Tanabe, H.
Osaka J. Math.
2 (1965), 191-204

NOTE ON UNIQUENESS OF SOLUTIONS OF DIFFERENTIAL INEQUALITIES OF PARABOLIC TYPE

HIROKI TANABE

(Received March 16, 1965)

Recently A. Friedman [3] proved some result on the uniqueness of solutions of ordinary differential inequalities

$$\left\| \frac{1}{i}\frac{du}{dt} - A(t)u \right\| \leqq \eta \| A(t)u \| + K\| u \| \tag{0.1}$$

in a Hilbert space, and as its application he showed that a certain uniqueness theorem holds for differential inequalities of parabolic type

$$\left\| \frac{\partial u}{\partial t} - P(x, t, D_x)u \right\|_0 \leqq \eta \| u(t) \|_{2m} + K \| u(t) \|_{2m-1} \tag{0.2}$$

in the class of functions satisfying some type of time-independent boundary conditions

$$B_j(x, D_x)u(x, t) = 0, \quad x \in \partial\Omega, \quad j = 1, \cdots, m, \tag{0.3}$$

where $\partial\Omega$ is the boundary of a bounded domain Ω and $\| \ \|_k$ stands for the usual norm of $H_k(\Omega) = W_2^k(\Omega)$. In [3] $A(t)$ is assumed to have a constant domain, and on this account it was required that the boundary conditions (0.3) did not depend on t. In this paper it will not be attempted to extend the above result concerning the inequalitites (0.1) in a Hilbert space to the case in which $A(t)$ has a variable domain; however, it will be shown that a similar result remains valid for more general differential inequalities of parabolic type with time-dependent boundary conditions

$$\| A(x, t, D_x, D_t)u \|_0 \leqq \eta \sum_{k=0}^{l} \| D_t^{l-k} u \|_{kd} + K \sum_{k=0}^{l-1} \| D_t^{l-k-1} u \|_{kd}, \tag{0.4}$$

$$B_j(x, t, D_x, D_t)u(x, t) = 0, \quad x \in \partial\Omega, \quad j = 1, \cdots, m. \tag{0.5}$$

Here $A(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}(x, t, D_x) D_t^k$ is a linear $d(=2m/l)$-parabolic differential operator in the sense of I. G. Petrowskii, or, what amounts to the same thing, for each t and θ with $-\pi \leqq \theta \leqq 0$ $A(x, t, D_x, e^{i\theta} D_y^d)$

is an elliptic operator of order $2m$ in $(x, y)\in\Omega\times(-\infty, \infty)$. Our main result is briefly described as follows: if for each t and θ with $-\pi\leqq\theta\leqq 0$ the Complementing Condition ([2]) is satisfied by $(A(x, t, D_x, e^{i\theta}D_y^\alpha)$, $\{B_j(x, t, D_x, e^{i\theta}D_y^\alpha)\})$ in $\bar{\Omega}\times(-\infty, \infty)$ and if η is sufficiently small, then any solution of (0.4)-(0.5) satisfying the initial conditions $D_t^k u(0)=0$, $k=0, 1, \cdots, l-1$, identically vanishes for $t>0$. The restriction on the smoothess in t of the coefficients is about the same as that of [3] and any Hölder contiunity of the coefficients of A will not be assumed. In the proof of the statement above essential use is made of an estimate which holds for solutions of some reduced weighted elliptic inhomogeneous boundary value problem (Lemma 2.1 below) and which can be proved in the same manner as Theorem 5.2 of [1]. As in [3] we shall use Fourier transforms with respect to t in order to obtain the necessary estimate, and in the present case we are obliged to treat such transforms of a class of functions which do not satisfy the homogeneous boundary conditions since the coefficients of the boundary operators are allowed to be dependent on t. Some results similar to Theorems 2 and 3 of [3] on the decreasing and uniqueness properties of solutions of (0.4)-(0.5) in an infinite domain $\Omega\times(-\infty, T]$ will also be obtained.

1. Assumptions. We denote by Ω a domain in the n-dimensional Euclidean space and by $\partial\Omega$ its boundary. For any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ we write $D_x^\alpha=\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and similarly $D_t=\frac{1}{i}\frac{\partial}{\partial t}$. $|\alpha|$ stands for the length of α: $|\alpha|=\alpha_1+\cdots+\alpha_n$. For any integer k we denote by $H_k(\Omega)$ the class of all complex valued functions whose distribution derivatives of order up to k are square integrable in Ω, the norm of $H_k(\Omega)$ being denoted by $\|\ \|_{k,\Omega}$:

$$\|u\|_{k,\Omega}^2 = \sum_{|\alpha|\leqq k}\int_\Omega |D_x^\alpha u(x)|^2 dx\,.$$

Let l and m be natural numbers and let $d=2m/l$. $A(x, t, D_x, D_t)$ is a linear differential operator of the form

$$A(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}(x, t, D_x)D_t^k$$

where for $0\leqq k<l$

$$A_{l-k}(x, t, D_x) = \sum_{|\alpha|\leqq 2m-kd} a_{l-k,\alpha}(x, t)D_x^\alpha$$

is a differential operator in x of order $2m-kd$ at the most with coefficients defined in $\bar{\Omega}\times(-\infty, T]$ and $A_0=1$, Let $m_j, j=1, \cdots, m$, be non-negative integers $<2m$ and let $l_j=[m_j/d]=$ the integral part of m_j/d. For $j=1, \cdots, m$, $B_j(x, t, D_x, D_t)$ is a linear differential operator of the form

$$B_j(x, t, D_x, D_t) = \sum_{k=0}^{l_j} B_{j,l_j-k}(x, t, D_x) D_t^k$$

where for $k=0, \cdots, l_j$

$$B_{j,l_j-k}(x, t, D_x) = \sum_{|\beta| \leqq m_j - kd} b_{j,l_j-k,\beta}(x, t) D_x^\beta$$

is a differential operator in x of order $m_j - kd$ at the most with coefficients defined on $\partial\Omega \times (-\infty, T]$. In what follows we suppose without restriction that the coefficients of $B_j(x, t, D_x, D_t)$ are defined in the whole of $\bar{\Omega} \times (-\infty, T]$.

Let y be an auxiliary real variable and by Γ we denote the infinite cylinder: $\Gamma = \Omega \times \{y : -\infty < y < \infty\}$. Then for each t and θ $A(x, t, D_x, e^{i\theta} D_y^\alpha)$ is a linear differential operator in $(x, y) \in \Gamma$ of order $2m$. Similarly $B_j(x, t, D_x, e^{i\theta} D_y^\alpha)$ is a linear differential operator in $(x, y) \in \Gamma$ of order m_j at the most.

We introduce the following assumptions.

(I) $A(x, t, D_x, D_t)$ is a d-parabolic operator in $\bar{\Omega} \times (-\infty, T]$, i.e. for each fixed $t \in (-\infty, T]$ and θ with $-\pi \leqq \theta \leqq 0$ $A(x, t, D_x, e^{i\theta} D_y^\alpha)$ is an elliptic operator in $(x, y) \in \bar{\Gamma}$.

(II) For $j=1, \cdots, m$, $B_j(x, t, D_x, e^{i\theta} D_y^\alpha)$ is a differential operator in $(x, y) \in \bar{\Gamma}$ of order m_j for each $t \in (-\infty, T]$ and θ with $-\pi \leqq \theta \leqq 0$.

(III) The Complementing Condition ([2]) is satisfied by $(A(x, t, D_x, e^{i\theta} D_y^\alpha), \{B_j(x, t, D_x, e^{i\theta} D_y^\alpha)\}, \Gamma)$ uniformly in $t \in (-\infty, T]$ and θ with $-\pi \leqq \theta \leqq 0$.

(IV) The coefficients of $A(x, t, D_x, D_t)$ are uniformly continuous in $\bar{\Omega} \times (-\infty, T]$. As regards the coefficients of $\{B_j(x, t, D_x, D_t)\}$

$$\left.\begin{array}{ll} D_x^\kappa b_{j,l_j-k,\beta}(x, t), & |\kappa| \leqq 2m - m_j \\ D_t^i b_{j,l_j-k,\beta}(x, t), & i = 0, \cdots, l+1 \end{array}\right\}, \quad \begin{array}{ll} |\beta| \leqq m_j - kd, & k = 0, \cdots, l_j, \\ j=1, \cdots, m & \end{array}$$

are uniformly continuous in $\bar{\Omega} \times (-\infty, T]$.

(V) Ω is a bounded domain of class C^{2m}.

From (I) it follows that d is an even integer. In what follows we shall denote by $C_0, C_1, \cdots$ constants depending only on the assumptions above and by M those dependent only on l and m. (IV) implies the existence of a continuous function $\varepsilon(r)$ satisfying $\varepsilon(r) > 0$ for $r > 0$, $\varepsilon(0) = 0$ such that

$$\left.\begin{array}{l}|a_{l-k,\alpha}(x,t)-a_{l-k,\alpha}(x,s)|\leqq\varepsilon(|t-s|)\\ \qquad |\alpha|\leqq 2m-kd,\quad k=0,\cdots,l-1,\\ |D_x^\kappa b_{j,l_j-k,\beta}(x,t)-D_x^\kappa b_{j,l_j-k,\beta}(x,s)|\leqq\varepsilon(|t-s|)\\ \qquad |\beta|\leqq m_j-kd,\quad k=0,\cdots,l_j,\quad |\kappa|\leqq 2m-m_j,\quad j=1,\cdots,m,\\ \qquad\qquad x\in\bar{\Omega},\quad -\infty<t,s<T.\end{array}\right\}$$

By a solution u of the boundary value problem of the form

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x\in\Omega,$$
$$B_j(x, t, D_x, D_t)u(x, t) = g_j(x,t), \quad x\in\partial\Omega, \quad j = 1, \cdots, m,$$

we shall always mean one such that $D_t^k u(t)$ belongs to $H_{2m-kd}(\Omega)$ and is strongly continuous in t there for $k=0, \cdots, l$.

2. Preliminary lemma. We begin this section with the following lemma.

Lemma 2.1. *Suppose that the differential operators* $A(x, D_x, D_t)$, $\{B_j(x, D_x, D_t)\}$ *with time-independent coefficients satisfy all the assumptions of section* 1. *If the function* $u(x, \lambda)$ *of* λ *with values belonging to* $H_{2m}(\Omega)$ *satisfies*

$$A(x, D_x, \lambda)u(x,\lambda) = f(x, \lambda), \quad x\in\Omega, \quad \operatorname{Im}\lambda\leqq 0,$$
$$B_j(x, D_x, \lambda)u(x,\lambda) = g_j(x, \lambda), \quad x\in\partial\Omega, \quad \operatorname{Im}\lambda\leqq 0, \quad j=1, \cdots, m.$$

where f *and* g_j, $j=1, \cdots, m$, *are functions of* λ *with values in* $L^2(\Omega)$ *and* $H_{2m-m_j}(\Omega)$ *respectively, then there exists a constant* $N>0$ *independent of* u *such that for* λ *with* $\operatorname{Im}\lambda\leqq 0$, $|\lambda|\geqq N$

$$\begin{aligned}\sum_{k=0}^{2m}|\lambda|^{(2m-k)/d}\|u(\lambda)\|_k&\leqq C_0\{\|f(\lambda)\|_0\\ &+\sum_{j=1}^{m}|\lambda|^{(2m-m_j)/d}\|g_j(\lambda)\|_0+\sum_{j=1}^{m}\|g_j(\lambda)\|_{2m-m_j}\}.\end{aligned}\tag{2.1}$$

Proof. (2.1) is a consequence of Agmon-Douglis-Nirenberg inquality applied to $(A(x, D_x, e^{i\theta}D_y^d), \{B_j(x, D_x, e^{i\theta}D_y^d)\}, \Gamma)$, $-\pi\leqq\theta\leqq 0$, and the function $\varphi(y)e^{i\mu y}u(x, \mu^d e^{i\theta})$ where φ is a smooth function such that $\varphi=1$ for $|y|\leqq 1/2$ and $\varphi=0$ for $|y|\geqq 1$ and μ is an arbitrary real number (cf. Theorem 5.2 of [1]).

Let u be a solution of

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x\in\Omega, \quad -\infty<t<T, \tag{2.2}$$
$$B_j(x, t, D_x, D_t)u(x, t) = 0, \quad x\in\partial\Omega, \quad -\infty<t<T, \quad j=1, \cdots, m. \tag{2.3}$$

Given $\delta>0$ let $\zeta(t)$ be a smooth function satisfying $\zeta(t)=0$ if $t\leqq 0$ or $t\geqq 5\delta$, $\zeta(t)=1$ if $\delta\leqq t\leqq 4\delta$, and $|D_t^k\zeta(t)|\leqq K_0\delta^{-k}$ for $k=0, \cdots, l+1$ and all t. Taking c, δ such that $c+5\delta<T$, we write $\zeta_c(t)=\zeta(t-c)$ and $v(x, t)=\zeta_c(t)u(x, t)$. Then v is a solution of

$$A(x, t, D_x, D_t)v(x, t) = F(x, t), \quad x\in\Omega, \quad -\infty<t<\infty, \tag{2.4}$$

$$B_j(x, t, D_x, D_t)v(x, t) = G_j(x, t), \quad x\in\partial\Omega, \quad -\infty<t<\infty, \quad j=1, \cdots, m, \tag{2.5}$$

where

$$F = \zeta_c f+\sum_{k=1}^{l} A_{l-k}(x, t, D_x)\sum_{p=0}^{k-1}\binom{k}{p}\zeta_c^{(k-p)}D_t^p u, \tag{2.6}$$

$$G_j = \sum_{k=1}^{l_j} B_{j,l_j-k}(x, t, D_x)\sum_{p=0}^{k-1}\binom{k}{p}\zeta_c^{(k-p)}D_t^p u. \tag{2.7}$$

For a function h of (x, t) we denote by $\hat{h}(x, \lambda)$ its Fourier transform with respect to t:

$$\hat{h}(x, \lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda t}h(x, t)dt.$$

Lemma 2.2. *If δ is sufficiently small and then if $-\sigma$ is sufficiently large depending on the choice of δ, we have*

$$\begin{aligned}&\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k)^2 d\lambda \\ &\leqq C_1\Big\{\int_{-\infty}^{\infty} e^{2\sigma t}\|F(t)\|_0^2 dt+\sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{G}_j(\lambda+i\sigma)\|_0)^2 d\lambda \\ &+\sum_{j=1}^{m}\int_{-\infty}^{\infty} e^{2\sigma t}\|G_j(t)\|_{2m-m_j}^2 dt\Big\}.\end{aligned} \tag{2.8}$$

Proof. Clearly v satisfies

$$A(x, c, D_x, D_t)v(x, t) = H(x, t), \quad x\in\Omega, \quad -\infty<t<\infty, \tag{2.9}$$

$$B_j(x, c, D_x, D_t)v(x, t) = \Phi_j(x, t), \quad x\in\partial\Omega, \quad -\infty<t<\infty, \quad j=1, \cdots, m, \tag{2.10}$$

where

$$H = F+\sum_{k=0}^{l}(A_{l-k}(x, c, D_x)-A_{l-k}(x, t, D_x))D_t^k v,$$

$$\Phi_j = G_j+\sum_{k=0}^{l_j}(B_{j,l_j-k}(x, c, D_x)-B_{j,l_j-k}(x, t, D_x))D_t^k v.$$

Taking the Fourier transforms of both sides of (2.9), (2.10) it follows that for any complex number λ

$$A(x, c, D_x, \lambda)\hat{v}(x, \lambda) = \hat{H}(x, \lambda), \quad x\in\Omega,$$

$$B_j(x, c, D_x, \lambda)\hat{v}(x, \lambda) = \hat{\Phi}_j(x, \lambda), \quad x\in\partial\Omega, \quad j=1, \cdots, m.$$

If λ, σ are real and $\sigma\leqq -N$, then by Lemma 2.1

$$\begin{aligned}&\sum_{k=0}^{2m}|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k\leqq C_0\{\|\hat{H}(\lambda+i\sigma)\|_0\\&+\sum_{j=1}^{m}|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{\Phi}_j(\lambda+i\sigma)\|_0+\sum_{j=1}^{m}\|\hat{\Phi}_j(\lambda+i\sigma)\|_{2m-mj}\}.\end{aligned} \tag{2.11}$$

Squaring and integrating both sides of (2.11) and using Plancherel theorem, we get

$$\begin{aligned}&\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k)^2d\lambda\\&\leqq C_2\Big\{\int_{-\infty}^{\infty}e^{2\sigma t}\|H(t)\|_0^2dt+\sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{\Phi}_j(\lambda+i\sigma)\|_0)^2d\lambda\\&+\sum_{j=1}^{m}\int_{-\infty}^{\infty}e^{2\sigma t}\|\Phi_j(t)\|_{2m-m_j}^2dt\}.\end{aligned} \tag{2.12}$$

From the following easily derived inequalities

$$\|H(t)\|_0\leqq\|F(t)\|_0+C_3\varepsilon(5\delta)\sum_{k=0}^{l}\|D_t^kv(t)\|_{2m-kd},$$

$$\|\Phi_j(t)\|_{2m-m_j}\leqq\|G_j(t)\|_{2m-m_j}+C_3\varepsilon(5\delta)\sum_{k=0}^{l_j}\|D_t^kv(t)\|_{2m-kd},$$

it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}e^{2\sigma t}\|H(t)\|_0^2dt&\leqq 2\int_{-\infty}^{\infty}e^{2\sigma t}\|F(t)\|^2dt\\&+C_4\varepsilon(5\delta)^2\sum_{k=0}^{l}\int_{-\infty}^{\infty}e^{2\sigma t}\|D_t^kv(t)\|_{2m-kd}^2dt,\end{aligned} \tag{2.13}$$

$$\begin{aligned}\int_{-\infty}^{\infty}e^{2\sigma t}\|\Phi_j(t)\|_{2m-m_j}^2dt&\leqq 2\int_{-\infty}^{\infty}e^{2\sigma t}\|G_j(t)\|_{2m-m_j}^2dt\\&+C_5\varepsilon(5\delta)^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}e^{2\sigma t}\|D_t^kv(t)\|_{2m-kd}^2dt.\end{aligned} \tag{3.14}$$

Let $\xi(t)$ be a smooth real valued function such that $\xi(t)=0$ if $t<-\delta$ or $t>6\delta$, $\xi(t)=1$ if $0<t<5\delta$, and

$$|D_t^k\xi(t)|\leqq K_0\delta^{-k}, \quad k=1, \cdots, l+1, \quad -\infty<t<\infty. \tag{2.15}$$

If we write $\xi_c(t)=\xi(t-c)$, then $\xi_c=1$ on the support of v, hence

$$\Phi_j = G_j+\sum_{k=0}^{l_j}\sum_{|\beta|\leqq m_j-kd}\gamma_{j,l_j-k,\beta}(x, c, t)D_x^\beta D_t^kv \tag{2.16}$$

where

$$\gamma_{j,l_j-k,\beta}(x, c, t) = \xi_c(t)(b_{j,l_j-k,\beta}(x, c)-b_{j,l_j-k,\beta}(x, t)).$$

As is easily seen

$$\int_{-\infty}^{\infty}|\hat{\gamma}_{j,l_j-k,\beta}(x, c, \lambda)|d\lambda \leqq C_6\delta\,, \tag{2.17}$$

$$\int_{-\infty}^{\infty}|\lambda|^{(2m-m_j)/d}|\hat{\gamma}_{j,l_j-k,\beta}(x, c, \lambda)|d\lambda \leqq C_7\delta^{1-(2m-m_j)/d}\,. \tag{2.18}$$

In view of (2.17), (2.18) as well as the following lemma:

Lemma 2.3. *If f and g are entire functions of exponential type, decreasing sufficiently rapidly on the real line, then for any real number σ and $\gamma>0$*

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{\gamma}|(f*g)(\lambda+i\sigma)|^2)d\lambda\right)^{1/2}\\ &\leqq 2^{\gamma}\int_{-\infty}^{\infty}|f(\lambda)|d\lambda\left(\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{\gamma}|g(\lambda+i\sigma)|)^2d\lambda\right)^{1/2}\\ &+2^{\gamma}\int_{-\infty}^{\infty}|\lambda|^{\gamma}|f(\lambda)|d\lambda\left(\int_{-\infty}^{\infty}|g(\lambda+i\sigma)|^2d\lambda\right)^{1/2};\end{aligned}$$

we get first integrating in λ and then in x

$$\begin{aligned}&\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}||\hat{\Phi}_j(\lambda+i\sigma)||_0)^2d\lambda\\ &\leqq 2\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}||\hat{G}_j(\lambda+i\sigma)||_0)^2d\lambda \\ &+C_7\delta^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d+k}||\hat{v}(\lambda+i\sigma)||_{m_j-kd})^2d\lambda\\ &+C_9\sum_{k=0}^{l_j}\delta^{2(1-(2m-m_j)/d)}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{k}||\hat{v}(\lambda+i\sigma)||_{m_j-kd})^2d\lambda\\ &=I+II+III\,.\end{aligned} \tag{2.19}$$

It is clear that II is domiated by $C_8\delta^2\times$the left side of (2.12). Substituting

$$\begin{aligned}&|\lambda+i\sigma|^{k}||\hat{v}(\lambda+i\sigma)||_{m_j-kd}\\ &\leqq\delta^{(2m-m_j)/d}|\lambda+i\sigma|^{(2m-m_j+kd)/d}||\hat{v}(\lambda+i\sigma)||_{m_j-kd}+\delta^{-k}||\hat{v}(\lambda+i\sigma)||_{m_j-kd}\\ &\leqq\delta^{(2m-m_j)/d}|\lambda+i\sigma|^{(2m-m_j+kd)/d}||\hat{v}(\lambda+i\sigma)||_{m_j-kd}\\ &+c_0\delta^{(2m-m_j)/d}||\hat{v}(\lambda+i\sigma)||_{2m}+\delta^{-m_j/d}||\hat{v}(\lambda+i\sigma)||_0\end{aligned}$$

in III we get

$$\begin{aligned}III\leqq&\,\delta^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d+k}||\hat{v}(\lambda+i\sigma)||_{m_j-kd})^2d\lambda\\ &+C_{12}\{\delta^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}||\hat{v}(\lambda+i\sigma)||_{2m}^2d\lambda+\delta^{2(1-l)}\int_{-\infty}^{\infty}||\hat{v}(\lambda+i\sigma)||_0^2d\lambda\}\,.\end{aligned} \tag{2.20}$$

Thus we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{\Phi}_j(\lambda+i\sigma)\|_0)^2d\lambda\\&\leqq 2\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{G}_j(\lambda+i\sigma)\|_0)^2d\lambda\\&+C_{13}\delta^2\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k)^2d\lambda\\&+C_{14}\delta^{2(1-l)}\int_{-\infty}^{\infty}\|\hat{v}(\lambda+i\sigma)\|_0^2d\lambda\,.\end{aligned}\tag{2.21}$$

It follows from (2.12), (2.13), (2.14) and (2.21) that if $\sigma\leqq -N$

$$\begin{aligned}&\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k)^2d\lambda\\&\leqq C_{15}\Big\{\int_{-\infty}^{\infty}e^{2\sigma t}\|F(t)\|_0^2dt+\sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-m_j)/d}\|\hat{G}_j(\lambda+i\sigma)\|_0)^2d\lambda\\&+\sum_{j=1}^{m}\int_{-\infty}^{\infty}e^{2\sigma t}\|G_j(t)\|_{2m-m_j}^2dt\Big\}\\&+C_{16}(\delta^2+\varepsilon(5\delta)^2)\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma)\|_k)^2d\lambda\\&+C(\delta)\int_{-\infty}^{\infty}\|\hat{v}(\lambda+i\sigma)\|_0^2d\lambda\,.\end{aligned}\tag{2.22}$$

Thus if δ is so small that $C_{16}(\delta^2+\varepsilon(5\delta)^2)\leqq 1/2$ and if $-\sigma\geqq\max(N, (4C(\delta))^{1/2l})$, then we get (2.8) from (2.2) choosing $C_1=4C_{15}$.

3. Main Theorem. Let u again be a solution of (2.2)-(2.3) and let us continue to use the notations of section 2. Suppose that for some constants η, $K>0$

$$\|f(t)\|_0\leqq\eta\sum_{k=0}^{l}\|D_t^{l-k}u(t)\|_{kd}+K\sum_{k=0}^{l-1}\|D_t^{l-k-1}u(t)\|_{kd},\quad -\infty<t\leqq T\,.\tag{3.1}$$

With the aid of Leibniz formula we get

$$\zeta_c\sum_{k=0}^{l}\|D_t^{l-k}u\|_{kd}\leqq\sum_{k=0}^{l}\|D_t^{l-k}v\|_{kd}+M\sum_{k=0}^{l-1}\sum_{p=0}^{l-k-1}\|\zeta_c^{(l-k-p)}D_t^pu\|_{kd}\tag{3.2}$$

as well as the one with $l-1$ in place of l, where ζ_c and v are functions defined in section 2. From the definition (2.6) of F it immediately follows that

$$\|F\|_0\leqq\zeta_c\|f\|_0+C_{17}\sum_{k=1}^{l}\sum_{p=0}^{k-1}\|\zeta_c^{(k-p)}D_t^pu\|_{2m-kd}\,;$$

hence with the aid of (3.1) and (3.2) we get

$$\begin{aligned}||F||_0 \leqq \eta \sum_{k=0}^{l} ||D_t^{l-k} v||_{kd} + M\eta \sum_{k=0}^{l-1} \sum_{p=0}^{l-k-1} ||\zeta_c^{(l-k-p)} D_t^p u||_{kd} \\ + K\sum_{k=0}^{l-2} ||D_t^{l-1-k} v||_{kd} + MK\sum_{k=0}^{l-1} \sum_{p=0}^{l-k-2} ||\zeta_c^{(l-1-k-p)} D_t^p u||_{kd} \\ + C_{18} \sum_{k=1}^{l} \sum_{p=0}^{k-1} ||\zeta_c^{(k-p)} D_t^p u||_{2m-kd} .\end{aligned} \tag{3.3}$$

Let γ be a real number and write $w = e^{\gamma t} u$. We shall use the notations

$$N_1(w) = \sum_{k=0}^{l-1} \int_c^{c+\delta} ||(D_t + i\gamma)^{l-1-k} w(t)||_{kd}^2 dt,$$

$$N_2(w) = \sum_{k=0}^{l-1} \int_{c+4\delta}^{c+5\delta} ||(D_t + i\gamma)^{l-1-k} w(t)||_{kd}^2 dt .$$

With the aid of (3.3) and $e^{\gamma t} D_t^p u = (D_t + i\gamma)^p w$ we get

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2(\sigma+\gamma)t} ||F||_0^2 dt \leqq C_{19} \Big\{ \eta^2 \sum_{k=0}^{l} \int_{-\infty}^{\infty} e^{2(\sigma+\gamma)t} ||D_t^{l-k} v||_{kd}^2 \\ + K^2 \sum_{k=0}^{l-1} \int_{-\infty}^{\infty} e^{2(\sigma+\gamma)t} ||D_t^{l-k-1} v||_{kd}^2 dt \Big\} \\ + K_0^2 (\eta + C_{19} + K)^2 \delta^{-2l} \{ e^{2\sigma c} N_1(w) + e^{2\sigma(c+4\delta)} N_2(w) \} .\end{aligned} \tag{3.4}$$

Arguing as in the proof of Lemma 2.2 we get

$$\begin{aligned}&\int_{-\infty}^{\infty} (|\lambda + i\sigma + i\gamma|^{(2m-m_j)/d} ||\hat{G}_j(\lambda + i\sigma + i\gamma)||_0)^2 d\lambda \\ &\leqq C_{20} \sum_{k=1}^{l_j} \sum_{p=0}^{k-1} \Big\{ \int_{-\infty}^{\infty} ||(\zeta_c^{(k-p)} D_t^p u)^\wedge (\lambda + i\sigma + i\gamma)||_{m_j-kd}^2 d\lambda \\ &+ \int_{-\infty}^{\infty} (|\lambda + i\sigma + i\gamma|^{(2m-m_j)/d} ||(\zeta_c^{(k-p)} D_t^p u)^\wedge (\lambda + i\sigma + i\gamma)||_{m_j-kd})^2 d\lambda \Big\} .\end{aligned} \tag{3.5}$$

If m_j/d is not an integer, we replace the last integrals in the above by other ones which have only integral powers of $|\lambda|$ so that we may make an immediate application of Plancherel theorem. Noting

$$\frac{2m - m_j}{d} = (l - l_j - 1) \frac{m_j - kd}{(l_j + 1 - k)d} + l \frac{(l_j + 1)d - m_j}{(l_j + 1 - k)d} ,$$

$$||w||_{m_j - kd} \leqq c_0 ||w||_{(l_j+1-k)d}^{(m_j - kd)/(l+1-k)d} ||w||_0^{((l_j+1)d - m_j)/(l_j+1-k)d}$$

for $k = 0, \cdots, l_j$ and any $w \in H_{(l_j+1-k)d}(\Omega)$, we can easily show that for any complex number μ and k, w as above

$$|\mu|^{(2m-m_j)/d} ||w||_{m_j-kd} \leqq c_0 (|\mu|^{l-l_j-1} ||w||_{(l_j+1-k)d} + |\mu|^{l-k} ||w||_0) .$$

Applying the above formula to (3.5) and then using Plancherel theorem we get

$$\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{(2m-m_j)/d}\|\hat{G}_j(\lambda+i\sigma+i\gamma)\|_0)^2 d\lambda$$
$$\leqq MC_{20}\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\left\{\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|\zeta_c^{(k-p)}(t)D_t^p u(t)\|_{2m-d}^2 dt\right.$$
$$+\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|D_t^{l-l_j-1}(\zeta_c^{(k-p)}(t)D_t^p u(t))\|_{(l_j+1-k)d}^2 dt$$
$$\left.+\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|D_t^{l-k}(\zeta_c^{(k-p)}(t)\,D_t^p u(t))\|_0^2 dt\right\}.$$

With the aid of Leibniz formula and by the argument in the derivation of (3.4) we find

$$\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{(2m-m_j)/d}\|\hat{G}_j(\lambda+i\sigma+i\gamma)\|_0)^2 d\lambda$$
$$\leqq C_{21}K_0^2\delta^{-2l}\{e^{2\sigma c}N_1(w)+e^{2\sigma(c+4\delta)}N_2(w)\}. \tag{3.4'}$$

In the same manner we can show that $\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|G_j(t)\|_{2m-m_j}^2 dt$ is also dominated by the right side of (3.4′) with C_{21} possibly replaced by another constant. Hence if δ is sufficiently small and γ is so large that $-\gamma\geqq\max(N, (4C(\delta))^{1/2l})$ (cf. the end of the proof of Lemma 2.2), then by Lemma 2.2 with σ replaced by $\sigma+\gamma$ we get for any $\sigma\leqq 0$

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma+i\gamma)\|_k)^2 d\lambda$$
$$\leqq C_{22}\left\{\eta^2\sum_{k=0}^{l}\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|D_t^{l-k}v\|_{kd}^2 dt\right.$$
$$\left.+K^2\sum_{k=0}^{l-1}\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|D_t^{l-k-1}v\|_{kd}^2 dt\right\} \tag{3.6}$$
$$+K_0^2(C_{23}+\eta+K)^2\delta^{-2l}\{e^{2\sigma c}N_1(w)+e^{2\sigma(c+4\delta)}N_2(w)\}.$$

If η is so small that $2C_{22}\eta^2<1$, (3.6) implies

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{(2m-k)/d}\|\hat{v}(\lambda+i\sigma+i\gamma)\|_k)^2 d\lambda$$
$$\leqq 2C_{22}K^2\sum_{k=0}^{l-1}\int_{-\infty}^{\infty}e^{2(\sigma+\gamma)t}\|D_t^{l-k-1}v\|_{kd}^2 dt \tag{3.7}$$
$$+2K_0^2(C_{23}+\eta+K)^2\delta^{-2l}\{e^{2\sigma c}N_1(w)+e^{2\sigma(c+4\delta)}N_2(w)\}.$$

The left side of (3.7) is not smaller than

$$\sum_{k=0}^{l-1}\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{l-k}\|\hat{v}(\lambda+i\sigma+i\gamma)\|_{kd})^2 d\lambda$$
$$\geqq\gamma^2\sum_{k=0}^{l-1}\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{l-k-1}\|\hat{v}(\lambda+i\sigma+i\gamma)\|_{kd})^2 d\lambda.$$

Therefore if $-\gamma\geqq\max(N, (4C(\delta))^{1/2l}, 4C_{22}K^2, 1)$ we get from (3.7)

$$\sum_{k=0}^{l-1}\int_{-\infty}^{\infty}(|\lambda+i\sigma+i\gamma|^{l-k-1}\|\hat{v}(\lambda+i\sigma+i\gamma)\|_{kd})^2 d\lambda$$
$$\leqq C'(e^{2\sigma c}N_1(w)+e^{2\sigma(c+4\delta)}N_2(w)), \tag{3.8}$$

where $C'=4K_0^2(C_{23}+\eta+K)^2/\delta^2$. If we apply Plancherel theorem to the left side of (3.8), we find that for any $\sigma\leqq 0$

$$\sum_{k=0}^{l-1}\int_{c+2\delta}^{c+3\delta}\|(D_t+i\gamma)^{l-k-1}w\|_{kd}^2 dt$$
$$\leqq C'(e^{-6\sigma\delta}N_1(w)+e^{2\sigma\delta}N_2(w)). \tag{3.9}$$

Theorem 3.1. *Suppose that u satisfy*

$$\|A(x,t,D_x,D_t)u(t)\|_0\leqq\eta\sum_{k=0}^{l}\|D_t^{l-k}u(t)\|_{kd}$$
$$+K\sum_{k=0}^{l-1}\|D_t^{l-k-1}u(t)\|_{kd},\qquad 0<t\leqq T, \tag{3.10}$$

$$B_j(x,t,D_x,D_t)u(x,t)=0,\quad x\in\partial\Omega,\quad 0<t\leqq T,\quad j=1,\cdots,m. \tag{3.11}$$

If η is sufficiently small and if $D_t^k u(0)=0$ for $k=0,\cdots,l-1$, then $u(t)\equiv 0$ for $0<t\leqq T$.

Proof. If we set $u(x,t)=0$ for $x\in\Omega$, $t<0$, u satisfies (3.10)-(3.13) for $-\infty<t<T$. Therefore $w=e^{\gamma t}u$ satisfies (3.9) if δ is sufficiently small and $-\gamma$ is sufficiently large depending on δ. Choosing $c=-\delta$, we obtain

$$\sum_{k=0}^{l-1}\int_{\delta}^{2\delta}\|(D_t+i\gamma)^{l-k-1}w\|_{kd}^2 dt$$
$$\leqq C'e^{2\sigma\delta}\sum_{k=0}^{l-1}\int_{3\delta}^{4\delta}\|(D_t+i\gamma)^{l-k-1}w\|_{kd}^2 dt\to 0\quad\text{if}\quad\sigma\to-\infty.$$

Thus $u(t)=0$ if $\delta<t<2\delta$. Varying δ in a sufficiently small interval $0<\delta<\delta_0$, we find that $u(t)=0$ for $0<t<2\delta_0$. We can now proceed step by step to show that $u(t)=0$ for $2\delta_0\leqq t\leqq 4\delta_0$, etc., and the proof is completed.

Let us again consider a function u satisfying (3.10)-(3.11) for $-\infty<t<T$. Then $w=e^{\gamma t}u$ satisfies (3.9). Choosing σ in such a way that the two terms on the right hand side of (3.9) become equal, we see that if $N_1(w)\leqq N_2(w)$ then

$$\sum_{k=0}^{l-1}\int_{c+2\delta}^{c+3\delta}\|(D_t+i\gamma)^{l-k-1}w\|_{kd}^2 dt\leqq 2C'N_1(w)^{1/4}N_2(w)^{3/4}.$$

Using a result of [1; pp. 241-233] one derives

Theorem 3.2. *If $u(\not\equiv 0)$ is a solution of* (3.10)-(3.11) *in* $\Omega\times(-\infty, T]$ *with* η *sufficiently small, then for any* $\varepsilon>0$ *there exist positive constants* L_0, L, ω *and* μ *such that*

$$\sum_{k=0}^{l-1}\int_t^{t+\varepsilon}\|D_s^{l-k-1}u(s)\|_{kd}^2 ds\geqq L_0L^{-t}\exp(-\omega\mu^{-t}),\quad -\infty<t<T-\varepsilon\,.$$

4. Uniquness at $t=-\infty$. In this section we again investigate solutions in $\Omega\times(-\infty, T]$. Writing B_{j,l_j-k} in such a form

$$B_{j,l_j-k}(x, t, D_x)u=\sum_{|\beta|\leqq m_j-kd}D_x^\beta(b_{j,l_j-k,\beta}^*(x, t)u)$$

we assume in addition to (I)$\sim$(IV)

(i) $\sup_x|a_{l-k,\alpha}(x, t)-a_{l-k,\alpha}(x, s)|\to 0$ as $t, s\to-\infty$

for $|\alpha|\leqq 2m-kd$, $k=0, \cdots, l$;

(ii) $$\left.\begin{aligned}&\sup_x|D_x^\kappa b_{j,l_j-k,\beta}^*(x, t)-D_x^\kappa b_{j,l_j-k,\beta}^*(x, s)|\to 0\\ &\qquad |\kappa|\leqq\max((l_j+1)d, 2m-kd)\\ &\sup_x|D_t^iD_x^\kappa b_{j,l_j-k,\beta}^*(x, t)|\to 0\\ &\qquad i=1, \cdots, l,\quad |\kappa|\leqq(l_j+1)d,\end{aligned}\right\}$$

as $t, s\to-\infty$ for $|\beta|\leqq m_j-kd$, $k=0, \cdots, l_j$, $j=1, \cdots, m$.

Theorem 4.1. *Under the assumptions above if* u *satisfies* (3.10), (3.11) *for* $-\infty<t<T$ *with* η *sufficiently small and if*

$$\lim_{t\to-\infty}e^{-\mu t}\sum_{k=0}^{l}\|D_t^{l-k}u(t)\|_{kd}=0\qquad\textit{for any}\quad\mu>0\,,$$

then $u(t)\equiv 0$ *for* $-\infty<t<T$.

Proof. In view of Theorem 3.1 it suffices to prove that $u(t)\equiv 0$ if t is smaller than some negative number s. Let $\zeta(t)$ be a smooth function satisfying $\zeta(t)=1$ if $t<s$, $\zeta(t)=0$ if $t>s+1$. We can proceed similarly to the proof of Theorem 2.1. $v=\zeta u$ is a solution of

$$\begin{aligned}A(x, t, D_x, D_t)v(x, t)&=F(x, t),\quad x\in\Omega,\quad -\infty<t<\infty\,,\\ B_j(x, t, D_x, D_t)v(x, t)&=G_j(x, t),\quad x\in\partial\Omega,\quad -\infty<t<\infty\,,\\ &\qquad j=1, \cdots, m\,,\end{aligned}$$

where F and G_j are functions defined similarly to (2.6) and (2.7) respectively. Furthermore we have

$$\begin{aligned}A(x, s, D_x, D_t)v(x, t)&=H(x, t),\quad x\in\Omega,\\ B_j(x, s, D_x, D_t)v(x, t)&=\Phi_j(x, t),\quad x\in\partial\Omega,\quad j=1, \cdots, m,\end{aligned}$$

with some function H and

$$\Phi_j = G_j + \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} D_x^\beta w_{j,k,\beta}(x, s, t) \tag{4.1}$$

where

$$w_{j,k,\beta}(x, s, t) = (b^*_{j,l_j-k,\beta}(x, s) - b^*_{j,l_j-k,\beta}(x, t)) D_t^k v(x, t) .$$

Taking the Fourier transforms of both sides of (4.1) and noting the last formula of p. 199

$$\begin{aligned} |\lambda|^{(2m-m_j)/d} \|\hat{\Phi}_j(\lambda)\|_0 &\leq |\lambda|^{(2m-m_j)/d} \|\hat{G}_j(\lambda)\|_0 \\ &+ c_0 \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \{ |\lambda|^{l-l_j-1} \|\hat{w}_{j,k,\beta}(s, \lambda)\|_{(l_j+1-k)d} \\ &+ |\lambda|^{l-k} \|\hat{w}_{j,k,\beta}(s, \lambda)\|_0 \} \end{aligned}$$

for any complex number λ. Hence using Plancherel theorem we get for any real number σ

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} (|\lambda + i\sigma|^{(2m-m_j)/d} \|\hat{\Phi}_j(\lambda + i\sigma)\|_0)^2 d\lambda \right)^{1/2} \\ &\leq \left(\int_{-\infty}^{\infty} (|\lambda + i\sigma|^{(2m-m_j)/d} \|\hat{G}_j(\lambda + i\sigma)\|_0)^2 d\lambda \right)^{1/2} \\ &+ c_0 \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \left\{ \left(\int_{-\infty}^{\infty} e^{2\sigma t} \| D_t^{l-l_j-1} w_{j,k,\beta}(s, t) \|^2_{(l_j+1-k)d} dt \right)^{1/2} \right. \\ &\left. + \left(\int_{-\infty}^{\infty} e^{2\sigma t} \| D_t^{l-k} w_{j,k,\beta}(s, t) \|_0^2 dt \right)^{1/2} \right\} . \end{aligned} \tag{4.2}$$

If we recall the assumptions here it is not difficult to verify that given $\varepsilon > 0$ we can choose $-s$ so large that the right side of (4.2) does not exceed

$$\left(\int_{-\infty}^{\infty} (|\lambda + i\sigma|^{(2m-m_j)/d} \|\hat{G}_j(\lambda + i\sigma)\|_0)^2 d\lambda \right)^{1/2} + M\varepsilon^2 \sum_{k=0}^{l} \left(\int_{-\infty}^{\infty} e^{2\sigma t} \|D_t^{l-k} v\|_{kd}^2 dt \right)^{1/2} .$$

Estimating $\int_{-\infty}^{\infty} (|\lambda + i\sigma|^{(2m-m_j)/d} \|\hat{G}_j(\lambda + i\sigma)\|_0)^2 d\lambda$ etc. in a similar manner we can show that if η is sufficiently small and if $-\sigma$ is sufficiently large we have

$$\sum_{k=0}^{l} \int_{-\infty}^{s} e^{2\sigma t} \|D_t^{l-k} u\|_{kd}^2 dt \leq C'' \sum_{k=0}^{l-1} \int_{s}^{s+1} e^{2\sigma t} \|D_t^{l-k-1} u\|_{kd}^2 dt . \tag{4.3}$$

From (4.3) it immediately follows that $u(t) = 0$ for $-\infty < t < s$.

OSAKA UNIVERSITY

References

[1] S. Agmon and L. Nirenberg: *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16** (1963), 121–239.

[2] S. Agmon, A. Douglis, and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[3] A. Friedman: *Uniqueness of solutions of ordinary differential inequalities in Hilbert space*, Arch. Rat. Mech. Anal. **17** (1964), 353–357.