
52 (2015), 929–945

HYPERELLIPTIC SURFACES WITH $K^2 < 4\chi - 6$

CARLOS RITO and MARÍA MARTÍ SÁNCHEZ

(Received April 3, 2013, revised June 12, 2014)

Abstract

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves of minimal genus g . We prove that if $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, then g is bounded. The surface S is determined by the branch locus of the covering $S \rightarrow S/i$, where i is the hyperelliptic involution of S . For $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we show how to determine the possibilities for this branch curve. As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$. This list of possibilities is sharp.

1. Introduction

For a smooth minimal hyperelliptic surface S of general type, Xiao [8, Theorem 1] has proved that if

$$K_S^2 < \frac{4g}{g+1}(\chi(\mathcal{O}_S) - \epsilon g - 2),$$

where either $\epsilon = 1$ if $\chi(\mathcal{O}_S) > (2g - 1)(g + 1) + 2$, or $\epsilon = 9/8$, then S has a pencil of hyperelliptic curves of genus $\leq g$. This result is not very useful for $g > 4$ and $\chi(\mathcal{O}_S)$ small. For example, in [1] Ashikaga and Konno consider surfaces S of general type with $K_S^2 = 3\chi(\mathcal{O}_S) - 10$. For these surfaces the canonical map is of degree 1 or 2. In the degree 2 case, the canonical image is a ruled surface, thus if S is regular, it has a pencil of hyperelliptic curves. By the above inequality, if $\chi(\mathcal{O}_S) \geq 47$, then S has such a hyperelliptic pencil of curves of genus ≤ 4 . But for $\chi(\mathcal{O}_S) \leq 46$ this result gives no information (for $\chi(\mathcal{O}_S) = 46$ the slope formula [7, Theorem 2] implies $g \leq 5 \vee g \geq 9$; we show that in this case S has a hyperelliptic pencil of minimal genus $g \leq 10$ and the cases $g = 9$, $g = 10$ do occur). Ashikaga and Konno study only the case $g \leq 4$ (there is an infinite number of possibilities). Nothing is said for the possibilities with $g \geq 5$ and $\chi(\mathcal{O}_S) \leq 46$. A similar situation occurs in [5].

In this paper we study smooth minimal surfaces S of general type which have a pencil of hyperelliptic curves (by *pencil* we mean a linear system of dimension 1). We say that S has such a pencil of *minimal genus* g if it has a hyperelliptic pencil of genus g and all hyperelliptic pencils of S are of genus $\geq g$. For S such that $K_S^2 < 4\chi(\mathcal{O}_S) - 6$,

we give bounds for the minimal genus g (Theorem 1), improving Xiao's inequality in the cases $g > 4$ and $\chi(\mathcal{O}_S)$ small.

The surface S is the smooth minimal model of a double cover of an Hirzebruch surface \mathbb{F}_e ramified over a curve \bar{B} (which determines S). We prove that if $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then \bar{B} has at most points of multiplicity 8 and we show how to determine the possibilities for \bar{B} (Proposition 2).

As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$; this list of possibilities is sharp (Theorem 3).

The paper is organized as follows. In Section 2 we present the main results of the paper. The hyperelliptic involutions of the fibres of S induce an involution i of S , so in Section 3 we review some general facts on involutions. Since the quotient S/i is a rational surface, a smooth minimal model of S/i is not unique. We make a choice for this minimal model in Section 4 (which is due to Xiao [9]) and we show some consequences of it. Section 5 contains the key result of the paper, which allow us to compute bounds for the minimal genus of the hyperelliptic fibration. We perform a careful analysis of the possibilities for the branch locus of the covering $S \rightarrow S/i$ considering the restrictions imposed by the choice of minimal model. Finally this is used in Section 6 to prove the main results, stated in Section 2.

Several calculations are made using a computational algebra system. The respective code lines are available at http://home.utad.pt/~crito/magma_code.html.

Notation. We work over the complex numbers; all varieties are assumed to be projective algebraic. A (-2) -curve or *nodal curve* A on a surface is a curve isomorphic to \mathbb{P}^1 such that $A^2 = -2$. An (m_1, m_2, \dots) -point of a curve, or point of type (m_1, m_2, \dots) , is a singular point of multiplicity m_1 , which resolves to a point of multiplicity m_2 after one blow-up, etc. By *double cover* we mean a finite morphism of degree 2. The rest of the notation is standard in algebraic geometry.

2. Main results

Theorem 1. *Let S be a smooth minimal surface of general type with a pencil of hyperelliptic curves of minimal genus g . If $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, then g is not greater than*

$$\max \left\{ -1 + \frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, 1 + \frac{8\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, \right. \\ \left. 1 + \frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 3}, \frac{3 + \sqrt{1 + 8\chi(\mathcal{O}_S)}}{2} \right\}.$$

Let $B \subset W$ be the branch locus of a double cover $V \rightarrow W$, where V and W are smooth surfaces (thus B is also smooth). Let $\rho: W \rightarrow P$ be the projection of W onto a minimal model and denote by \bar{B} the projection $\rho(B)$.

REMARK 4. This result gives three examples where Theorem 1 is almost sharp: in the cases $(g, K^2 - 3\chi) = (10, -10), (9, -13), (8, -15)$ we have $\chi \leq 46, 37, 29$, thus Theorem 1 implies $g \leq 11, 10, 9$, respectively (cf. Remark 10).

There is at least one case where Theorem 1 is sharp: a double plane with branch locus a curve of degree 18 with 8 points of multiplicity 6. In this case $\chi = 5, K^2 = 8$ and $g = 5$.

3. Involutions

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves. This hyperelliptic structure induces an involution (i.e. an automorphism of order 2) i of S . The quotient S/i is a rational surface.

Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points P_1, \dots, P_t . Let $p: S \rightarrow S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i), i = 1, \dots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points $Q_i, i = 1, \dots, t$. Let now $h: V \rightarrow S$ be the blow-up of S at P_1, \dots, P_t and set $R' = h^*(R'')$. The involution i induces a biregular involution \tilde{i} on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow p \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi: V \rightarrow W$ is the projection onto the quotient and $g: W \rightarrow S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \dots, t,$$

are (-2) -curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since π is a double cover, its branch locus $B' + \sum_1^t A_i$ is even, i.e. there is a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

4. Choice of minimal model

Part of this section may be found in [9]. We use the notation introduced so far. As above, W is a rational surface, thus either it is isomorphic to \mathbb{P}^2 or its minimal model is an Hirzebruch surface \mathbb{F}_e .

(*). Blowing-up, if necessary, \mathbb{P}^2 at a point, we can suppose that $W \neq \mathbb{P}^2$.

Notice that in this case the map $h: V \rightarrow S$ is the contraction of two (-1) -curves. With this assumption we do not need to consider the case $W = \mathbb{P}^2$ separately.

Thus there is a birational morphism

$$\rho: W \rightarrow \mathbb{F}_e.$$

Let $\bar{B} := \rho(B)$ and consider the double cover $S' \rightarrow \mathbb{F}_e$ with branch locus \bar{B} . If \bar{B} is singular then S' is also singular and S is isomorphic to the minimal smooth resolution of S' .

We can define k and l such that

$$\bar{B} \equiv: kC_0 + \left(\frac{ek}{2} + l\right)F,$$

where C_0 and F are, respectively, the negative section and a ruling of \mathbb{F}_e (thus $C_0^2 = -e$, $C_0F = 1$, $F^2 = 0$). Notice that $\bar{B}^2 = 2kl$ and $K_p \bar{B} = -2k - 2l$.

(*). Among all the possibilities for the map ρ , we choose one satisfying, in this order:

- 1) the degree k of \bar{B} over a section is minimal;
- 2) the greatest order of the singularities of \bar{B} is minimal;
- 3) the number of singularities with greatest order is also minimal.

Recall that a $(2r - 1, 2r - 1)$ singularity of \bar{B} is a pair (x_j, x_k) such that x_k is infinitely near to x_j and $r_j = 2r - 2$, $r_k = 2r$.

Let

$$r_m := \max\{r_i\}$$

or $r_m := 0$ if \bar{B} is smooth.

By *elementary transformation* over $x_i \in \mathbb{F}_e$ we mean the blow-up of x_i followed by the blow-down of the strict transform of the ruling of \mathbb{F}_e that contains x_i .

The following is a consequence of the two assumptions (*) on the map ρ .

Proposition 5 ([9]). *We have:*

- a) *If $k \equiv 0 \pmod{4}$, then $r_m \leq k/2 + 2$ and the equality holds only if x_m belongs to a singularity $(k/2 + 1, k/2 + 1)$. In this last case $l \geq k + 2$ and all the branches of the singularity are tangent to the ruling of \mathbb{F}_e that contains it.*
- b) *If $k \equiv 2 \pmod{4}$, then $r_m \leq k/2 + 1$ and the equality holds only if x_m belongs to a singularity $(k/2, k/2)$. In this case $l \geq k$.*

In a similar vein:

Proposition 6. *We have that:*

- a) *if $l = k + 2$ and $k > 8$, there are at most two $(k/2 + 1, k/2 + 1)$ -points;*
- b) *$l \geq k/2$ and $l \geq k/2 + r_m - 2$;*
- c) *if $l = k/2 + r_m - 2$, then either:*
 - *$e = 2, l = k - 2$, the branch locus \bar{B} has a $(k/2 - 1, k/2 - 1)$ -point and all singularities are of multiplicity $< k/2$, or*
 - *we can suppose $e = 1$, the negative section C_0 of \mathbb{F}_1 is contained in \bar{B} , \bar{B} has a point of multiplicity r_m contained in C_0 and the remaining singularities are of multiplicity $< r_m$.*

Proof. a) This is due to Borrelli ([3]). Suppose that there are three singularities $(k/2 + 1, k/2 + 1)$. The rulings of \mathbb{F}_e through these points are contained in \bar{B} and then $\bar{B}C_0 = l - ek/2 \geq 4$ ($\bar{B}C_0$ is even). This implies $e \leq 1$. Making, if necessary, an elementary transformation over one of these points, we can suppose that $e = 1$.

Let ρ be as above and $E_i, E'_i, i = 1, 2, 3$, be the exceptional divisors corresponding to three singularities $(k/2 + 1, k/2 + 1)$ of \bar{B} . The general element of the linear system $|\rho^*(4C_0 + 5F) - \sum_1^3(2E_i + 2E'_i)|$ is a smooth and irreducible rational curve C such that $CB < k$. This contradicts the choice (*) of the map ρ .

b) If $r_m > k/2$ then the result follows from Proposition 5. Suppose now $r_m \leq k/2$. We have $\bar{B}C_0 \geq -e$, i.e. $l - ek/2 \geq -e$. Therefore if $e \geq 2$, then

$$l \geq k - 2 \geq \frac{k}{2} \quad \text{and} \quad l \geq k - 2 \geq \frac{k}{2} + r_m - 2.$$

When $e = 0$ we obtain immediately $l \geq k$, by the choice of the map ρ , thus $l \geq k/2 + r_m$.

If $e = 1$ then $\bar{B}C_0 = l - k/2 \geq 0$. Blowing-down C_0 we obtain a singularity of order at most $l - k/2 + 1$, hence the choice of the minimal model implies $r_m \leq l - k/2 + 2$ (notice that the equality happens only if the order of the singularity is $(r_m - 1, r_m - 1)$).

c) Assume that $l = k/2 + r_m - 2$. Proposition 5 implies $r_m \leq k/2$. From $\bar{B}C_0 \geq -e$ we obtain $k/2 + r_m - 2 = l \geq ek/2 - e$, thus either $e = 1$ or $e = 2$ and $r_m = k/2$ (notice that $e = 0$ implies $l \geq k$).

In the case $e = 1$ we can, as in the proof of b), contract the section with self-intersection (-1) to obtain a branch curve in \mathbb{P}^2 with at most singularities of type $(l - k/2 + 1, l - k/2 + 1)$.

Suppose now that $e = 2$ and there is a point x_i of multiplicity $k/2$. In this case $\bar{B}C_0 = -2$, hence $x_i \notin C_0$. We make an elementary transformation over x_i to obtain the case $e = 1$ also with $l = k - 2$. □

5. Bound of genus

In this section we prove the key result to establish bounds for the minimal genus of the hyperelliptic fibrations.

From [6] (cf. also [4]), we get the following:

Proposition 7. *Let $S'' \rightarrow S'$ be the canonical resolution of a double cover $S' \rightarrow \mathbb{F}_e$ with branch locus $\bar{B} \equiv kC_0 + (ek/2 + l)F$. Let S be the minimal model of S'' and $t := K_S^2 - K_{S''}^2$. If S is of general type, then:*

- a) $\sum(r_i - 2)(k - r_i - 2) = H$;
- b) $2l = G + \sum(r_i - 2)$,

where

$$H = 2k^2 - k(4\chi(\mathcal{O}_S) + t - K_S^2 + 8) + 16\chi(\mathcal{O}_S) + 2t - 2K_S^2$$

and

$$G = -2k + 4\chi(\mathcal{O}_S) + t - K_S^2 + 8.$$

Proof. From [6, Propositions 2 and 3, a)] one gets:

- a) $2kl = -48 + 12l + 12k - 8\chi(\mathcal{O}_S) + 4K_S^2 - 4t + \sum(r_i - 2)(r_i - 4)$;
- b) $2k + 2l = 8 + 4\chi(\mathcal{O}_S) + t - K_S^2 + \sum(r_i - 2)$.

The result is obtained replacing (a) by (a) + (6 - k)(b). □

The motivation for Lemma 8 and Proposition 9 below is the following. Among all the solutions of the equations of Proposition 7, the ones with biggest l correspond to the solutions with singularities of maximal order. This gives an upper bound for l . But we also have a lower bound for l , implied by the assumptions (*) on the map ρ (Propositions 5 and 6). We note that the arguments used in the proofs are mostly formal.

Lemma 8. *Suppose that $k > 8$. With the above notation, we have*

- a) $2l \leq G + H/(k - r_m - 2)$, and
- b) if r_m is obtained only from singularities of type $(r_m - 1, r_m - 1)$, then

$$2l \leq G + \frac{H}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}(2r_m - 6).$$

Proof. a) Proposition 5 implies $r_m \leq k/2 + 2$. If $k - r_m - 2 \leq 0$, we get from $k - 2 \leq r_m \leq k/2 + 2$ that $k \leq 8$. Hence $k - r_m - 2 > 0$ and the statement follows from Proposition 7.

b) By the assumptions, if x_i does not belong to a $(r_m - 1, r_m - 1)$ singularity, we have $r_i < r_m$. Let $n \geq 1$ be the number of singularities of type $(r_m - 1, r_m - 1)$ and $s \geq 0$ be the number of singular points x_j of another type. As seen in Section 4, each

singularity $(r_m - 1, r_m - 1)$ corresponds to two infinitely near singular points x_k, x_{k+1} with $r_k = r_m - 2, r_{k+1} = r_m$. Therefore

$$\sum_{i=1}^{2n+s} (r_i - 2) = n(2r_m - 6) + \sum_{j=1}^s (r_j - 2),$$

with $r_j < r_m$. Thus from Proposition 7, b) we get

$$2l = G + n(2r_m - 6) + \sum_{j=1}^s (r_j - 2).$$

By Proposition 7, a),

$$H = n((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) + \sum_{j=1}^s (r_j - 2)(k - r_j - 2),$$

hence

$$n = \frac{H - \sum_{j=1}^s (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}$$

and then

$$(1) \quad 2l = G + \frac{H - \sum_{j=1}^s (r_j - 2)(k - r_j - 2)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}(2r_m - 6) + \sum_{j=1}^s (r_j - 2).$$

Since $r_j < r_m, j = 1, \dots, s,$

$$(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2) \leq (2r_m - 6)(k - r_j - 2).$$

This implies

$$\sum_{j=1}^s (r_j - 2) \leq \sum_{j=1}^s \frac{(r_j - 2)(k - r_j - 2)(2r_m - 6)}{(r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)}$$

and the result follows from (1). □

The next result will allow us to give bounds for k . Notice that, since \bar{B} is even and $\bar{B}C_0 = l - ek/2,$

$$k \equiv 0 \pmod{4} \implies l \equiv 0 \pmod{2}.$$

Proposition 9. *In the conditions of Proposition 7, suppose that $k > 8$. If $k \equiv 0 \pmod{4},$ one of the following holds:*

a) $r_m = k/2 + 2$, $l = k + 2$ and

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k \leq 16\chi(\mathcal{O}_S) - 16, \quad \text{with } t \geq 2;$$

b) $r_m = k/2 + 2$, $l \geq k + 4$ and

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 - 16\chi(\mathcal{O}_S)k + 32\chi(\mathcal{O}_S) \leq 0, \quad \text{with } t \geq 2;$$

c) $r_m = k/2$, $l = k - 2$ and

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 4)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 32)k \\ + 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 96 \leq 0, \quad \text{with } t \geq 1,$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 2)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8, \quad \text{with } t \geq 1,$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 5)k^2 + (-48\chi(\mathcal{O}_S) - 8t + 8K_S^2 + 44)k \\ + 160\chi(\mathcal{O}_S) + 16t - 16K_S^2 - 128 \leq 0, \quad \text{with } t \geq 2;$$

d) $r_m = k/2$, $l = k + j$, $j \geq 0$, and

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n,$$

with $n \leq j + 7$, where n is the number of points x_i (possibly infinitely near) such that $r_i = k/2$;

e) $r_m \leq k/2 - 2$ and

$$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)},$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2.$$

If $k \equiv 2 \pmod{4}$, one of the following holds:

f) $r_m = k/2 + 1$ and

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 2)k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 20, \quad \text{with } t \geq 1,$$

or

$$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 + (-32\chi(\mathcal{O}_S) - 4t + 4K_S^2 + 48)k \\ + 80\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 96 \leq 0, \quad \text{with } t \geq 2;$$

g) $r_m \leq k/2 - 1$ and

$$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)},$$

or

$$2(4\chi(\mathcal{O}_S) + t - K_S^2 - 6)k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 28.$$

REMARK 10. As noted in Remark 4, there are examples where cases e) and g) fail to be sharp by 1. The reason for not having a sharp result is the following: in these examples we have $r_m = 0$, thus we are using $l \geq k/2 - 2$ in the proof of e) and g). But in fact we have $l \geq k/2$ in these cases, from Proposition 6, b).

The last example referred in Remark 4 shows that case d) with $k = 12$, $j = 0$, $n = 7$ is sharp.

Proof of Proposition 9. Let H, G be as defined in Proposition 7 and let

$$P_1(l, r_m, G, H, k) := (2l - G)(k - r_m - 2) - H,$$

$$P_2(l, r_m, G, H, k) := (2l - G)((r_m - 4)(k - r_m) + (r_m - 2)(k - r_m - 2)) - H(2r_m - 6).$$

From Lemma 8,

$$P_1 \leq 0 \quad \text{and} \quad P_2 \leq 0.$$

a) Let n be the number of $(k/2 + 1, k/2 + 1)$ points. From Propositions 5, a) and 6, a), $n = 1$ or 2 . From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n(k/2(k/2 - 4) + (k/2 - 2)^2), \quad G' = G + n(k - 2)$$

and $r_i \leq k/2, \forall i$.

The result follows from

$$P_1(k + 2, k/2, G', H', k) \leq 0.$$

Notice that $t \geq 2n$.

b) From Proposition 5, there are at most $(k/2 + 1, k/2 + 1)$ singularities. The inequality

$$P_2(k + 4, k/2 + 2, G, H, k) \leq 0$$

gives the result.

c) Let n be the number of points of multiplicity $k/2$ and m be the number of $(k/2 - 1, k/2 - 1)$ singularities. From Proposition 6, c), $n = 0$ or 1 .

If $n = 0$, then $r_m = k/2$ implies $m \geq 1$ (thus $t \geq 1$). From

$$P_2(k - 2, k/2, G, H, k) \leq 0$$

one gets the first inequality.

Suppose $n = 1$. Notice that, as shown in the proof of Proposition 6, c), the point of multiplicity $k/2$ is obtained from the blow-up of \mathbb{P}^2 at a point of type $(k/2 - 1, k/2 - 1)$. Hence $t \geq 1$.

Let

$$H' := H - (k/2 - 2)^2, \quad G' = G + k/2 - 2$$

(we remove the contribution of the point of multiplicity $k/2$).

If $m = 0$, then

$$P_1(k - 2, k/2 - 2, G', H', k) \leq 0$$

implies the second inequality.

If $m > 0$, then

$$P_2(k - 2, k/2, G', H', k) \leq 0$$

gives the third inequality. In this case $t \geq 2$.

d) Let $j := l - k$ and let n be the number of points x_i (possibly infinitely near) such that $r_i = k/2$. From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n(k/2 - 2)^2, \quad G' = G + n(k/2 - 2)$$

and $r_i \leq k/2 - 2, \forall i$.

The inequality

$$P_1(k + j, k/2 - 2, G', H', k) \leq 0$$

gives

$$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n.$$

It only remains to show that $n \leq j + 7$.

One can verify, using the double cover formulas (see e.g. [2, V. 22]), that $n \geq j + 8$ implies $\chi(\mathcal{O}_S) < 1$, except for $n = 8, l = k$ and $n = 10, k = 12, l = 14$. We claim that in these cases $K_S^2 \leq 0$. This is impossible because S is of general type.

Proof of the claim. From the double cover formulas one gets that $\chi(\mathcal{O}_S) \leq 2$ and there is at least a (-2) -curve A contained in the branch curve B , otherwise $K_S^2 \leq 0$. One has

$$B \equiv -\frac{k}{2}K_W + (l - k)\tilde{F} + \sum \left(\frac{k}{2} - r_i\right)E_i,$$

where \tilde{F} is the total transform of F and each E_i is an exceptional divisor with self-intersection -1 . Since $AB = -2, AK_W = 0, l \geq k$ and $r_i \leq k/2 \forall i$, we have $AE_i < 0$

for some i such that $r_i < k/2$. The only possibility is the existence of a $(3, 3)$ -point in \bar{B} and $\chi(\mathcal{O}_S) = 1$. But the imposition of such a singularity in the branch locus decreases the self-intersection of the canonical divisor by 1, thus $K_S^2 \leq 0$.

e) From Proposition 6, b), $l \geq k/2 + r_m - 2$. Let

$$f(r_m) := P_1(k/2 + r_m - 2, r_m, G, H, k).$$

We have

$$f(r_m) = -2r_m^2 + br_m + c \leq 0,$$

where

$$b = 4\chi(\mathcal{O}_S) + t - K_S^2 - k + 8$$

and

$$c = k^2 - 10k - 8\chi(\mathcal{O}_S) + 24.$$

Suppose that $c = f(0) > 0$ (i.e. $k > 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$). Then $f(r_m)$ has exactly one positive root x . One has

$$4x - b = \sqrt{b^2 + 8c}$$

and $k/2 - 2 \geq r_m \geq x$ implies that

$$(4(k/2 - 2) - b)^2 \geq b^2 + 8c.$$

This inequality gives the result.

f) Let n be the number of points of type $(k/2, k/2)$.

If $n = 1$, we proceed as in a), with $l \geq k$.

If $n > 1$, the inequality is given by

$$P_2(k, k/2 + 1, G, H, k) \leq 0.$$

g) It is analogous to the proof of e): in this case the result follows from $k/2 - 1 \geq r_m \geq x$. □

6. Proof of main results

Proof of Theorem 1. Consider the parabola given by $f(x) = ax^2 + bx + c$, with $a > 0$. If $f(k) \leq 0$, $f(z) \geq 0$ and $z \geq -b/2a$ (the first coordinate of the vertex), then $k \leq z$.

This fact and Proposition 9 imply that, if $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, one of the following holds:

a) $k \leq (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 6)$;

- b) $k \leq (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8), t \geq 2;$
- c) $k \leq 4 + (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) + t - K_S^2 - 4), t \geq 1;$
- c') $k \leq 4 + (16\chi(\mathcal{O}_S) - 4)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 5), t \geq 2;$
- d) $k \leq 4 + (16\chi(\mathcal{O}_S) - 32)/(4\chi(\mathcal{O}_S) - K_S^2 - 6);$
- e) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)};$
- e') $k \leq 4 + (16\chi(\mathcal{O}_S))/(4\chi(\mathcal{O}_S) - K_S^2);$
- f) $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 1);$
- f') $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) + t - K_S^2 - 8), t \geq 2;$
- g) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)};$
- g') $k \leq 2 + (16\chi(\mathcal{O}_S) - 16)/(4\chi(\mathcal{O}_S) - K_S^2 - 6).$

We want to show that k is not greater than

$$\max \left\{ \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}, \right. \\ \left. 4 + \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 3}, 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)} \right\}.$$

The result follows easily. Just notice that

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \leq 8 \implies 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}$$

and

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \geq 8 \implies 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}. \quad \square$$

Proof of Proposition 2. Let $(\alpha), (\beta)$ be the equations of Proposition 7, a), b), respectively. One has that $[(\alpha) + (k - 10)(\beta)]/8$ is equivalent to

$$(2) \quad \frac{1}{8} \sum (r_i - 2)(8 - r_i) = 15 + K_S^2 - t - 3\chi(\mathcal{O}_S) - \frac{1}{4}(k - 10)(l - 10)$$

and $(\beta) + (2)$ is equivalent to

$$(3) \quad \chi(\mathcal{O}_S) = 1 + \frac{1}{4}(k - 2)(l - 2) - \frac{1}{8} \sum r_i(r_i - 2).$$

Now it suffices to show that $r_m \leq 8$.

Suppose that $K_S^2 < 3\chi(\mathcal{O}_S) - 6$.

From [8, Theorem 1] one gets that if $\chi(\mathcal{O}_S) \geq 54$, then S has a pencil of hyperelliptic curves of genus ≤ 6 . In this case $k \leq 14$, thus $r_m \leq k/2 + 2$ implies $r_m \leq 8$.

From the proof of Theorem 1 we obtain that if $\chi(\mathcal{O}_S) \leq 31$, then one of the possibilities below occur. In all cases $r_m \leq 8$.

- a) and b) $k < 16, r_m < 8$;
- c), c') and d) $k \leq 18, r_m = k/2 \leq 8$;
- e) $k \leq 20, r_m \leq k/2 - 2 \leq 8$;
- e') $k \leq 16, r_m \leq k/2 - 2 \leq 6$;
- f) $k \leq 14, r_m = k/2 + 1 \leq 8$;
- f') $k \leq 16, r_m = k/2 + 1 \leq 8$;
- g) $k \leq 18, r_m \leq k/2 - 1 \leq 8$;
- g') $k \leq 14, r_m \leq k/2 - 1 \leq 6$.

Suppose now that $32 \leq \chi(\mathcal{O}_S) \leq 53$. From Theorem 1 we get that $k \leq 18$ or $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$. In this last case $k \leq 24$ and $r_m \leq k/2 - 1$ (see Proposition 9 e), g)). Thus we have $r_m \leq 18/2 + 2$ or $r_m \leq 24/2 - 1$. Since r_m is even, $r_m \leq 10$.

Let N_j be the number of points x_i such that $r_i = j$. We have

$$\sum (r_i - 2) \geq 8N_{10} + 6N_8$$

and, from (2),

$$8N_{10} \geq (k - 10)(l - 10) - 32.$$

Using Proposition 7, b) and the assumption $\chi(\mathcal{O}_S) \geq 32$, this implies

$$2l + 2k \geq 15 + (k - 10)(l - 10) + 6N_8,$$

or equivalently

$$(4) \quad (k - 12)(l - 12) \leq 29 - 6N_8.$$

Suppose $r_m = 10$. Then Propositions 5 and 6 give two possibilities:

- $k = 16, l \geq k + 2 = 18$, there is a singularity of type (9, 9) ($N_8 \geq 1$);
- $k \geq 18, l \geq k/2 + r_m - 2 \geq 17$.

Both cases contradict (4). We conclude that $r_m \leq 8$. □

Proof of Theorem 3. First we claim that if A is a (-2) -curve contained in the branch curve B , the image \bar{A} of A in \mathbb{F}_e does not intersect a negligible singularity of \bar{B} , unless \bar{A} is the negative section of \mathbb{F}_1 and the only singularity of \bar{B} is a double point in C_0 (this corresponds to a smooth branch curve in \mathbb{P}^2). In fact otherwise there is a (-1) -curve E such that $AE = 1$ or 2 . If $AE = 1$, then $A + E$ can be contracted to a smooth point of the branch curve $\bar{B} \subset \mathbb{F}_e$. This is a contradiction because the canonical resolution blows-up only singular points of \bar{B} . Suppose $AE = 2$. The inverse image of A is a (-1) -curve which contracts to a smooth point of S . The inverse image of E is then contracted to a curve \hat{E} with arithmetic genus 1 and $\hat{E}^2 = 2$. We obtain from the adjunction formula that $K_S \hat{E} = -2$, which is impossible because S is of general type.

Recall that $t := K_S^2 - K_{S^v}^2$. The following holds:

- (1) $l \geq k/2$ (Because $l - ek/2 = \bar{B}C_0 \geq -e$ and $\bar{B}C_0$ is even.);
- (2) $l = k/2 \iff (t = 2 \wedge N_4 = N_6 = N_8 = 0)$ (In this case $e = 1$ and $\bar{B}C_0 = 0$.);
- (3) $l = k/2 + 2 \iff (N_6 = N_8 = 0 \wedge t \geq N_4 \wedge (t = N_4 \vee N_4 > 1))$; (If $N_4 \neq 0$, this corresponds to a branch curve in \mathbb{P}^2 with N_4 points of type (3, 3) (see Proposition 6, b), c).);
- (4) $l = k - 2 \wedge t = 0 \iff k/2$ even; (As in (1), $l \geq ek/2 - e$, thus $e \leq 2$. If $e = 2$, $\bar{B}C_0 = -2$ implies $t \geq 1$. Hence $e = 1$ and then l even implies $k/2$ even.);
- (5) $l < k - 2 \iff l - k/2$ even; (As in (1), $l \geq ek/2 - e$, thus $e = 1$ and then $l - k/2 = \bar{B}C_0$ is even.)
- (6) $t = 1 \wedge N_4 = N_6 = N_8 = 0 \iff l = k - 2$. (If there are only negligible singularities, $t = 1$ is only possible if the negative section of \mathbb{F}_2 is an isolated component of the branch locus.)

For given values of $K_S^2 - 3\chi(\mathcal{O}_S)$ and k , we want to choose the solution of the equation given in Proposition 2, b) which maximizes the value of $\chi(\mathcal{O}_S)$, given by the equation in Proposition 2, c). We can assume $N_6 = N_8 = 0$.

It suffices to compute the numerical possibilities for Proposition 2, b), c) which satisfy conditions (1), ..., (6). We note the following: since $k \geq 12$, [8, Theorem 1] implies $\chi(\mathcal{O}_S) \leq 69$, then Theorem 1 gives $k \leq 28$; $l \geq k/2$, $k \geq 12$ and (2) imply $-7 \geq K_S^2 - 3\chi(\mathcal{O}_S) \geq -18 + t + N_4$, thus $K_S^2 - 3\chi(\mathcal{O}_S) \geq -18$, $t \leq 11$ and $N_4 \leq 11$.

A simple algorithm is available at http://home.utad.pt/~crito/magma_code.html.

It remains to prove the existence. All cases can be constructed as double covers of \mathbb{P}^2 , \mathbb{F}_0 , \mathbb{F}_1 or \mathbb{F}_2 . The table below contains information about l or the degree of the branch curve in \mathbb{P}^2 and about the singularities of the branch curve, if any.

$K^2 - 3\chi$	g	-7	-8	-9	-10
5		$\mathbb{F}_0, l = 26$	$\mathbb{F}_0, l = 24$	$\mathbb{F}_0, l = 22$	$\mathbb{F}_1, l = 20$
6		$\mathbb{F}_0, l = 18$	$\mathbb{F}_1, l = 17$	$\mathbb{F}_0, l = 16$	$\mathbb{F}_1, l = 15$
7		$\mathbb{F}_1, l = 14, (3, 3)$	$\mathbb{F}_2, l = 14$	$\mathbb{F}_1, l = 14$	$\mathbb{F}_1, l = 12, (3, 3)$
8		$\mathbb{F}_1, l = 13, (3, 3)$	$\mathbb{F}_1, l = 13, (4)$	$\mathbb{F}_1, l = 13$	
9			$\mathbb{P}^2, 22, (3, 3)$		$\mathbb{F}_1, l = 12$
10					$\mathbb{P}^2, 22$

g	-11	-12	-13	-14	-15	-16
5	$\mathbb{F}_0, l = 18$	$\mathbb{F}_0, l = 16$	$\mathbb{F}_0, l = 14$	$\mathbb{F}_0, l = 12$	$\mathbb{F}_1, l = 10$	$\mathbb{F}_1, l = 8$
6	$\mathbb{F}_0, l = 14$	$\mathbb{F}_1, l = 13$	$\mathbb{F}_1, l = 11, (4)$	$\mathbb{F}_1, l = 11$		$\mathbb{F}_1, l = 9$
7	$\mathbb{F}_1, l = 12, (4)$	$\mathbb{F}_1, l = 12$	$\mathbb{P}^2, 18, (3, 3)$		$\mathbb{F}_1, l = 10$	$\mathbb{P}^2, 16$
8	$\mathbb{P}^2, 20, (3, 3)$		$\mathbb{F}_1, l = 11$		$\mathbb{P}^2, 18$	
9			$\mathbb{P}^2, 20$			
10						

Suppose first that S is smooth and is the double cover of an Hirzebruch surface \mathbb{F}_e with branch locus $B \equiv 2L \equiv kC_0 + (ek/2 + l)F$. We get from the double cover formulas (see e.g. [2]) that

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_{\mathbb{F}_e}) + \frac{1}{2}L(K_{\mathbb{F}_e} + L) = 2 + \frac{1}{4}kl - \frac{1}{2}(k + l)$$

and

$$K_S^2 = 2(K_{\mathbb{F}_e} + L)^2 = 16 - 4(k + l) + kl.$$

Now we compute χ and K^2 for the cases given in the table above taking in account that a 4-uple point in the branch locus decreases K^2 by 2 and χ by 1 and a (3,3)-point decreases both K^2 and χ by 1. Notice that $k = 2g + 2$.

Finally if S is a double cover of \mathbb{P}^2 with branch locus a smooth curve of degree d , then

$$\chi(\mathcal{O}_S) = 2 + \frac{1}{8}d(d - 6) \quad \text{and} \quad K_S^2 = \frac{1}{2}(d - 6)^2.$$

The result follows by computing χ and K^2 for $d = 16, 18, 20$ and 22 . □

ACKNOWLEDGEMENTS. The authors wish to thank Margarida Mendes Lopes, for all the support, Ana Bravo and Orlando Villamayor, for their hospitality at the Universidad Autónoma de Madrid.

This research was partially supported by the Fundação para a Ciência e a Tecnologia (Portugal) through Projects PEst-OE/MAT/UI4080/2011 (Center for Mathematics of the University of Trás-os-Montes e Alto Douro, UTAD), PTDC/MAT/099275/2008 and PTDC/MAT-GEO/0675/2012. Presently the first author is a member of the Center for Mathematics of the University of Porto (project PEst-C/MAT/UI0144/2013). Both authors were collaborators of the Center for Mathematical Analysis, Geometry and Dynamical Systems of Instituto Superior Técnico, Universidade Técnica de Lisboa.

References

- [1] T. Ashikaga and K. Konno: *Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$* , Tohoku Math. J. (2) **42** (1990), 517–536.
- [2] W.P. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven: *Compact Complex Surfaces*, second edition, Springer, Berlin, 2004.
- [3] G. Borrelli: *The classification of surfaces of general type with nonbirational bicanonical map*, J. Algebraic Geom. **16** (2007), 625–669.
- [4] C. Ciliberto and M. Mendes Lopes: *On surfaces with $p_g = q = 2$ and non-birational bi-canonical maps*, Adv. Geom. **2** (2002), 281–300.
- [5] K. Konno: *Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$* , Math. Ann. **290** (1991), 77–107.

- [6] C. Rito: *Involutions on surfaces with $p_g = q = 1$* , Collect. Math. **61** (2010), 81–106.
- [7] G. Xiao: *Fibered algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.
- [8] G. Xiao: *Hyperelliptic surfaces of general type with $K^2 < 4\chi$* , Manuscripta Math. **57** (1987), 125–148.
- [9] G. Xiao: *Degree of the bicanonical map of a surface of general type*, Amer. J. Math. **112** (1990), 713–736.

Carlos Rito
Departamento de Matemática
Universidade de Trás-os-Montes e Alto Douro
5001-801 Vila Real
Portugal
e-mail: crito@utad.pt

María Martí Sánchez
Nebrija Universidad
Calle Pirineos 55
28040 Madrid
Spain
e-mail: mmartisanc@gmail.com