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A DECOMPOSITION OF NORMAL SUBGROUPS IN A GROUP

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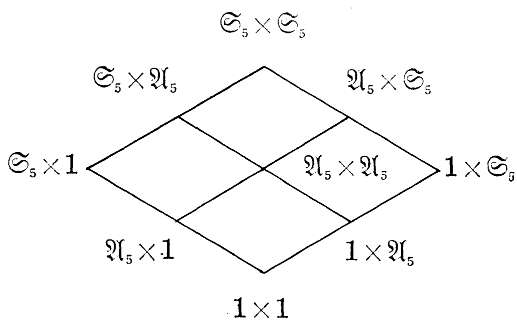
In his paper [5], E. Schenkman pointed out the similarity between the properties of the set of normal subgroups of a group with the maximal condition for normal subgroups and the properties of ideals of a commutative Noetherian ring. This was based on making the analogy in the following way: the sum of ideals corresponds to the product of normal subgroups; the product of ideals corresponds to the commutator of normal subgroups; and the residual quotient of ideals has an analogue introduced there. The concepts of prime ideals, irreducible ideals, and radical of an ideal have analogues for normal subgroups.

From such a point of view, we can consider naturally a decomposition of normal subgroups of a group on the analogy of the primary decomposition of ideals in a commutative Noetherian ring. Let us consider, for example, the direct product of the symmetric group $\mathfrak{S}_5$ of degree 5 with itself. Then it can be easily seen that the lattice of its normal subgroups is as follows:



In this group the normal subgroup $\mathfrak{A}_5 \times 1$ can be written, in two ways, as intersections of primary subgroups:

$$\begin{aligned}\mathfrak{A}_5 \times 1 &= (\mathfrak{S}_5 \times 1) \cap (\mathfrak{A}_5 \times \mathfrak{S}_5) \\ &= (\mathfrak{S}_5 \times 1) \cap (\mathfrak{A}_5 \times \mathfrak{A}_5).\end{aligned}$$

This suggests problems to us that, for any normal subgroup A in an arbitrary group G with a certain finiteness condition for normal subgroups, whether A can be written as an intersection of primary subgroups of G or not, and that if there exists such a decomposition, then whether this decomposition is determined uniquely by A as in the case of ideals in commutative Noetherian rings or not.

It is the main purpose of this paper to present necessary and sufficient conditions that such a decomposition exists, and to discuss the question of the uniqueness theorem.

We shall define, in § 1, the concepts of prime subgroups and radicals of normal subgroups in an arbitrary group along the same line as ideals in an associative ring due to N. H. McCoy [3]. The concept of an m-system will be introduced here as an analogue for associative rings. The radical of a normal subgroup A, denoted by $r(A)$, will be defined, in this paper, as the intersection of all minimal prime subgroups belonging to A and is somewhat different from Schenkman's one [5].

In § 2, we shall define the concept of primary subgroups in an arbitrary group along the same line as ideals in a commutative Noetherian ring. This is also different from Schenkman's one [5]. For any two normal subgroups A and B, the residual quotient of A by B will be defined and some useful properties concerning the residual quotients will be considered.

In § 3, the concepts of tertiary radicals and tertiary subgroups in an arbitrary group will be introduced by the analogy of R. Croisot [1]. We shall show here that every normal subgroup in a group with the maximal condition for normal subgroups can be written always as an intersection of a finite number of tertiary subgroups (Theorem 3.9).

We shall collect, in § 4, certain results of normal subgroups in a group G with the maximal condition for normal subgroups. One of the interesting results is as follows: for any normal subgroup A, there exists a suitable integer n such that $r(A)^{(n)}\subseteqq A$, where $r(A)^{(n)}$ denotes n-th derived group of $r(A)$ (Proposition 4.5). We may remark from this that the Schenkman's radical coincides with our radical, and that every primary subgroup in our sense is always one of the primary subgroups in Schenkman's sense.

In § 5, applying the notion of the Artin-Rees property of J. A. Riley [4] to our case, we shall give necessary and sufficient conditions that every normal subgroup in a group with the maximal condition for normal subgroups can be represented as an intersection of a finite number of primary subgroups (Theorem 5.4). This is one of the main theorems.

In § 6, we shall prove another main theorem that if a normal subgroup A has a primary decomposition, then in any two normal decompositions of A the number of primary components is the same and the radicals of these coincide in some order (Theorem 6.1). This is also a generalization of a theorem due to E. Schenkman [5]. And in §7, we shall show that a similar result holds in the tertiary decompositions (Theorem 7.2).

Finally, in §8, we shall prove the second uniqueness theorem for normal decompositions. Suppose that A is a decomposable subgroup, and let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a normal decomposition. If $\{r(Q_1), r(Q_2),\cdots, r(Q_m)\}$ is an isolated set of A, then $Q_1\cap Q_2\cap\cdots\cap Q_m$ depends only on $r(Q_1), r(Q_2),\cdots,r(Q_m)$ and not on the particular normal decomposition considered (Theorem 8.7).

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1. Prime subgroups and radicals

Let G be an arbitrary group and let a be an element in G. Throughout this paper, (a) will denote the normal subgroup in G generated by a, that is, the smallest normal subgroup which contains a. For any two normal subgroups A and B in G, we shall define the *commutator subgroup* $[A, B]$ of these normal subgroups as the subgroup generated by all commutators of the form

$$[a, b] = aba^{-1}b^{-1},$$

where a is in A and b is in B. More generally, for any n normal subgroups $A_1, A_2,\cdots,A_n$ in G, we shall define by recurrence a *complex commutator* of weight m in the components $A_1, A_2,\cdots,A_n$. The complex commutators of weight 1 are the normal subgroups $A_1, A_2,\cdots,A_n$ themselves. Suppose that the complex commutators of all weight less than m have been defined already. Then those of weight m consist of all the commutator subgroups $[A, B]$, where A and B are any complex commutators of weight m_1 and m_2 in the components $A_1, A_2,\cdots,A_n$ respectively, such that $m_1+m_2=m$.

DEFINITION 1.1. A normal subgroup P in G is a *prime subgroup* of G or simply P is *prime* in G, if whenever $[(a), (b)]\subseteqq P$ at least one of a and b belongs to P.

From this definition we have

Proposition 1.2. *In order that a normal subgroup P in G is a prime* *subgroup of G, it is necessary and sufficient that if whenever* $[A, B] \subseteqq P$ *at least one of* A *and* B *is contained in* P.

More generally, applying this proposition, we can prove easily the following by the induction on the weight of the complex commutators.

Proposition 1.3. *Let* $A_1, A_2, \cdots, A_n$ *be any n normal subgroups in G, and let* $C[A_1, A_2, \cdots, A_n]$ *be a complex commutator of weight m in the components* $A_1, A_2, \cdots, A_n$. *Then any prime subgroup of G which contains* $C[A_1, A_2, \cdots, A_n]$ *must contain at least one of the* A_i.

Corollary 1.4. *Let* $A_1, A_2, \cdots, A_n$ *be any n normal subgroups in G. Then any prime subgroup of G which contains the intersection* $A_1 \cap A_2 \cap \cdots \cap A_n$ *must contain at least one of the* A_i.

Proposition 1.5. (E. Schenkman [5]) *Let A be a normal subgroup in G and let* $P_1, P_2, \cdots, P_n$ *be prime subgroups none of which contains A, then there exists an element a in A such that no* P_i *contains a.*

Proof. We use induction on the number n of prime subgroups. If $n=1$, the assertion is trivial. Let us now assume that the proposition is true for $n-1$ prime subgroups, then for each i, $1 \leqq i \leqq n$, there exists an element a_i in A, which is not contained in any of $P_1, P_2, \cdots, P_{i-1}, P_{i+1}, \cdots, P_n$. It is clear that we need only to consider the case in which $a_i \in P_i$ for all i. For a fixed prime subgroup P_k, there exist in (a_1) and (a_2) two elements a_1' and a_2' respectively such that $[a_1', a_2']$ is contained in $P_1 \cap P_2 \cap A$ but not in P_k. Then there exist in $([a_1', a_2'])$ and (a_3) two elements a_2'' and a_3' respectively such that $[a_2'', a_3']$ is contained in $P_1 \cap P_2 \cap P_3 \cap A$ but not in P_k. Continuing an exactly similar argument, we obtain two elements a_{k-2}'' and a_{k-1}' such that $[a_{k-2}'', a_{k-1}']$ is contained in $P_1 \cap P_2 \cap \cdots \cap P_{k-1} \cap A$ but not in P_k. Therefore there exist in $([a_{k-2}'', a_{k-1}'])$ and (a_{k+1}) two elements a_{k-1}'' and a_{k+1}' respectively such that $[a_{k-1}'', a_{k+1}']$ is contained in $P_1 \cap P_2 \cap \cdots \cap P_{k-1} \cap P_{k+1} \cap A$ but not in P_k. Finally, using an exactly similar argument repeatedly, we obtain two elements a_{n-1}'' and a_n' such that $b_k = [a_{n-1}'', a_n']$ is contained in $P_1 \cap P_2 \cap \cdots \cap P_{k-1} \cap P_{k+1} \cap \cdots \cap P_n \cap A$ but not in P_k. If we put $a = b_1 b_2 \cdots b_n$, then a is contained in A but not contained in any of $P_1, P_2, \cdots, P_n$. This completes the proof.

To give another characterization of a prime subgroup P, we shall consider the set theoretic complement $C(P)$ of P in G. This is an m-system in the following sense:

DEFINITION 1.6. A subset M $(\neq \phi)$ of G is an *m-system*, if for any two elements m_1 and m_2 in M there exist in (m_1) and (m_2) two elements m_1' and m_2' respectively such that $[m_1', m_2']$ is in M. The empty set ϕ to be considered as an m-system.

This concept plays in the present paper the same role as those defined by N. H. McCoy [3] in associative rings, and so we can translate some of results due to McCoy into ours. First we have from the definition of prime subgroups the following:

Proposition 1.7. *A normal subgroup P in G is a prime subgroup of G if and only if its complement $C(P)$ is an m-system.*

Moreover, in this proposition we can weaken the assumption that P is a normal subgroup. We shall prepare a definition and several lemmas.

DEFINITION 1.8. A prime subgroup P is called a *minimal prime subgroup* belonging to a normal subgroup A, if it contains A, and if there is no prime subgroup containing A which is strictly contained in P.

Lemma 1.9. *Let A be a normal subgroup in G, and let M be an m-system which does not meet A. Then M is contained in an m-system M^* which is maximal in the class of m-systems which do not meet A.*

This is, of course, an immediate consequence of Zorn's lemma and is merely stated in the form of a lemma for convenience of reference.

Now let A and B be two normal subgroups in G, then AB will denote the subgroup generated by A and B.

Lemma 1.10. *Let M be an m-system in G and let A be a normal subgroup in G which does not meet M. Then A is contained in a normal subgroup P^* in G which is maximal in the class of normal subgroups which do not meet M. The normal subgroup P^* is necessarily a prime subgroup of G.*

Proof. The existence of P^* follows at once from Zorn's lemma. We now show that P^* is a prime subgroup of G. If M is empty, then P^* coincides with G and P^* is a prime subgroup of G. Suppose that M is not empty, and that, for $i=1, 2$, $a_i \notin P^*$. Then the maximal property of P^* implies that $P^*(a_i)$ contains an element m_i of M. Thus there exists an element b_i in (a_i) such that $b_i \equiv m_i \pmod{P^*}$. Since M is an m-system, we can choose m_i' in (m_i) such that $[m_1', m_2']$ is in M, and hence $[m_1', m_2'] \notin P^*$ since P^* does not meet M. Since each m_i' is contained in (m_i), it has an expression of the form $m_i' = \prod_k x_k m_i^{n_k} x_k^{-1}$. Then $b_i' = \prod_k x_k b_i^{n_k} x_k^{-1}$ is in (b_i) and $b_i' \equiv m_i' \pmod{P^*}$. Thus b_i' is in (a_i) and $[b_1', b_2'] \equiv [m_1', m_2'] \not\equiv 1 \pmod{P^*}$, and hence $[(a_1), (a_2)]$ is not contained in P^*. This shows that P^* is a prime subgroup of G.

We now prove

Proposition 1.11. *A set P of elements of G is a minimal prime subgroup belonging to a normal subgroup A in G if and only if its complement $C(P)$ is maximal in the class of m-systems which do not meet A.*

Proof. Let P be a set of elements of G with a property that $M=C(P)$ is a maximal m-system which does not meet A. If M is the empty set, then $P=G$ is a prime subgroup of G and the maximal property of M implies that P is a minimal prime subgroup belonging to A. Suppose that M is not empty. By Lemma 1.10, A is contained in a prime subgroup P^* of G which is maximal in the class of normal subgroups which do not meet M. P^* does not equal to G and $C(P^*)$ is an m-system which contains M and does not meet A. The maximal property of M implies that $C(P^*)=M$, and hence $P^*=P$. Thus P is a prime subgroup of G containing A. Clearly, there can exist no prime subgroup P' of G such that $A\subseteqq P'\subsetneqq P$, since this would imply that $C(P')$ is an m-system which does not meet A and properly contains M. This is impossible because of the maximal property of M. Hence P is a minimal prime subgroup belonging to A.

Conversely, if P is a minimal prime subgroup belonging to A, then $M=C(P)$ is an m-system which does not meet A, and Lemma 1.9 shows the existence of a maximal m-system M^* which contains M and does not meet A. By the part of the proposition just proved, $C(M^*)=P^*$ is a minimal prime subgroup belonging to A. Since $M^*\supseteqq M$, it follows that $P^*\subseteqq P$. Thus $A\subseteqq P^*\subseteqq P$, from which it follows that $P^*=P$ and $M=M^*$. This shows that M is a maximal m-system which does not meet A, and completes the proof of the proposition.

If P is any prime subgroup of G containing a normal subgroup A in G, then $M=C(P)$ is an m-system which does not meet A. By Lemma 1.9, M is contained in an m-system M^* which is maximal in the class of m-systems which do not meet A. Proposition 1.11 shows that $C(M^*)$ is a minimal prime subgroup belonging to A. Since $C(P)\subseteqq M^*$, it follows that $A\subseteqq C(M^*)\subseteqq P$. This proves that any prime subgroup which contains A contains a minimal prime subgroup belonging to A. As a special case in which $P=G$, for any normal subgroup A in G there exists at least one minimal prime subgroup belonging to A.

Applying this remark, we now define

DEFINITION 1.12. The *radical* $r(A)$ of a normal subgroup A in G is the intersection of all minimal prime subgroups belonging to A. If A is its own radical, then A will be called a *radical subgroup* or simply a *radical.*

From the above remark, we have

Proposition 1.13. *The radical $r(A)$ of a normal subgroup A in G is the intersection of all prime subgroups containing A.*

Let A be a normal subgroup in G. Then we shall define

$$A^{(0)} = A, A^{(1)} = [A, A] \quad \text{and} \quad A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$$

for any positive integer $n>1$. The Schenkman's radical of A, denoted by Rad A, is the group union of all normal subgroups B in G such that for some n, $B^{(n)}$ is contained in A. It follows from Proposition 1.3 that Rad A is contained in $r(A)$. However, in the later section (§ 4) we shall show that Rad A coincides with $r(A)$ under the assumption that G satisfies the maximal condition for normal subgroups.

Proposition 1.14. *Let A and B be any two normal subgroups in G. Then*

(1) $r(A)\supseteqq A$,
(2) $A\supseteqq B$ *implies* $r(A)\supseteqq r(B)$,
(3) $r(r(A))=r(A)$,
(4) $r([A, B])=r(A\cap B)=r(A)\cap r(B)$.

Proof. (1) and (2) follow from Definition 1.12 and Proposition 1.13 respectively.

(3) By (1) and (2) above we have $r(r(A))\supseteqq r(A)$. Since any minimal prime subgroup P belonging to A is a prime subgroup containing $r(A)$, we have $r(r(A))\subseteqq r(A)$, and hence we have $r(r(A))=r(A)$.

(4) Since $[A, B]\subseteqq A\cap B$, we have by (2) above $r([A, B])\subseteqq r(A\cap B)\subseteqq r(A)\cap r(B)$. Conversely, since any prime subgroup containing $[A, B]$ contains $r(A)\cap r(B)$, it follows that $r([A, B])\supseteqq r(r(A)\cap r(B))$, and hence by (1) above we have $r([A, B])\supseteqq r(A)\cap r(B)$, which completes the proof.

We shall say that a normal subgroup A in G is *meet-irreducible,* if whenever $A=B\cap C$, where B and C are normal subgroups in G, then either $A=B$ or $A=C$.

It is evident that a prime subgroup is always meet-irreducible. On the other hand, if a normal subgroup A is meet-irreducible and is a radical such that $[B, C]$ is contained in A, then we have $[AB, AC]=[A, A][A, C][B, A][B, C]\subseteqq A$. $AB\cap AC$ contains A and is abelian mod $[AB, AC]$, and hence is abelian mod A. Since A is a radical, it follows from Proposition 1.3 that $AB \cap AC = A$, and since A is meet-irreducible, either AB or AC is equal to A, and hence either B or C is contained in A, whence it follows that A is prime in G. We may conclude from this remark that a normal subgroup P in G is prime if and only if it is a radical and is meet-irreducible.

Proposition 1.15. *The radical $r(A)$ of a normal subgroup A consists of those elements a of G with the property that every m-system which contains a contains an element of A.*

Proof. Suppose that there exists an m-system M which contains a but does not meet A. By Lemma 1.9, M is contained in an m-system M^* which is maximal in the class of m-systems which do not meet A. By Proposition 1.11, $C(M^*)$ is a minimal prime subgroup belonging to A, and clearly $C(M^*)$ does not contain a. Hence a can not be in the radical $r(A)$.

Conversely, let P be any minimal prime subgroup belonging to A. Then $C(P)$ is an m-system which does not meet A, and hence $C(P)$ does not contain a by our assumption, that is, a is contained in P. Thus a is in $r(A)$, which completes the proof.

DEFINITION 1.16. The *radical* of G will be defined as the radical $r(E)$ of the unit subgroup E of G.

Proposition 1.17. *Let A be a normal subgroup in G. Then the radical of the factor group G/A is equal to $r(A)/A$. In particular, if A is a radical, the radical of the factor group G/A is the unit subgroup.*

Proof. It follows from Proposition 1.13 that the radical of G/A is the intersection $\bigcap \{P/A : P$ is a prime subgroup of G containing $A\}$, and hence is equal to the factor group $(\bigcap \{P : P$ is a prime subgroup of G containing $A\})/A$. Thus it coincides with $r(A)/A$.

2. Primary subgroups and residual quotients

DEFINITION 2.1. A normal subgroup Q in G is called a *primary subgroup* of G if the conditions $[(a), (b)] \subseteqq Q$ and $a \notin Q$ always imply that $b \in r(Q)$.

Let us note that prime subgroups are always primary subgroups. Slightly similar to Proposition 1.2 we have

Proposition 2.2. *A normal subgroup Q in G is a primary subgroup of G if and only if $[A, B] \subseteqq Q$ and $A \nsubseteq Q$, then $B \subseteqq r(Q)$.*

In the later section (§ 4), we shall prove that if G satisfies the maximal condition for normal subgroups, then there exists a suitable integer n such that $Q \supseteqq r(Q)^{(n)}$. Hence, if this is the case, a condition that B is contained in $r(Q)$ is equivalent to a condition that $B^{(n)}$ is contained in Q for some integer n.

As is easily seen, P is a prime subgroup of G if and only if P is a primary subgroup and is a radical. Hence by the remark in the preceding section, if P is a radical, then the following conditions are equivalent:

(1) P is a prime subgroup of G,
(2) P is a primary subgroup of G,
(3) P is meet-irreducible.

There are some properties of primary subgroups that we shall need. First, we prove

Proposition 2. 3. *If Q_1 and Q_2 are primary subgroups of G such that $r(Q_1)=r(Q_2)$, then $Q=Q_1 \cap Q_2$ is also a primary subgroup of G such that $r(Q)=r(Q_1)=r(Q_2)$.*

Proof. By Proposition 1. 14 (4), $r(Q)=r(Q_1 \cap Q_2)=r(Q_1)\cap r(Q_2)$, and hence $r(Q)=r(Q_1)=r(Q_2)$. We now show that Q is a primary subgroup. Assume that $[(a),(b)]\subseteqq Q$ and that $b\notin r(Q)$, then for each $i=1, 2$, $[(a),(b)]\subseteqq Q_i$ and $b\notin r(Q_i)$, and hence a is in Q_i. Thus a is in Q. This completes the proof.

If a normal subgroup A in G can be expressed in the form

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n ,$$

where each Q_i is a primary subgroup of G, we shall say that we have a *primary decomposition* of A, and the individual Q_i will be called the *primary components* of the decomposition. Those normal subgroups which can be written in the form above will be called *decomposable subgroups*. A decomposition in which no Q_i contains the intersection of the remaining Q_j is called *irredundant*.

Proposition 2. 4. *Let A be a normal subgroup in G. If $A=Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is an irredundant primary decomposition of A such that not all of Q_i have the same radical, then A is not a primary subgroup of G.*

Proof. For each j, $1\leqq j\leqq n$, we have $[\bigcap_{k\neq j} Q_k, Q_j]\subseteqq A$. Since the decomposition is irredundant, $\bigcap_{k\neq j} Q_k \not\subseteqq Q_j$, and hence we have $\bigcap_{k\neq j} Q_k \not\subseteqq A$. Let us now suppose that A is a primary subgroup of G. Then it follows from these relations that $Q_j \subseteqq r(A)$, and hence $r(Q_j)\subseteqq$

$r(r(A))=r(A)=\bigcap_i r(Q_i)$. Thus $r(Q_j)\subseteqq r(Q_i)$ for all i and j, which is impossible because of our assumption.

Proposition 2.5. *Suppose that a normal subgroup A is a decomposable subgroup, $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ is its primary decomposition, and the radical $r(Q_i)$ of each primary component Q_i is expressed in the form*

$$r(Q_i)=P_{i1}\cap P_{i2}\cap\cdots\cap P_{in_i},$$

where P_{ij} are prime subgroups containing Q_i for $1\leqq j\leqq n_i$, $1\leqq i\leqq n$. Then,

(1) *any prime subgroup which contains A must contain at least one of the P_{ij},*

(2) *the minimal prime subgroups belonging to A are just those prime subgroups P_{ij} which do not strictly contain any other P_{kr}.*

Proof. (1) Let P be a prime subgroup of G containing A, then, by Corollary 1.4, we can choose i so that $Q_i\subseteqq P$, and hence $r(Q_i)\subseteqq r(P)=P$. Again by Corollary 1.4 we can choose some j so that $P_{ij}\subseteqq P$. This proves the first assertion.

(2) If, in particular, P is a minimal prime subgroup belonging to A, then by (1) just proved we can choose some i and j so that $P_{ij}=P$. Conversely, suppose that P_{ij} does not strictly contain any other P_{kr}. Since P_{ij} contains A, it must contain a minimal prime subgroup P belonging to A, as we have remarked in §1. P contains, by (1) above, at least one of the P_{kr}. Hence $P_{ij}\supseteqq P\supseteqq P_{kr}$ and consequently, from the choice of P_{ij}, $P_{ij}=P=P_{kr}$, which completes the proof.

Now we make the following definition.

DEFINITION 2.6. Let A and B be any two normal subgroups in G. The *residual quotient* of A by B, denoted by $A:B$, will be defined to be the set of all elements x in G such that $[(x), B]$ is contained in A.

Lemma 2.7. *The residual quotient of A by B is a normal subgroup in G containing A.*

Proof. Suppose that x and y are in $A:B$, then $[(xy), B]\subseteqq[(x)(y), B]=[(x), B][(y), B]\subseteqq A$. This proves that xy is in $A:B$. Since $(x^{-1})=(x)$ and $(cxc^{-1})=(x)$ for all c in G, we see that $A:B$ is a normal subgroup in G. Moreover, if a is in A, then certainly $[(a), B]\subseteqq A$, which gives us the relation $A\subseteqq A:B$.

As is easily seen from Definition 2.6, we have

(1) $[A:B, B]=[B, A:B]\subseteqq A$,
(2) $(\bigcap_i A_i):B=\bigcap_i(A_i:B)$.

Now we shall give a characterization of prime subgroups and that of primary subgroups by means of residual quotients.

Proposition 2.8. (1) *A normal subgroup P in G is a prime subgroup of G if and only if $P:A=P$ for all normal subgroups A in G such that $A\nsubseteq P$.*

(2) *A normal subgroup Q in G is a primary subgroup of G if and only if $Q:A=Q$ for all normal subgroups A in G such that $A\nsubseteq r(Q)$.*

Proof. As we have remarked above, (1) is a special case of (2), and so we shall prove only (2). Now, let us suppose that Q is a primary subgroup of G and that A is a normal subgroup in G such that $A\nsubseteq r(Q)$. Then from the fact that $[Q:A, A]\subseteqq Q$, we have $Q:A\subseteqq Q$, whence it follows that $Q:A=Q$.

Conversely, suppose that $[(a), (b)]\subseteqq Q$ and that $b\notin r(Q)$. Then (b) is not contained in $r(Q)$, and hence by our assumption we have $Q:(b)=Q$, which shows that a is in Q, This proves that Q is a primary subgroup of G.

Let A and B be any two normal subgroups in G. If $A\supseteqq B$, then from the definition of residual quotients we see that $A:B=G$. From this and Proposition 2.8 (1), if P is a prime subgroup of G, then $P:A$ is G or P according as A is or is not contained in P.

Lemma 2.9. *Let A $(\neq G)$ be a normal subgroup in G such that $A=P_1\cap P_2\cap\cdots\cap P_n$, where each P_i is a minimal prime subgroup belonging to A and the decomposition is irredundant. Then $A:B=A$ if and only if B is contained in no P_i.*

Proof. The sufficiency follows from Proposition 2.8 (1). To prove the necessity, suppose that $A:B=A$. Then $P_1\cap P_2\cap\cdots\cap P_n=(P_1\cap P_2\cap\cdots\cap P_n):B=(P_1:B)\cap(P_2:B)\cap\cdots\cap(P_n:B)$, and by Corollary 1.4 for each i, $1\leqq i\leqq n$, there exists some j, $1\leqq j\leqq n$, such that $P_j:B\subseteqq P_i$. Since P_j is contained in $P_j:B$ and since P_i is a minimal prime subgroup belonging to A, P_j is equal to P_i and hence $P_i:B=P_i$ for all i. Hence B is contained in no P_i.

More generally from this lemma, we have

Proposition 2.10. *Suppose that A is as in Lemma 2.9 and suppose that $B=B_1'\cap B_2'\cap\cdots\cap B_m'$, where each B_j' is a normal subgroup in G. Then $A:B=A$ if and only if no B_j' is contained in any P_i.*

3. Tertiary radicals and tertiary subgroups

We now make the following definition.

DEFINITION 3.1. The *tertiary radical* of a normal subgroup A in G, denoted by $t(A)$, will be defined to be the set of all elements a in G satisfying the condition

$$b \notin A \Rightarrow \exists c \in (b) \text{ such that } c \notin A \text{ and } [(a), (c)] \subseteqq A\,.$$

Lemma 3.2. *The tertiary radical $t(A)$ of A is a normal subgroup in G which contains A.*

Proof. Let a_1 and a_2 be any two elements in $t(A)$. Suppose that $b \notin A$. Then there exists c_1 in (b) such that $c_1 \notin A$ and $[(a_1), (c_1)] \subseteqq A$ and, since $c_1 \notin A$, there exists c_2 in (c_1) such that $c_2 \notin A$ and $[(a_2), (c_2)] \subseteqq A$. Then $[(a_1a_2), (c_2)] \subseteqq [(a_1), (c_1)][(a_2), (c_2)] \subseteqq A$. Hence a_1a_2 is in $t(A)$. Since $(a^{-1})=(a)$ and $(cac^{-1})=(a)$ for all c in G, we see that $t(A)$ is a normal subgroup in G. Moreover, if $b \notin A$, then b is in (b) but not in A and certainly $[(a), (b)] \subseteqq A$ for all a in A, which gives us the relation $A \subseteqq t(A)$.

DEFINITION 3.3. A normal subgroup T in G is called a *tertiary subgroup* of G if the conditions $[(a), (b)] \subseteqq T$ and $a \notin T$ always imply that b is in $t(T)$.

From this definition, we have

Proposition 3.4. *A normal subgroup T in G is a tertiary subgroup of G if and only if $[A, B] \subseteqq T$ and $A \nsubseteq T$, then B is contained in $t(T)$.*

Now we prove

Proposition 3.5. *Every meet-irreducible normal subgroup in G is a tertiary subgroup.*

Proof. Suppose that a normal subgroup A is meet-irreducible but not a tertiary subgroup. Then there exist a and b such that $[(a), (b)] \subseteqq A$, $a \notin A$ and $b \notin t(A)$, and hence there exists $c \notin A$ such that

$$[(b),(c')] \subseteqq A \quad \text{and} \quad c' \in (c) \Rightarrow c' \in A\,.$$

Let us now consider the normal subgroup $A' = A(a) \cap A(c)$ and let $x \in A'$. Then $x = a_1a' = a_2c'$ with a_1, $a_2 \in A$, $a' \in (a)$ and $c' \in (c)$. It follows from this that $c' = a_3a'$ for some $a_3 \in A$. Since $[(b), (c')] \subseteqq [(b), (a_3)][(b), (a)] \subseteqq A$, we have $c' \in A$, and hence $x \in A$. Whence it follows that $A = A'$, which contradicts the fact that A is meet-irreducible.

By means of usual methods, we can show that every normal subgroup in a group with the maximal condition for normal subgroups can be expressed as an intersection of a finite number of meet-irreducible normal subgroups. Hence, by Proposition 3.5, we have

Proposition 3.6. *If G satisfies the maximal condition for normal subgroups, then every normal subgroup in G can be represented as an intersection of a finite number of tertiary subgroups of G.*

Proposition 3.7. *Let $A=T_1\cap T_2\cap\cdots\cap T_n$ be an irredundant decomposition of a normal subgroup A, where each T_i is a tertiary subgroup of G. Then the tertiary radical $t(A)$ of A is equal to*

$$t(T_1)\cap t(T_2)\cap\cdots\cap t(T_n).$$

Proof. Let $a\in t(T_1)\cap t(T_2)\cap\cdots\cap t(T_n)$ and let $b\notin A$. Then we may assume that $b\notin T_1$ and, since $a\in t(T_1)$, there exists $b_1\in(b)$ such that $b_1\notin T_1$ and $[(a),(b_1)]\subseteqq T_1$. If b_1 is contained in T_2, we have $b_1\notin T_1\cap T_2$ and $[(a),(b_1)]\subseteqq T_1\cap T_2$. On the other hand, since a is in $t(T_2)$, if b_1 is not contained in T_2, then there exists an element $b_1'\in(b_1)$ such that $b_1'\notin T_2$ and $[(a),(b_1')]\subseteqq T_2$. We have therefore that $[(a),(b_1')]\subseteqq T_1\cap T_2$. Thus we can choose, in either case, an element $b_2\in(b)$ such that $b_2\notin T_1\cap T_2$ and $[(a),(b_2)]\subseteqq T_1\cap T_2$. Using an exactly similar argument repeatedly, we obtain an element $b_n\in(b)$ such that $b_n\notin T_1\cap T_2\cap\cdots\cap T_n$ and $[(a),(b_n)]\subseteqq T_1\cap T_2\cap\cdots\cap T_n$. Thus $a\in t(A)$ and hence $t(T_1)\cap t(T_2)\cap\cdots\cap t(T_n)\subseteqq t(A)$.

Conversely, suppose that a is in $t(A)$. Let b be an element in $T_2\cap T_3\cap\cdots\cap T_n$ but not in T_1. Then b is not contained in A and hence there exists $c\in(b)$ such that $c\notin A$ and $[(a),(c)]\subseteqq A$. On one hand, we have $c\notin T_1$, for otherwise we should have $c\in T_1\cap(b)\subseteqq A$. Since T_1 is a tertiary subgroup, a is in $t(T_1)$. Similarly, a is in $t(T_i)$ for $2\leqq i\leqq n$. Thus $t(A)\subseteqq t(T_1)\cap t(T_2)\cap\cdots\cap t(T_n)$, which completes the proof.

By a similar method as in Proposition 2.3, we have

Proposition 3.8. *If T_1 and T_2 are tertiary subgroups of G such that $t(T_1)=t(T_2)$, then $T=T_1\cap T_2$ is also a tertiary subgroup of G such that $t(T)=t(T_1)=t(T_2)$.*

If a normal subgroup A in G can be expressed in the form

$$A=T_1\cap T_2\cap\cdots\cap T_n,$$

where each T_i is a tertiary subgroup of G, we shall say that we have a *tertiary decomposition* of A, and the individual T_i will be called the *tertiary components* of the decomposition. An irredundant tertiary decomposition, in which the tertiary radicals of the various tertiary components are all different, is called a *normal decomposition.* Each tertiary decomposition, as is easily seen from Proposition 3. 8, can be refined into one which is normal.

Finally, we apply Proposition 3. 6 to obtain the following

Theorem 3. 9. *If G satisfies the maximal condition for normal subgroups, then every normal subgroup in G has a normal tertiary decomposition.*

4. The maximal condition for normal subgroups

We shall collect in this section certain results of normal subgroups under the assumption that G satisfies the maximal condition for normal subgroups.

Proposition 4. 1. *Suppose that G satisfies the maximal condition for normal subgroups. Then for any normal subgroup A in G there exist at most a finite number of minimal prime subgroups belonging to A. Thus the radical $r(A)$ of A is an intersection of a finite number of minimal prime subgroups belonging to A.*

Proof. If A is a prime subgroup of G, then the assertion is trivial. We may suppose therefore that A is not a prime subgroup of G. Then we can choose $a \notin A$ and $b \notin A$ such that $[(a), (b)] \subseteq A$. Let us suppose that A has an infinite number of minimal prime subgroups P_i belonging to A. Then since $[A(a), A(b)] = [A, A][A, (b)]\ [(a), A]\ [(a), (b)] \subseteq A$, at least one of $A(a)$ and $A(b)$ must be contained in an infinite number of P_i. Without loss of generality we may suppose that the one which is contained in an infinite number of P_i is $A(a)$. It is easily seen that those P_i which contain $A(a)$ are minimal prime subgroups belonging to $A(a)$ and moreover $A(a) \supsetneqq A$. Continuing an exactly similar argument, we obtain a strictly increasing infinite sequence $A \subsetneqq A(a) \subsetneqq \cdots$ of normal subgroups in G, which is impossible from our assumption.

Proposition 4. 2. *Suppose that G satisfies the maximal condition for normal subgroups. Let A be a normal subgroup in G and let $P_1, P_2, \cdots, P_n$ be the set of all minimal prime subgroups belonging to A. Then there exist some $P_{i_1}, P_{i_2}, \cdots, P_{i_m}$, $1 \leqq i_k \leqq n$, and some complex commutator $C[P_{i_1}, P_{i_2}, \cdots, P_{i_m}]$ in the components $P_{i_1}, P_{i_2}, \cdots, P_{i_m}$, such that $C[P_{i_1}, P_{i_2}, \cdots, P_{i_m}] \subseteq A$.*

Proof. We shall prove by "Teilerinduktion" (Cf. B. L. van der

Waerden [6], § 84). If A is a prime subgroup of G, then the assertion is trivial. We may suppose therefore that A is not a prime subgroup of G. Then we can choose $a \notin A$ and $b \notin A$ such that $[(a), (b)] \subseteqq A$. From Proposition 4. 1, there exist a finite number of minimal prime subgroups $P'_1, P'_2, \cdots, P'_r$ belonging to $A(a)$ and likewise a finite number of minimal prime subgroups $P''_1, P''_2, \cdots, P''_s$ belonging to $A(b)$. From the induction hypothesis there exist some $P'_{j_1}, P'_{j_2}, \cdots, P'_{j_u}$, $1 \leqq j_k \leqq r$, and some complex commutator $C'[P'_{j_1}, P'_{j_2}, \cdots, P'_{j_u}]$ which is contained in $A(a)$ and likewise there exist some $P''_{h_1}, P''_{h_2}, \cdots, P''_{h_v}$, $1 \leqq h_t \leqq s$, and some complex commutator $C''[P''_{h_1}, P''_{h_2}, \cdots, P''_{h_v}]$ which is contained in $A(b)$. Then we have

$$[C'[P'_{j_1}, P'_{j_2}, \cdots, P'_{j_u}], C''[P''_{h_1}, P''_{h_2}, \cdots, P''_{h_v}]] \subseteqq [A(a), A(b)] \subseteqq A .$$

P'_{j_k} and P''_{h_t} are prime subgroups of G containing $A(a)$ and $A(b)$ respectively, and hence are those prime subgroups of G which contain A. As we have remarked in § 1, P'_{j_k} and P''_{h_t} contain some minimal prime subgroups P_{j_k} and P_{h_t} belonging to A respectively, and hence we have

$$\begin{aligned}&[C'[P_{j_1}, P_{j_2}, \cdots, P_{j_u}], C''[P_{h_1}, P_{h_2}, \cdots, P_{h_v}]] \\ \subseteqq &[C'[P'_{j_1}, P'_{j_2}, \cdots, P'_{j_u}], C''[P''_{h_1}, P''_{h_2}, \cdots, P''_{h_v}]] \subseteqq A .\end{aligned}$$

This completes the proof of the proposition.

From this proposition, we see at once

Corollary 4. 3. *Suppose that G satisfies the maximal condition for normal subgroups. Then there exists a complex commutator $C[r(A)]$ of some weight in the single component $r(A)$ which is contained in A.*

Now we prove

Lemma 4. 4. *Every complex commutator $C[A]$ of weight n, $n \geqq 1$, in the component A contains the $(n-1)$-th derived group $A^{(n-1)}$.*

Proof. We use induction on the weight n of the complex commutator. If $n=1$, the assertion is trivial. Let us now assume that the lemma has already been established when the weight of complex commutators is less than n. Then those of weight n has the expression of the form $C[A]=[C'[A], C''[A]]$, where $C'[A]$ and $C''[A]$ are complex commutators of weight n_1 and n_2 in the single component A respectively such that $n_1+n_2=n$. Let us now suppose, for example, that $n_1 \leqq n_2$. Then from this assumption and the induction hypothesis that

$$[C'[A], C''[A]] \supseteqq [A^{(n_1-1)}, A^{(n_2-1)}] \supseteqq [A^{(n_2-1)}, A^{(n_2-1)}] = A^{(n_2)} \supseteqq A^{(n-1)}.$$

This completes the proof.

Combining Lemma 4.4 with Corollary 4.3, we obtain

Proposition 4.5. *Suppose that G satisfies the maximal condition for normal subgroups. Then there exists some integer n such that $r(A)^{(n)} \subseteqq A$.*

As we have remarked in § 1, the Schenkman's radical Rad A of a normal subgroup A in G is contained in $r(A)$. On the other hand, it follows from Proposition 4.5 that Rad A contains $r(A)$. Thus the Schenkman's radical Rad A coincides with $r(A)$ under the maximal condition for normal subgroups.

Moreover, Schenkman defined a primary subgroup Q of G to be such that if $[A, B] \subseteqq Q$ then for some n either $A^{(n)}$ or $B^{(n)}$ is contained in Q. Suppose that G satisfies the maximal condition for normal subgroups. Then it is easily seen from Proposition 4.5 that if Q is a primary subgroup in our sense then Q is also a primary subgroup in Schenkman's sense. However, the converse is not true, even if G satisfies the maximal condition for normal subgroups, as the example mentioned in the introduction illustrates. In this group, $\mathfrak{A}_5 \times 1$ is a primary subgroup in Schenkman's sense, while, on the contrary, that is not a primary subgroup in our sense.

Now we shall consider a characterization of tertiary subgroups in a group with the maximal condition for normal subgroups by means of residual quotients. To do this, we shall apply the notion of essential residuals of L. Lesieur [2] to our case. We make the following definition.

DEFINITION 4.6. Let A be a normal subgroup in G. A normal subgroup R in G is called an *essential residual* of A, if there exists a normal subgroup $B \supsetneqq A$ in G with $R = A : B$, and

$$A \subsetneqq C \subseteqq B \Rightarrow A : C = A : B .$$

Proposition 4.7. *Let A be a normal subgroup in G. Then the following conditions are equivalent*:

(1) *R is an essential residual of A,*

(2) *there exists a normal subgroup $B \nsubseteqq A$ with $R = A : B$, and*

$$C \subseteqq B,\ C \nsubseteqq A \Rightarrow A : C = A : B ,$$

(3) *there exists an element $b \notin A$ with $R = A : (b)$, and*

$$c \notin A,\ c \in (b) \Rightarrow A:(b) = A:(c).$$

Proof. (1)$\Rightarrow$(2). Let $R=A:B$, $A \subsetneqq B$, be an essential residual of A. Then certainly $B \nsubseteq A$ and if $C \subseteqq B$ and $C \nsubseteq A$, we have $A \subsetneqq AC \subseteqq B$. Hence, by our assumption, we have $A:C=A:AC=A:B$.

(2)$\Rightarrow$(3). Suppose that $R=A:B$, where $B \nsubseteq A$ is a normal subgroup in G such that

$$C \subseteqq B,\ C \nsubseteq A \Rightarrow A:C = A:B.$$

Let us consider an element b in B but not in A. Then it follows from our assumption that $A:(b)=A:B$, and likewise, if $c \notin A$ and $c \in (b)$, we have $A:(c)=A:B$.

(3)$\Rightarrow$(1). Suppose that $R=A:(b)$, where $b \notin A$ is an element such that

$$c \notin A,\ c \in (b) \Rightarrow A:(b) = A:(c).$$

Then $A \subsetneqq A(b)$ and $A:A(b)=A:(b)=R$. We now show that $A:A(b)=R$ is an essential residual of A. Let us consider a normal subgroup C such that $A \subsetneqq C \subseteqq A(b)$. Then $C=A\{(b)\cap C\}$ and $A:C=A:A\{(b)\cap C\}$ $=A:\{(b)\cap C\}$. Moreover, we have $(b)\cap C \nsubseteq A$, for otherwise we should have $C=A\{(b)\cap C\} \subseteqq A\cdot A=A$. Thus we can obtain an element c in $(b)\cap C$ but not in A and, by our assumption above, we have $A:A(b)$ $=A:(b)=A:(c)$. It follows from this that $A:C=A:\{(b)\cap C\} \supseteqq A:(b)$ $=A:A(b)=A:(c) \supseteqq A:C$, which show that $A:C=A:A(b)=R$. This completes the proof of the proposition.

In the rest of this section, an essential residual $A:(b)$, which satisfies the condition (3) of the proposition just proved, will be called for convenience of reference an *essential residual of A with respect to* (b).

Lemma 4.8. *Let A be a normal subgroup in G. Then every essential residual of A contains the tertiary radical $t(A)$ of A.*

Proof. Suppose that $R=A:(b)$ is an essential residual of A with respect to (b) and that $a \in t(A)$. Then, since $b \notin A$, we can choose an element $c \in (b)$ such that $c \notin A$ and $[(a), (c)] \subseteqq A$. It follows therefore that $a \in A:(c)=A:(b)=R$. This completes the proof.

A residual quotient $A:B$ of A by B, in which $A \not\supseteqq B$ holds, is called *proper*. As is easily seen from Proposition 4.7 (2), a residual quotient of A, which is maximal in the class of proper residual quotients of A, is an essential residual of A.

We now prove

Proposition 4.9. *Suppose that G satisfies the maximal condition for normal subgroups. Then a normal subgroup A in G is a tertiary subgroup of G if and only if A has the only its essential residual.*

Proof. Suppose that A is a tertiary subgroup of G and that $R=A:(b)$ is an essential residual of A with respect to (b). We now show that $R=t(A)$ holds. Let us suppose that $a\in R$. Then we have $[(a),(b)]\subseteqq A$ and, since $b\notin A$, we have $a\in t(A)$. It follows from this and Lemma 4.8 that $R=t(A)$ holds.

Conversely, suppose that a normal subgroup A has the only its essential residual and that $[(a),(b)]\subseteqq A$ and $b\notin A$. We now show that $a\in t(A)$. If $a\notin t(A)$, then from the definition of tertiary radicals we can choose an element $c\notin A$ such that

$$(*)\qquad [(a),(c')]\subseteqq A \quad\text{and}\quad c'\in(c)\Rightarrow c'\in A.$$

Let us consider the class of residual quotients of A, each of which has the form $A:(x)$ where x is in (c) but not in A. A maximal element $A:(b_0)$ of this class is, as is easily seen, an essential residual of A. Since $[(a),(b)]\subseteqq A$ and $b\notin A$, $A:(b)$ contains a and is proper residual quotient of A. Hence there exists a maximal one in the class of residual quotients of A, each of which is proper and contains $A:(b)$ and, by our assumption, it coincides with $A:(b_0)$. Thus we have $a\in A:(b)\subseteqq A:(b_0)$ and, by the relation (*), we have $b_0\in A$. This is a contradiction.

Let $A=T_1\cap T_2\cap\cdots\cap T_n$ be a normal decomposition of A as an intersection of tertiary subgroups T_i, $1\leqq i\leqq n$. We shall show that the essential residuals of A are precisely the tertiary radicals $t(T_i)$. This is a generalization of the proposition just proved. To show this, we now prove the following lemma.

Lemma 4.10. *Let $R=A:(b)$ be an essential residual of A with respect to (b) and let $A=T_1\cap T_2\cap\cdots\cap T_n$ be a normal decomposition of A as an intersection of tertiary subgroups T_i, $1\leqq i\leqq n$. Then there exists the only T_i, say T_1, such that $A\cap(b)=T_1\cap(b)$. Moreover, we have $R=t(T_1)$ and $T_2\cap T_3\cap\cdots\cap T_n\cap(b)\supsetneqq A\cap(b)$.*

Proof. Let $\{T_1, T_2,\cdots,T_m\}$ be a minimal subset of $\{T_1, T_2,\cdots,T_n\}$ such that

$$A\cap(b)=T_1\cap T_2\cap\cdots\cap T_m\cap(b).$$

Then $m\geqq 1$, since $b\notin A$, and we have $b\notin T_1$, for otherwise we should have $A\cap(b)=T_2\cap T_3\cap\cdots\cap T_m\cap(b)$. Suppose that $a\in R=A:(b)$. Then $[(a),(b)]\subseteqq A\subseteqq T_1$ and, since T_1 is a tertiary subgroup of G, we have $a\in t(T_1)$, which shows that $R\subseteqq t(T_1)$. We now show that $R\supseteqq t(T_1)$. Since $A\cap(b)\subsetneqq T_2\cap T_3\cap\cdots\cap T_m\cap(b)$, we can choose an element c in $T_2\cap T_3\cap\cdots\cap T_m\cap(b)$ but not in $A\cap(b)$. If x is an arbitrary element in $t(T_1)$, then, since $c\notin T_1$, there exists an element $c'\in(c)$ such that $c'\notin T_1$ and $[(x),(c')]\subseteqq T_1$. It follows from this that $[(x),(c')]\subseteqq T_1\cap T_2\cap\cdots\cap T_m\cap(b)=A\cap(b)\subseteqq A$. Thus we have $x\in A:(c')$, and hence, by virtue of Proposition 4.7 (3), we have $x\in A:(b)=R$, which shows that $t(T_1)\subseteqq R$. Whence it follows that $R=t(T_1)$. At the same time, this shows that $m=1$, since if $m>1$ then by a similar method we have $R=t(T_2)$, which contradicts the fact that the tertiary radicals of the various tertiary components of A are all different. Moreover, we have $T_1\cap(b)=A\cap(b)$ and $T_2\cap T_3\cap\cdots\cap T_n\cap(b)\supsetneqq A\cap(b)$. This completes the proof of this lemma.

As an application of this lemma, we now prove

Proposition 4.11. *Suppose that G satisfies the maximal condition for normal subgroups and let*

$$A = T_1\cap T_2\cap\cdots\cap T_n$$

be a normal decomposition of A as an intersection of tertiary subgroups T_i, $1\leqq i\leqq n$. *Then the n tertiary radicals* $t(T_1), t(T_2),\cdots,t(T_n)$ *exhaust all essential residuals of A.*

Proof. By virtue of the lemma just proved, an essential residual of A coincides necessarily with some of the $t(T_i)$, $1\leqq i\leqq n$.

Conversely, we show that $t(T_1)$, for instance, is an essential residual of A. Let $B=T_2\cap T_3\cap\cdots\cap T_n$ and let $R=A:(b)$ be a maximal element in the class of residual quotients of A, each of which has the form $A:(x)$ where x is in B but not in A. As is easily seen, $A:(b)$ is an essential residual of A. Furthermore, we have $A\cap(b)=T_1\cap B\cap(b)=T_1\cap(b)$, and hence we have, again by Lemma 4.10, $R=t(T_1)$. This completes the proof of the proposition.

5. The Artin-Rees property

We now make the following definition.

DEFINITION 5.1. G is said to have the *Artin-Rees property* for normal subgroups if for any two normal subgroups A and B in G and any non-negative integer n, there is a non-negative integer $h(n)$ such that

$$A^{(h(n))} \cap B \subseteqq [A^{(n)}, B].$$

Especially, we shall call that G has the *property* (P) for normal subgroups, if for any two normal subgroups A and B in G, there is a non-negative integer h such that $A^{(h)} \cap B \subseteqq [A, B]$.

Proposition 5.2. *If G has the property (P) for normal subgroups, then every tertiary subgroup of G is always a primary subgroup.*

Proof. Let T be a tertiary subgroup of G and let $[(a), (b)] \subseteqq T$ and $a \notin T$. Then we have $b \in t(T)$. We shall now show that, with a suitable non-negative integer h, $(b)^{(h)} \subseteqq T$. Let us consider the residual quotient $T:(b)$. By the assumption that G has the property (P) for normal subgroups, there exists a non-negative integer h such that $(b)^{(h)} \cap (T:(b)) \subseteqq [(b), T:(b)]$. If $(b)^{(h)} \nsubseteq T$, we can choose an element b_1 in $(b)^{(h)}$ but not in T. Since b is in $t(T)$, there exists b_2 in (b_1) such that $b_2 \notin T$ and $[(b), (b_2)] \subseteqq T$. Thus b_2 is in $T:(b)$ and hence $b_2 \in (b)^{(h)} \cap (T:(b)) \subseteqq [(b), T:(b)] \subseteqq T$, which is a contradiction. The proposition then follows from Proposition 1.3.

Proposition 5.3. *Suppose that G satisfies the maximal condition for normal subgroups and that every normal subgroup in G can be represented as an intersection of a finite number of primary subgroups of G. Then G has the Artin-Rees property for normal subgroups.*

Proof. Let A and B be any two normal subgroups in G and let n be any non-negative integer. Suppose that

$$[A^{(n)}, B] = Q_1 \cap Q_2 \cap \cdots \cap Q_m$$

is a primary decomposition of $[A^{(n)}, B]$. If $B \subseteqq Q_i$ for $1 \leqq i \leqq m$, the assertion is trivial, since $A^{(n)} \cap B = B = [A^{(n)}, B]$. On the other hand, if, for at least one i, we have $B \nsubseteq Q_i$, then we may assume without loss of generality that there exists some m' $(\leqq m)$ such that $B \nsubseteq Q_i$ for $1 \leqq i \leqq m'$ and

$$[A^{(n)}, B] = Q_1 \cap Q_2 \cap \cdots \cap Q_{m'} \cap B.$$

Since $[A^{(n)}, B] \subseteqq Q_i$ for $1 \leqq i \leqq m'$ and since each Q_i is a primary subgroup, there exists, for each i, a non-negative integer s_i such that $(A^{(n)})^{(s_i)} \subseteqq Q_i$. If we put $s = \max\{s_1, s_2, \cdots, s_{m'}\}$, we have $A^{(n+s)} = (A^{(n)})^{(s)} \subseteqq Q_i$ for $1 \leqq i \leqq m'$. Thus we have $A^{(n+s)} \cap B \subseteqq Q_1 \cap Q_2 \cap \cdots \cap Q_{m'} \cap B = [A^{(n)}, B]$, which completes the proof of the proposition.

From Propositions 3.5, 5.2, and 5.3, we have the following main theorem.

Theorem 5.4. *Suppose that G satisfies the maximal condition for normal subgroups. Then the following conditions are equivalent:*

(1) *every normal subgroup in G can be represented as an intersection of a finite number of primary subgroups of G,*

(2) *G has the Artin-Rees property for normal subgroups,*

(3) *G has the property (P) for normal subgroups,*

(4) *every tertiary subgroup of G is a primary subgroup,*

(5) *every meet-irreducible normal subgroup in G is a primary subgroup.*

As is easily seen, the group $\mathfrak{S}_5 \times \mathfrak{S}_5$ in the example mentioned in the introduction has the Artin-Rees property for normal subgroups.

Finally, we shall close this section with the following proposition which corresponds to Krull's "intersection theorem".

Proposition 5.5. *Suppose that G satisfies both the maximal condition for normal subgroups and one of the conditions stated in Theorem 5.4. Let A be a normal subgroup in G and let $B=\bigcap_{n\geqq 0} A^{(n)}$. Then $[A, B]=B$. If, in particular, A is contained in the radical of G, then $\bigcap_{n\geqq 0} A^{(n)}=E$, the unit subgroup in G.*

Proof. By the property (P) for normal subgroups, there exists an integer h such that $A^{(h)}\cap B\subseteqq[A, B]$. Then $B=A^{(h)}\cap B\subseteqq[A, B]\subseteqq B$, and hence $[A, B]=B$. If A is contained in the radical of G, then there exists a suitable integer m such that $A^{(m)}\subseteqq E$. Thus we have $B=E$.

6. Uniqueness theorem for primary decompositions

Let A be a decomposable subgroup in G, and let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be its primary decomposition. If Q_i contains the intersection of the remaining Q_j it may be left out altogether, and hence we obtain an irredundant primary decomposition of A. An irredundant primary decomposition, in which the radicals of the various primary components are all different, is called a *normal decomposition.* Each primary decomposition, as is easily seen from Proposition 2.3, can be refined into one which is normal. Furthermore, we can prove that the number of primary components and the radicals of primary components of a normal decomposition of A depend only on A and not on the particular normal decomposition considered. This is one of our main theorems and is also a generalization of a theorem due to Schenkman [5].

Theorem 6.1. *Suppose that a normal subgroup A in G has a primary decomposition and let*

$$A = Q_1\cap Q_2\cap\cdots\cap Q_m = Q_1'\cap Q_2'\cap\cdots\cap Q_n'$$

be two normal decompositions of A. Then $m=n$, and it is possible to number the components in such a way that $r(Q_i)=r(Q'_i)$ for $1\leqq i\leqq m=n$.

Proof. We use induction on the number m of primary components. If $m=1$, then $Q_1=Q'_1\cap Q'_2\cap\cdots\cap Q'_n$, and moreover if $n>1$, then it follows from Proposition 2.4 that the right side of the equality above is not a primary subgroup, which is impossible since Q_1 is a primary subgroup. Similarly, $n=1$ implies $m=1$, and if this is the case, the assertion is trivial. We may suppose therefore that $m>1$, in which case all the primary subgroups $Q_1, Q_2,\cdots,Q_m, Q'_1, Q'_2,\cdots,Q'_n$ are proper subgroups. Among the radicals $r(Q_1), r(Q_2),\cdots,r(Q_m), r(Q'_1), r(Q'_2),\cdots,r(Q'_n)$ choose one which is not strictly contained in any of the others. Without loss of generality we may assume that it is $r(Q_1)$. We will now show that $r(Q_1)$ must occur among $r(Q'_1), r(Q'_2),\cdots,r(Q'_n)$. Otherwise we could form the residual quotients by Q_1:

$$(Q_1:Q_1)\cap(Q_2:Q_1)\cap\cdots\cap(Q_m:Q_1)=(Q'_1:Q_1)\cap(Q'_2:Q_1)\cap\cdots\cap(Q'_n:Q_1).$$

Now, for $1<i\leqq m$, $Q_1\nsubseteq r(Q_i)$, since otherwise $r(Q_1)\subsetneqq r(Q_i)$ contrary to the choice of $r(Q_1)$. Similarly, for $1\leqq j\leqq n$, it follows that $Q_1\nsubseteq r(Q'_j)$. By Proposition 2.8 (2), we have

$$Q_i:Q_1=Q_i \quad \text{for} \quad 1<i\leqq m, \qquad Q'_j:Q_1=Q'_j \quad \text{for} \quad 1\leqq j\leqq m.$$

Furthermore, since $Q_1:Q_1=G$, it follows that

$$Q_2\cap Q_3\cap\cdots\cap Q_m=Q'_1\cap Q'_2\cap\cdots\cap Q'_n=A.$$

This contradicts the assumption that the given decomposition is normal.

Let us now assume that $m\leqq n$. We shall show that $m=n$ and by a suitable ordering $r(Q_i)=r(Q'_i)$ for $1\leqq i\leqq m=n$. Let us assume that these results are valid for normal subgroups which may be represented by fewer than m primary subgroups. We arrange the Q_i and Q'_j so that $r(Q_1)=r(Q'_1)$. Put $Q=Q_1\cap Q'_1$ then, by Proposition 2.3, Q is a primary subgroup of G such that $r(Q)=r(Q_1)=r(Q'_1)$. Also,

$$\begin{aligned} Q_i:Q&=Q_i \quad \text{for} \quad 1<i\leqq m,\\ Q_1:Q&=G. \end{aligned}$$

For the first relation follows from the fact that, since $r(Q_1)\nsubseteq r(Q_i)$, Q is not contained in $r(Q_i)$, while the second follows from $Q\subseteqq Q_1$. Consequently $A:Q=Q_2\cap Q_3\cap\cdots\cap Q_m$. An exactly similar argument shows that $A:Q=Q'_2\cap Q'_3\cap\cdots\cap Q'_n$, and hence

$$Q_2\cap Q_3\cap\cdots\cap Q_m=Q'_2\cap Q'_3\cap\cdots\cap Q'_n,$$

and moreover both decompositions are normal. Hence by the induction hypothesis we have $m-1=n-1$, that is, $m=n$. Furthermore, by a suitable ordering we have $r(Q_i)=r(Q'_i)$ for $1<i\leqq m=n$. Since $r(Q_1)=r(Q'_1)$, the proof is completed.

7. Uniqueness theorem for tertiary decompositions

Lemma 7.1. *Let $A=T\cap B=T'\cap B'$, where T and T' are tertiary subgroups of G with $t(T)\neq t(T')$. Then $A=B\cap B'$.*

Proof. It is sufficient to prove that $B\cap B'\subseteqq A$. Suppose that $a\in B\cap B'$, and that, for instance, $t(T)\nsubseteq t(T')$. Then there exists an element b in $t(T)$ but not in $t(T')$. If $a\notin A$, then $a\notin T$ and therefore we can choose an element $a'\in(a)$ such that $a'\notin T$ and $[(b),(a')]\subseteqq T$. It follows that $[(b),(a')]\subseteqq T\cap B=T'\cap B'$. But, on one hand, we have $a'\notin T'$, for otherwise we should have $a'\in T'\cap B'=T\cap B\subseteqq T$. Since T' is a tertiary subgroup, b is in $t(T')$, which is a contradiction.

Theorem 7.2. *Let A be a normal subgroup in G and let*

$$A = T_1\cap T_2\cap\cdots\cap T_n = T'_1\cap T'_2\cap\cdots\cap T'_m$$

be two normal tertiary decompositions of A. Then $n=m$, and it is possible to number the components in such a way that $t(T_i)=t(T'_i)$ for $1\leqq i\leqq n=m$.

Proof. It is sufficient to show that $t(T_1)$, for example, is equal to some $t(T'_j)$. If it is not, we have the following relations:

$$\begin{array}{ll} (1) & t(T_1)\neq t(T'_1), \\ (2) & t(T_1)\neq t(T'_2), \\ \cdots & \cdots\cdots\cdots\cdots \\ (m) & t(T_1)\neq t(T'_m). \end{array}$$

Using the relation (1), we obtain from Lemma 7.1 that

$$A = T'_2\cap T'_3\cap\cdots\cap T'_m\cap T_2\cap T_3\cap\cdots\cap T_n.$$

From the relation (2) and the equality

$$A = T'_2\cap\cdots\cap T'_m\cap T_2\cap\cdots\cap T_n = T_1\cap T_2\cap\cdots\cap T_n,$$

we have by the same method

$$A = T'_3\cap\cdots\cap T'_m\cap T_2\cap\cdots\cap T_n.$$

Continuing an exactly similar argument, we obtain after a finite number of steps that

$$A = T_2 \cap \cdots \cap T_n ,$$

which contradicts the fact that T_1 is not superfluous.

8. The isolated components of a normal subgroup

We now make the following definition.

DEFINITION 8.1. Let A be a normal subgroup in G and let M be an m-system. The *isolated component* of A determined by M, or more simply, the *M-component* of A, denoted by A_M, will be defined, if M is not empty, to be the set of all elements x in G such that $[(x), (c)] \subseteqq A$ for at least one $c \in M$. On the other hand, if M is empty, we shall define that $A_M = A$.

Lemma 8.2. *Let A be a normal subgroup in G and let M be an m-system. Then the M-component of A is a normal subgroup in G which contains A.*

Proof. If M is empty, then the assertion is trivial. We may now assume that M is not empty. Let x and y be any two elements in A_M. Then from the definition of the M-component of A there exist some c and d in M such that $[(x), (c)] \subseteqq A$ and $[(y), (d)] \subseteqq A$ respectively. Since M is an m-system, there exist c' in (c) and d' in (d) such as $[c', d']$ is in M. Then

$$\begin{aligned} [(xy), ([c', d'])] &\subseteqq [(x)(y), ([c', d'])] \\ &= [(x), ([c', d'])][(y), ([c', d'])] \\ &\subseteqq [(x), (c)][(y), (d)] \subseteqq A . \end{aligned}$$

Hence xy is in A_M. Since $(x^{-1}) = (x)$ and $(cxc^{-1}) = (x)$ for all c in G, we see that A_M is a normal subgroup in G. Moreover, if a is in A, then certainly $[(a), (c)] \subseteqq A$ for any c in M, which gives us the relation $A \subseteqq A_M$.

Now we shall give a characterization of primary subgroups by means of isolated components.

Proposition 8.3. *Let Q be a normal subgroup in G. Then Q is a primary subgroup of G if and only if for any m-system M either $Q_M = Q$ or $Q_M = G$ holds.*

Proof. Let us suppose that for any m-system M we have either $Q_M = Q$ or $Q_M = G$, and that Q is not a primary subgroup of G. Then we can choose $b \notin Q$ and $c \notin r(Q)$ such that $[(b), (c)] \subseteqq Q$. Since $c \notin r(Q)$, there exists a prime subgroup P which contains Q and does not contain c. If we denote the complement $C(P)$ of P in G by M, then M is an m-system and b is contained in Q_M, and hence $Q \subsetneqq Q_M$ because b is not contained in Q. From our assumption, it follows that Q_M is equal to G, and hence c is contained in Q_M. Thus there exists at least one $d \in M$ such that $[(c), (d)] \subseteqq Q$. Since P is a prime subgroup containing Q, $[(c), (d)] \subseteqq P$, and hence we have either $c \in P$ or $d \in P$, which is impossible in any case.

Conversely, let us suppose that Q is a primary subgroup of G. If M is empty, then the assertion is trivial. Now we may suppose that there exists a non-empty m-system M such that $Q \subsetneqq Q_M \subsetneqq G$. Let b be an element which is contained in Q_M but not in Q. Then there exists at least one $c \in M$ such that $[(b), (c)] \subseteqq Q$. Since Q is a primary subgroup, we have $c \in r(Q)$, and it follows from Proposition 1.15 that $M \cap Q$ is not empty. This shows that $G \subseteqq Q_M$, which is a contradiction.

Let A be a normal subgroup in G. If A has a primary decomposition, then the isolated component of A can be expressed in terms of the decomposition as follows:

Proposition 8.4. *Let A be a normal subgroup in G and let M be an m-system. Suppose that $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, where Q_i is a primary subgroup of G. If, for $m+1 \leqq i \leqq n$, $r(Q_i)$ meet M but not for $1 \leqq i \leqq m$, then $A_M = Q_1 \cap Q_2 \cap \cdots \cap Q_m$.*

Proof. If M is empty, then the assertion is trivial. We may therefore assume that M is not empty. Let x be in A_M. Then $[(x), (c)] \subseteqq A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, where c is a suitable element of M. Consequently, if $1 \leqq i \leqq m$ we have $[(x), (c)] \subseteqq Q_i$ and $c \notin r(Q_i)$, which shows that x is in Q_i, and hence $A_M \subseteqq Q_1 \cap Q_2 \cap \cdots \cap Q_m$.

Now let $y \in Q_1 \cap Q_2 \cap \cdots \cap Q_m$. For each $j > m$, we can choose $c_j \in Q_j \cap M$, since, from Proposition 1.15, $M \cap r(Q_j) \neq \phi$ implies $M \cap Q_j \neq \phi$. Since M is an m-system, there exist c'_{m+1} in (c_{m+1}) and c'_{m+2} in (c_{m+2}) such that $c''_{m+2} = [c'_{m+1}, c'_{m+2}]$ is in $Q_{m+1} \cap Q_{m+2} \cap M$. Similarly, there exist c^*_{m+2} in (c''_{m+2}) and c'_{m+3} in (c_{m+3}) such that $c''_{m+3} = [c^*_{m+2}, c'_{m+3}]$ is in $Q_{m+1} \cap Q_{m+2} \cap Q_{m+3} \cap M$. Continuing an exactly similar argument, we obtain after a finite number of steps an element c''_n which is in $Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n \cap M$. Then

$$[(y), (c''_n)] \subseteqq (Q_1 \cap Q_2 \cap \cdots \cap Q_m) \cap (Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n) = A .$$

Since c''_n is in M, y is in A_M, and hence $Q_1 \cap Q_2 \cap \cdots \cap Q_m \subseteqq A_M$. This completes the proof of the proposition.

Let M be an m-system. Combining this proposition with Proposition 8.3, we obtain that if Q is a primary subgroup of G, then Q_M is G or Q according as $r(Q)$ meets or does not meet M.

From Proposition 8.4, we see at once

Corollary 8.5. *A decomposable subgroup has at most a finite number of isolated components.*

Suppose that a normal subgroup A in G has a primary decomposition, and let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a normal decomposition of A. Then as we have proved in Theorem 6, 1, the number of primary components and the radicals of primary components depend only on A and not on the particular normal decomposition considered. A subset $\{r(Q_1), r(Q_2),\cdots, r(Q_m)\}$ of the radicals is called an *isolated set* of A, if for $m+1\leqq j\leqq n$, each $r(Q_j)$ is not contained in any of $r(Q_i)$ for $1\leqq i\leqq m$.

For examples, each minimal element of the set $\{r(Q_1), r(Q_2),\cdots,r(Q_n)\}$ forms on its own an isolated set of A, and the $r(Q_i)$ which do not meet a given set of elements will also form an isolated set of A.

We now prove

Proposition 8.6. *Suppose that a normal subgroup A in G has a primary decomposition. Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a normal decomposition of A, and let $r(Q_i)=\bigcap_k P_{ik}$ be the expression of $r(Q_i)$ as the intersection of all minimal prime subgroups belonging to Q_i. Then the following conditions are equivalent:*

(1) *for $m+1\leqq j\leqq n$, each $r(Q_j)$ is not contained in any of $r(Q_i)$ for $1\leqq i\leqq m$,*

(2) *for each Q_i, $1\leqq i\leqq m$, there exists at least one minimal prime subgroup $P_{ik_i}=P_i^*$ such that P_i^* does not contain P_{jk} for all j, $m+1\leqq j\leqq n$, and for all k,*

(3) *each $r(Q_i)$, $1\leqq i\leqq m$, does not contain the intersection $Q_{m+1}\cap Q_{m+2}\cap\cdots\cap Q_n$.*

Proof. (1)$\Rightarrow$(2). Assume that there exists some i, $1\leqq i\leqq m$, such that for some j, $m+1\leqq j\leqq n$ and some k_j, P_{jk_j} is contained in P_{ik} for all k. Then $P_{ik}\supseteqq P_{jk_j}\supseteqq r(Q_j)$ for all k. Hence $r(Q_j)$ is contained in $r(Q_i)$, which is a contradiction.

(2)$\Rightarrow$(3). Assume that there exists some i, $1\leqq i\leqq m$, such that $r(Q_i)$ contains the intersection $Q_{m+1}\cap Q_{m+2}\cap\cdots\cap Q_n$. Then by Corollary 1.4 we can choose some j, $m+1\leqq j\leqq n$, such that $P_i^*\supseteqq Q_j$. As we have remarked in §1, P_i^* contains at least one of minimal prime subgroups

P_{js} belonging to Q_j, which is a contradiction.

(3)$\Rightarrow$(1). Assume that for some i, $1 \leqq i \leqq m$, and some j, $m+1 \leqq j \leqq n$, $r(Q_i)$ contains $r(Q_j)$. Then $r(Q_i) \supseteqq r(Q_j) \supseteqq Q_j \supseteqq Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n$. This is a contradiction.

We come now to the second uniqueness theorem for normal decompositions.

Theorem 8.7. *Suppose that a normal subgroup A in G has a primary decomposition, and let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a normal decomposition of A. If $\{r(Q_1), r(Q_2), \cdots, r(Q_m)\}$ is an isolated set of A, then $Q_1 \cap Q_2 \cap \cdots \cap Q_m$ depends only on $r(Q_1), r(Q_2), \cdots, r(Q_m)$ and not on the particular normal decomposition of A.*

Proof. Let

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n$$

be two normal decompositions of A such that $r(Q_i) = r(Q'_i)$ for all i, $1 \leqq i \leqq n$. If we denote $Q_{m+1} \cap Q_{m+2} \cap \cdots \cap Q_n$ and $Q'_{m+1} \cap Q'_{m+2} \cap \cdots \cap Q'_n$ by Q and Q' respectively, then, by Proposition 8.6 (3), Q is not contained in any of $r(Q_i)$ for $1 \leqq i \leqq m$, and hence, it follows from Proposition 2.8 (2) that $Q_i : Q = Q_i$ and $Q'_i : Q = Q'_i$ for all i, $1 \leqq i \leqq m$. But on one hand, since $Q_j \supseteqq Q$ for $m+1 \leqq j \leqq n$, $Q_j : Q = G$. These relations show that $A : Q = (Q_1 \cap Q_2 \cap \cdots \cap Q_n) : Q = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ and $A : Q = (Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n) : Q = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m \cap (Q' : Q)$. Thus we have $Q_1 \cap Q_2 \cap \cdots \cap Q_m \subseteqq Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m$, and similarly $Q_1 \cap Q_2 \cap \cdots \cap Q_m \supseteqq Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m$, which completes the proof.

Corollary 8.8. *Let $r(Q)$ be a minimal element in the set $\{r(Q_1), r(Q_2), \cdots, r(Q_n)\}$ of the radicals of the primary components of A. Then the primary component corresponding to $r(Q)$ is the same for all normal decompositions of A.*

We shall show by means of an example that the corresponding assertion for a non-minimal element $r(Q)$ is false. Let us consider the group $\mathfrak{S}_5 \times \mathfrak{S}_5$ mentioned in the introduction. Then $\mathfrak{A}_5 \times 1$ has two normal decompositions:

$$\begin{aligned} \mathfrak{A}_5 \times 1 &= (\mathfrak{S}_5 \times 1) \cap (\mathfrak{A}_5 \times \mathfrak{S}_5) \\ &= (\mathfrak{S}_5 \times 1) \cap (\mathfrak{A}_5 \times \mathfrak{A}_5), \end{aligned}$$

and both $\mathfrak{A}_5 \times \mathfrak{S}_5$ and $\mathfrak{A}_5 \times \mathfrak{A}_5$ have the same radical G which is not a minimal element in the set of the radicals of the primary components of $\mathfrak{A}_5 \times 1$.

Finally, from Theorem 8.7, we prove

Proposition 8.9. *Let A be a decomposable subgroup with uniquely determined radicals $R_1, R_2, \cdots, R_n$ of the primary components of its normal decomposition. Suppose that for each i, $1 \leqq i \leqq n$, there exists a normal decomposition of A,*

$$A = Q_1^{(i)} \cap Q_2^{(i)} \cap \cdots \cap Q_n^{(i)},$$

such that the radical of $Q_j^{(i)}$ is R_j for $1 \leqq j \leqq n$. Then A has a normal decomposition:

$$A = Q_1^{(1)} \cap Q_2^{(2)} \cap \cdots \cap Q_n^{(n)}.$$

Proof. Let R_{i_1} be a minimal element of the set $\{R_1, R_2, \cdots, R_n\}$. Then R_{i_1} forms on its own an isolated set of A, and hence by Theorem 8.7

$$(1) \qquad Q_{i_1}^{(1)} = Q_{i_1}^{(2)} = \cdots = Q_{i_1}^{(n)}.$$

Next, from the set $\{R_1, R_2, \cdots, R_{i_1-1}, R_{i_1+1}, \cdots, R_n\}$, we select R_{i_2} which is minimal in this set. Then the set $\{R_{i_1}, R_{i_2}\}$ forms also an isolated set of A, and hence again by Theorem 8.7, we have

$$(2) \qquad Q_{i_1}^{(1)} \cap Q_{i_2}^{(1)} = Q_{i_1}^{(2)} \cap Q_{i_2}^{(2)} = \cdots = Q_{i_1}^{(n)} \cap Q_{i_2}^{(n)}.$$

Using an exactly similar argument repeatedly, we obtain that for each k, $1 \leqq k \leqq n$,

$$\begin{aligned} Q_{i_1}^{(1)} \cap Q_{i_2}^{(1)} \cap \cdots \cap Q_{i_k}^{(1)} &= Q_{i_1}^{(2)} \cap Q_{i_2}^{(2)} \cap \cdots \cap Q_{i_k}^{(2)} \\ &= \cdots\cdots\cdots \\ &= Q_{i_1}^{(n)} \cap Q_{i_2}^{(n)} \cap \cdots \cap Q_{i_k}^{(n)}. \end{aligned}$$

Then $Q_1^{(1)} \cap Q_2^{(2)} \cap \cdots \cap Q_n^{(n)} = Q_{i_1}^{(i_1)} \cap Q_{i_2}^{(i_2)} \cap \cdots \cap Q_{i_n}^{(i_n)}$ is, by the relation (1) above, equal to $Q_{i_1}^{(i_2)} \cap Q_{i_2}^{(i_2)} \cap Q_{i_3}^{(i_3)} \cap \cdots \cap Q_{i_n}^{(i_n)}$, and by the relation (2) above this is equal to $Q_{i_1}^{(i_3)} \cap Q_{i_2}^{(i_3)} \cap Q_{i_3}^{(i_3)} \cap Q_{i_4}^{(i_4)} \cap \cdots \cap Q_{i_n}^{(i_n)}$. Continuing an exactly similar argument, we obtain $Q_1^{(1)} \cap Q_2^{(2)} \cap \cdots \cap Q_n^{(n)} = Q_{i_1}^{(i_n)} \cap Q_{i_2}^{(i_n)} \cap \cdots \cap Q_{i_n}^{(i_n)} = A$. It is easily seen from Theorem 6.1 that this decomposition is normal.

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