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Osaka J. Math.
3 (1966), 163-182

ON ESTIMATES FOR DERIVATIVES OF SOLUTIONS OF WEIGHTED ELLIPTIC BOUNDARY VALUE PROBLEMS

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(Received March 2, 1966)

It is well known that any solution u of the parabolic differential equation

$$\partial u/\partial t - \partial^2 u/\partial x^2 = 0$$

belongs to a Gevrey's class, namely for any compact set K there exist constants M_0 and M such that

$$\sup_{(x,t)\in K} |(\partial/\partial t)^m (\partial/\partial x)^n u(x, t)| \leqq M_0 M^{m+n} \Gamma(n+1)\Gamma(2m+1)$$

for any non-negative integers m and n. In this paper this result is generalized to more general weighted elliptic boundary value problems of first order in t (cf. [2] for the definition of weighted ellipticity):

$$D_t u(x, t) + A(x, t, D_x)u(x, t) = f(x, t), \; x\in\Omega, \tag{0.1}$$

$$B_j(x, t, D_x)u(x, t) = g_j(x, t), \; x\in\partial\Omega, \; j=1, \cdots, m, \tag{0.2}$$

where $A(x, t, D_x)$ is a linear differential operator of order $2m$ and Ω is a bounded domain in the n-dimensional Euclidean space. The boundary system $\{B_j(x, t, D_x)\}$ is assumed to consist of differential operators of order $<2m$; however, it need not be normal. Throughout this paper a function h (resp. ϕ) of (x, t) (resp. x) is said to belong to Gevrey's class $G(\sigma, \tau)$ (resp. $G(\tau)$), $\sigma, \tau \geqq 1$, if with some constants M_0 and M

$$\sup_{x\in\Omega, t} |D_t^l D_x^\kappa h(x, t)| \leqq M_0 M^{l+|\kappa|} \Gamma(|\kappa|\sigma+1)\Gamma(l\tau+1)$$

$$(\text{resp. } \sup_{x\in\Omega} |D_x^\kappa \phi(x)| \geqq M_0 M^{|\kappa|} \Gamma(|\kappa|\sigma+1))$$

for any l and κ, and the boundary $\partial\Omega$ is said to be of Gevrey's class $G(\sigma)$ if some open part of $\partial\Omega$ containing each point of $\partial\Omega$ is mapped onto a part of a hyperplane by means of a one-to-one mapping of the class $G(\sigma)$. It will always be assumed that the elliptic boundary systems $(\pm D_y^{2m} + A(x, t, D_x), \; \{B_j(x, t, D_x)\}, \; \Omega\times\{y: -\infty<y<\infty\})$ satisfy the

Complementing Condition ([3]) for each fixed t. Under this assumption it will be shown that any solution of (0.1)−(0.2) belongs to Gevery's class $G(\sigma, 2m\sigma)$ provided that all the coefficients of A, $\{B_j\}$ and f, $\{g_j\}$ belong to the class $G(\sigma, 2m\sigma)$ and $\partial\Omega$ is of the class $G(\sigma)$. This result gives an affirmative answer to the conjecture of J.L. Lions and E. Magenes [4].

In section 3 the property of the solution considered as a function of t with values in $H_{2m}(\Omega)$ will be investigated. The main result in that section is that if all the coefficients of A, $\{B_j\}$, and f, $\{g_j\}$ belong to Gevrey's class $G(\tau)$, $\tau \geqq 1$, as functions of t, then so does the solution u of (0.1)−(0.2) as a function of t with values in $H_{2m}(\Omega)$, namely with some constants L_0, L

$$||D_t^k u(t)||_{2m} \leqq L_0 L^k \Gamma(k\tau+1)$$

for all integers $k \geqq 0$, where $||\ ||_{2m}$ is the norm of $H_{2m}(\Omega)$. In this result the known functions need not belong to Gevrey's class in x.

In the last section it will be shown that u belongs to the class $G(\sigma, 2m\sigma)$ with the aid of the result in section 3. As in [7] it will first be proved that the Cauchy data of u on the boundary belong to $G(\sigma, 2m\sigma)$. Unlike the case of analyticity Cauchy-Kowalevskii theorem and Holmgren's theorem cannot be used, therefore we estimate all derivatives of the solution following the technique of C.B. Morrey and L. Nirenberg [5].

It is quite probable that the same result remains valid for problems of arbitrary order in t:

$$A(x, t, D_x, D_t)u(x, t) = f(x, t),\ x \in \Omega, \tag{0.3}$$

$$B_j(x, t, D_x, D_t)u(x, t) = g_j(x, t),\ x \in \partial\Omega,\ j=1, \cdots, m; \tag{0.4}$$

however, the computation in that case would be extremely lengthy, so we shall investigate only the simpler situation.

1. Notations and assumptions. We denote by Ω a domain in the n-dimensional Eucidean space E_n and by $\partial\Omega$ its boundary. Let $(x, t) = (x_1, \cdots, x_n, t)$ be the generic point in E_{n+1}. We write $D_x = (D_1, \cdots, D_n) = (-1)^{-1/2}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $D_t = (-1)^{-1/2}\partial/\partial t$ and denote by D_x^α, $\alpha = (\alpha_1, \cdots, \alpha_n)$, the x-derivative $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. $|\alpha|$ stands for the length of the multi-index of α: $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For any non-negative integer k we denote by $H_k(\Omega)$ the class of all complex valued functions whose distribution derivatives of order up to k are square integrable in Ω, the norm of $H_k(\Omega)$ being denoted by

$$||u||_{k,\Omega}^2 = \sum_{|\alpha|\leqq k}\int_\Omega |D_x^\alpha u(x)|^2 dx\,.$$

$H_{k-1/2}(\partial\Omega)$ is to be the class of functions ϕ which are the boundary values of functions belonging to $H_k(\Omega)$. In this class of functions we introduce the norm

$$\langle\phi\rangle_{k,\partial\Omega} = \inf||v||_{k,\Omega},$$

where the infimum is taken over all functions v in $H_k(\Omega)$ which equal ϕ on $\partial\Omega$.

$A(x, t, D_x)$ is a linear differential operator in x of order $2m$ with coefficients defined in $\bar{\Omega}\times\{t: -\infty<t<\infty\}$:

$$A(x, t, D_x) = \textstyle\sum_{|\alpha|\leqq 2m} a_\alpha(x, t) D_x^\alpha\,.$$

For each $j=1, \cdots, m$, $B_j(x, t, D_x)$ is a linear differential operator in x of order m_j with coefficients defined on $\partial\Omega\times\{t: -\infty<t<\infty\}$:

$$B_j(x, t, D_x) = \textstyle\sum_{|\beta|\leqq m_j} b_{j,\beta}(x, t) D_x^\beta\,.$$

$\{B_j(x, t, D_x)\}$ is a system of operators which defines boundary conditions, and in what follows we shall assume that all the coefficients of $\{B_j\}$ are defined in the whole of $\bar{\Omega}\times\{t: -\infty<t<\infty\}$. Let y be an auxiliary real variable and we denote by Q the infinite cylinder: $Q=\{(x, y): x\in\Omega, -\infty<y<\infty\}$. For each fixed t, $\pm D_y^{2m}+A(x, t, D_x)$ are differential operators in (x, y) of order $2m$ with coefficients defined in Q.

Assumptions (I). For each fixed t, $\pm D_y^{2m}+A(x, t, D_x)$ is an elliptic operator of order $2m$ in Q.

(II) The order m_j of B_j is smaller than $2m$ for each j.

(III) The Complementing Condition ([2]) is satisfied by the system $(\pm D_y^{2m}+A(x, t, D_x), \{B_j(x, t, D_x)\}_{j=1}^m, Q)$ for each fixed t.

The assumption concerning the smoothness of the coefficients will be stated in each of the following sections and in the last section $\partial\Omega$ will be required to satisfy a more restrictive assumption. By a solution of (0.1)–(0.2) we always mean a function u with the properties that (i) $u(t)=u(x, \cdot)\in H_{2m}(\Omega)$ for each t, (ii) $u(t)$ is continuous in t in the strong topology of $H_{2m}(\Omega)$ and (iii) u satisfies (0.1)–(0.2).

Let τ and σ be real numbers such that $\tau\geqq 1$, $\sigma\geqq 1$.

DEFINITION 1. A function $u(t)$, $-\infty<t<\infty$, with values in a Hilbert space X (in many cases in what follows X will be the set of all complex numbers) is said to belong to *Gevrey's class* $G(\tau)$ if for any positive constant R there exist constants H_0 and H such that

$$\sup_{-R<t<R}\|D_t^q u(t)\| \leqq H_0 H^q \Gamma(\tau q+1)$$

for all integers $q \geqq 0$, where $\|\ \|$ is the norm of X.

DEFINITION 2. A numerical valued function $u(x, t)$ defined in $\bar{\Omega} \times \{t: -\infty<t<\infty\}$ is said to belong to *Gevrey's class* $G(\sigma, \tau)$ if for any positive constant R there exist constants H_0 and H such that

$$\sup_{x \in \Omega, -R<t<R} |D_t^q D_x^\kappa u(x, t)| \leqq H_0 H^{q+|\kappa|} \Gamma(\sigma|\kappa|+1) \Gamma(\tau q+1).$$

for any κ and $q \geqq 0$.

From now on we shall write $\|\ \|_k$, $\langle\ \rangle_k$ omitting Ω and $\partial\Omega$ if there is no fear of confusion.

2. Preliminary lemmas. In this section we assume that all the coefficients of A, $\{B_j\}$ have derivatives in t of all orders which are continuous in $\bar{\Omega} \times \{t: -\infty<t<\infty\}$ and that f, g_j, $j=1, \cdots, m$, are infinitely differentiable functions of t with values in $L^2(\Omega)$ and $H_{2m-m_j}(\Omega)$ respectively. Let ρ be a positive number satisfying $\rho \leqq 1$, and r, δ be positive numbers such that $r+\delta<\rho$. φ is to be a smooth function such that $\varphi(t)=1$ for $-r<t<r$, $\varphi(t)=0$ for $|t|>r+\delta$, and $|\varphi'(t)| \leqq K/\delta$ where K is a positive number independent of r and δ. In what follows in this section we denote by $C_1, C_2, \cdots$ constants depending only on the assumptions stated in the preceding and the present sections. If h is a function of (x, t), we denote by $\hat{h}$ its Fourier transform with respect to t:

$$\hat{h}(x, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(-1)^{1/2}\lambda t} h(x, t) dt$$

and by $h^{(q)}$ its derivative in t of order q:

$$h^{(q)}(x, t) = D_t^q h(x, t).$$

We shall use the following notations

$$e_q(f, r) = \left(\int_{-r}^{r} \|f^{(q)}(t)\|_0^2 dt\right)^{1/2}, \tag{2.1}$$

$$e_{j,q}(g_j, r+\delta) = \left(\int_{-r-\delta}^{r+\delta} \|g_j^{(q)}(t)\|_{2m-m_j}^2 dt\right)^{1/2} + \left(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|(\varphi g_j^{(q)})^\wedge(\lambda)\|_0)^2 d\lambda\right)^{1/2}, \tag{2.2}$$

$$d_q(u, r) = \left(\int_{-r}^{r} \|u^{(q+1)}(t)\|_0^2 dt\right)^{1/2} + \left(\int_{-r}^{r} \|u^{(q)}(t)\|_{2m}^2\, dt\right)^{1/2} \tag{2.3}$$

for $q=0, 1, 2, \cdots$.

Let η be a smooth function of t such that $\eta(t)=1$ for $-1<t<1$, $\eta(t)=0$ for $|t|>2$. Let M_q, $q=0, 1, \cdots$, be positive numbers such that for all α, β, κ, j with $|\alpha|\leqq 2m$, $|\beta|\leqq m_j$, $|\kappa|\leqq 2m-m_j$, $j=1, \cdots, m$,

$$|D_t^q a_\alpha(x, t)|\leqq M_q, \tag{2.4}$$

$$|D_t^q D_x^\kappa b_{j,\beta}(x, t)|\leqq M_q, \tag{2.5}$$

$$\int_{-\infty}^{\infty} |(\eta D_t^q b_{j,\beta})^\wedge(x, \lambda)|\, d\lambda\leqq M_q, \tag{2.6}$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |(\eta D_t^q b_{j,\beta})^\wedge(x, \lambda)|\, d\lambda\leqq M_q \tag{2.7}$$

in $\Omega\times\{t: -1<t<1\}$ or Ω.

Lemma 2.1. *If ρ is sufficiently small, then for any positive numbers r, δ such that $r+\delta<\rho$ and for any non-negative integer q the following inequality holds for any solution u of* (0.1)-(0.2):

$$\begin{aligned} d_q(u, r)\leqq C_1\Big\{ & e_q(f, r+\delta)+\textstyle\sum_{j=1}^m e_{j,q}(g_j, r+\delta) \\ & + \frac{1}{\delta}\Big(\int_{-r-\delta}^{r+\delta} ||D_t^q u(t)||_0^2 dt\Big)^{1/2} + \textstyle\sum_{p=0}^{q-1}\binom{q}{p} M_{q-p} d_p(u, r+\delta) \\ & + \frac{1}{\delta}\textstyle\sum_{p=0}^{q-1}\binom{q}{p} M_{q-p}\Big(\int_{-r-\delta}^{r-\delta} ||D_t^p u(t)||_0^2 dt\Big)^{1/2}\Big\}. \end{aligned} \tag{2.8}$$

This lemma is essentially proved in [6] and [7]. However, for the sake of convenience we give below an outline of the proof.

Lemma 2.2. *If v is a solution of*

$$D_t v(x, t)+A(x, 0, D_x)v(x, t) = f(x, t),\ x\in\Omega,\ -\infty<t<\infty, \tag{2.9}$$

$$B_j(x, 0, D_x)v(x, t) = g_j(x, t),\ x\in\partial\Omega,\ -\infty<t<\infty,\ j=1, \cdots, m, \tag{2.10}$$

and if the support of v considered as a function of t with values in $H_{2m}(\Omega)$ is compact, then

$$\begin{aligned} & \int_{-\infty}^{\infty} ||D_t v(t)||_0^2 dt+\int_{-\infty}^{\infty} ||v(t)||_{2m}^2 dt \\ & \leqq C_2\Big\{\int_{-\infty}^{\infty} ||f(t)||_0^2 dt+\textstyle\sum_{j=1}^m\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m}||\hat g_j(\lambda)||_0)^2 d\lambda \\ & +\textstyle\sum_{j=1}^m\int_{-\infty}^{\infty} ||g_j(t)||_{2m-m_j}^2 dt+\int_{-\infty}^{\infty} ||u(t)||_0^2\, dt\Big\}. \end{aligned} \tag{2.11}$$

Proof. The Fourier transform $\hat v$ of v with respect to t satisfies

$$\lambda\hat v(x, \lambda)+A(x, 0, D_x)\hat v(x, \lambda) = \hat f(x, \lambda),\ x\in\Omega, \tag{2.12}$$

$$B_j(x, 0, D_x)\hat{v}(x, \lambda) = \hat{g}_j(x, \lambda), \ x \in \partial\Omega, \ j=1, \cdots, m. \tag{2.13}$$

Following S. Agmon [1] let us consider the functions

$$w_{\pm}(x, y, \mu) = \zeta(y) \exp((-1)^{1/2}\mu y)\hat{v}(x, \ \pm \mu^{2m}),$$

where ζ is a smooth function such that $\zeta(y)=1$ for $|y| \leqq 1/2$ and $\zeta(y)=0$ for $|y| \geqq 1$, and μ is an arbitrary real number. Due to (2.12) and (2.13) $w_{\pm}$ satisfies

$$(\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}(x, y, \mu) = \zeta(y) \exp((-1)^{1/2}\mu y)\hat{f}(x, \ \pm\mu^{2m})$$
$$\pm \sum_{k=0}^{2m-1}\binom{2m}{k}D_y^{2m-k}\zeta(y)\mu^k \exp((-1)^{1/2}\mu y)\hat{v}(\pm\mu^{2m}, x), \ x \in \Omega, \tag{2.14}$$

$$B_j(x, 0, D_x)w_{\pm}(x, y, \mu) = \zeta(y) \exp((-1)^{1/2}\mu y)\hat{g}_j(x, \ \pm\mu^{2m}),$$
$$x \in \partial\Omega, \quad j=1, \cdots, m. \tag{2.15}$$

It is easy to show that

$$\|(\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}\|_{0,Q}$$
$$\leqq C_3\{\|\hat{f}(\pm\mu^{2m})\|_0 + (1+|\mu|^{2m-1})\|\hat{u}(\pm\mu^{2m})\|_0\}, \tag{2.16}$$

$$\langle B_j(x, 0, D_x)w_{\pm}\rangle^2_{2m-m_j,\partial Q} = \langle\zeta \exp((-1)^{1/2}\mu y)\hat{g}_j(\pm\mu^{2m})\rangle^2_{2m-m_j,\partial Q}$$
$$\leqq \|\zeta \exp((-1)^{1/2}\mu y)\hat{g}_j(\pm\mu^{2m})\|^2_{2m-m_j,Q}$$
$$\leqq C_4\sum_{k=0}^{2m-m_j}(1+|\mu|)^{2k}\|\hat{g}_j(\pm\mu^{2m})\|^2_{2m-m_j-k}.$$

Hence with the aid of the well known inequality

$$\|w\|_{2m-m_j-k} \leqq c_0\|w\|_{2m-m_j}^{(2m-m_j-k)/(2m-m_j)}\|w\|_0^{k/(2m-m_j)},$$

we get

$$\langle B_j(x, 0, D_x)w_{\pm}\rangle_{2m-m_j,\partial Q}$$
$$\leqq C_5\{\|\hat{g}_j(\pm\mu^{2m})\|_{2m-m_j} + (1+|\mu|)^{2m-m_j}\|\hat{g}_j(\pm\mu^{2m})\|_0\}. \tag{2.17}$$

As is easily seen

$$\|w_{\pm}\|^2_{2m,Q} \geqq \sum_{k=0}^{2m}|\mu|^{2k}\|\hat{v}(\pm\mu^{2m})\|^2_{2m-k}. \tag{2.18}$$

Using (2.16), (2.17) and (2.18) in the Agmon-Douglis-Nirenberg inequality

$$\|w_{\pm}\|_{2m,Q} \leqq C_6\{\|(\pm D_y^{2m} + A(x, 0, D_x))w_{\pm}\|_{0,Q}$$
$$+ \sum_{j=1}^{m}\langle B_j(x, 0, D_x)w_{\pm}\rangle_{2m-m_j,\partial Q} + \|w_{\pm}\|_{0,Q}\}$$

which can be applied to $w_{\pm}$ in Q by assumption and then putting $\lambda = \pm\mu^{2m}$, we get

$$\begin{aligned}&|\lambda|\,\|\hat{v}(\lambda)\|_0+\|\hat{v}(\lambda)\|_{2m}\\&\leqq C_7\{\|\hat{f}(\lambda)\|_0+\textstyle\sum_{j=1}^m|\lambda|^{(2m-m_j)/2m}\|\hat{g}_j(\lambda)\|_0\\&+\textstyle\sum_{j=1}^m\|\hat{g}_j(\lambda)\|_{2m-m_j}+\|\hat{u}(\lambda)\|_0\}\end{aligned}\tag{2.19}$$

for any real number λ. Integrating the squares of both sides of (2.19) over $-\infty<\lambda<\infty$ and then applying Plancherel's theorem, we get (2.11).

Lemma 2.3. *Let v be a solution of* (0.1)-(0.2). *If the support of v considered as a function of t is contained in a sufficiently small neighbourhood of the origin, then the same estimate as* (2.11) *holds replacing C_2 by another constant if necessary.*

Proof. The lemma is easily proved considering $\psi(t)v(x, t)$ where ψ is a smooth function which has a small compact support and identically equals 1 on the support of v.

Lemma 2.4. *If ρ is sufficiently small, then for the solution u of* (0.1)-(0.2)

$$\begin{aligned}d_0(u, r)\leqq C_8\Big\{&e_0(f, r+\delta)\\&+\textstyle\sum_{j=1}^m e_{j,0}(g_j, r+\delta)+\dfrac{1}{\delta}\displaystyle\int_{-r-\delta}^{r+\delta}\|u(t)\|_0^2\,dt\Big\}\end{aligned}$$

whenever $r+\delta<\rho$.

Proof. The lemma is easily proved if we apply Lemma 2.3 to $\varphi(t)u(x, t)$.

Lemma 2.1 can be obtained if we differentiate both sides of (0.1)-(0.2) q times in t and applying Lemma 2.4 to $D_t^q u$.

Lemma 2.5. *If $\alpha\geqq 1$ and $\beta>0$, then*

$$\Gamma(\alpha+\beta)\geqq\Gamma(\alpha)\Gamma(\beta+1)\,.\tag{2.20}$$

If $\alpha\geqq 1$ and $\beta\geqq 1$, then

$$2^{\alpha+\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\geqq\Gamma(\alpha+\beta+1).\tag{2.21}$$

If $0\leqq\alpha'\leqq\alpha$ and $0\leqq\beta'\leqq\beta$, then

$$\frac{\Gamma(\alpha'+\beta'+1)}{\Gamma(\alpha'+1)\Gamma(\beta'+1)}\leqq\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,\frac{1+\alpha+\beta}{1+\alpha'+\beta'}\,.\tag{2.22}$$

If $\tau\geqq 1$, then for any pair of non-negative integers p and q satisfying $p\leqq q$

$$\binom{q}{p} \leqq \frac{\tau\Gamma(\tau q+1)}{\Gamma(\tau p+1)\Gamma(\tau(q-p)+1)}. \tag{2.23}$$

Proof. (2.20), (2.21) and (2.22) are all simple consequences of

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

(2.22) implies

$$\binom{q}{p} = \frac{\Gamma(q+1)}{\Gamma(p+1)\Gamma(q-p+1)} \leqq \frac{\Gamma(\tau q+1)}{\Gamma(\tau p+1)\Gamma(\tau(q-p)+1)} \frac{\tau q+1}{q+1}. \tag{2.24}$$

(2.23) is a direct consequence of (2.24).

3. Estimates for derivatives in t. In this section we assume that

(V) the coefficients of A and the derivatives in x of the coefficients of B_j of order up to $2m-m_j$, $j=1, \cdots, m$, all belong to Gevery's class $G(\tau)$ as functions of t uniformly. Hence there exist positive constants M_0 and M such that

$$M_q \leqq M_0 M^q \Gamma(\tau q+1) \tag{3.1}$$

for all integers $q \geqq 0$ (cf. (2.4)~(2.7) for the meaning of M_q);

(VI) f and g_j, $j=1, \cdots, m$, belong to Gevrey's class $G(\tau)$ when they are considered as functions of t with values in $L^2(\Omega)$ and $H_{2m-m_j}(\Omega)$ respectively. Hence there exist positive constants N_0 and N such that

$$\|f^{(q)}(t)\|_0 \leqq N_0 N^q \Gamma(\tau q+1), \quad -1 \leqq t \leqq 1, \tag{3.2}$$

$$\|g_j^{(q)}(t)\|_{2m-m_j} \leqq N_0 N^q \Gamma(\tau q+1), \quad -1 \leqq t \leqq 1, \ j=1, \cdots, m. \tag{3.3}$$

We introduce the notation

$$N_{\rho,q}(u) = \Gamma(\tau q+1)^{-1} \sup_{\rho/2 \leqq r < \rho} d_q(u, r)(\rho-r)^{q+1} \tag{3.4}$$

for $q=0, 1, 2, \cdots$.

Theorem 3.1. *Under the assumptions* (I)~(VI) *any solution of* (0.1)-(0.2) *considered as a function of t with values in $H_{2m}(\Omega)$ belongs to Gevrey's class $G(\tau)$.*

Proof. Let us multiply both sides of (2.8) with $\delta=(\rho-r)(q+1)^{-1}$ by $\Gamma(\tau q+1)^{-1}(\rho-r)^{q+1}$. If we notice $\delta=(\rho-r-\delta)/q$ we get

$$\begin{aligned}
&\Gamma(\tau q+1)^{-1}d_q(u,r)(\rho-r)^{q+1}\\
&\quad\leqq C_9 e(1+q^{-1})\Big\{\Gamma(\tau q+1)^{-1}e_q(f,r+\delta)(\rho-r-\delta)^{q+1}\\
&\quad+\Gamma(\tau q+1)^{-1}\textstyle\sum_{j=1}^m e_{j,q}(g_j,r+\delta)(\rho-r-\delta)^{q+1}\\
&\quad+q\Gamma(\tau q+1)^{-1}d_{q-1}(u,r+\delta)(\rho-r-\delta)^q\\
&\quad+\tau M_0\textstyle\sum_{p=0}^{q-1}\Gamma(\tau p+1)^{-1}M^{p-q}d_p(u,r+\delta)(\rho-r-\delta)^{q+1}\\
&\quad+\tau M_0 q\textstyle\sum_{p=0}^{q-1}\Gamma(\tau p+1)^{-1}M^{p-q}d_{p-1}(u,r+\delta)(\rho-r-\delta)^q\\
&\quad+M_0 qM^q\Big(\int_{-r-\delta}^{r+\delta}\|u(t)\|_0^2\,dt\Big)^{1/2}(\rho-r-\delta)^q\Big\}\,.
\end{aligned}\tag{3.5}$$

From (3.2) it follows that

$$\Gamma(\tau q+1)^{-1}e_q(f,r+\delta)(\rho-r-\delta)^{q+1}\leqq\sqrt{2\rho}\,\rho N_0(\rho N)^q\,.\tag{3.6}$$

Noting $|\lambda|^{(2m-m_j)/2m}\leqq|\lambda|+1$ and

$$\Gamma(\tau q+\tau+1)\leqq 2^{\tau q+\tau-1}\Gamma(\tau q+1)\Gamma(\tau+1)\tag{3.7}$$

which follows from (2.21), we can easily show

$$\begin{aligned}
&\Gamma(\tau q+1)^{-1}e_{j,q}(g_j,r+\delta)(\rho-r-\delta)^{q+1}\\
&\quad\leqq\sqrt{2\rho}\{2^{\tau-1}\Gamma(\tau+1)N_0\rho N(2^\tau\rho N)^q+(2\rho+Kq)N_0(\rho N)^q\}\,.
\end{aligned}\tag{3.8}$$

Using (3.4) and

$$\begin{aligned}
\Gamma(\tau q+1)&=\tau q\Gamma(\tau q)=\tau q\Gamma(\tau(q-1)+1+\tau-1)\\
&\geqq\tau q\Gamma(\tau(q-1)+1)\Gamma(\tau)=\Gamma(\tau+1)q\Gamma(\tau(q-1)+1
\end{aligned}$$

which follows from (2.20), we get

$$q\Gamma(\tau q+1)^{-1}d_{q-1}(u,r+\delta)(\rho-r-\delta)^q\leqq\Gamma(\tau+1)^{-1}N_{\rho,q-1}(u)\,.\tag{3.9}$$

Similarly

$$\Gamma(\tau p+1)^{-1}M^{q-p}d_p(u,r+\delta)(\rho-r-\delta)^{q+1}\leqq(\rho M)^{q-p}M_{\rho,p}(u)\,,\tag{3.10}$$

$$\begin{aligned}
&\tau q\Gamma(\tau p+1)^{-1}M^{q-p}d_{p-1}(u,r+\delta)(\rho-r-\delta)^q\\
&\quad\leqq\Gamma(\tau)^{-1}qp^{-1}(\rho M)^{q-p}N_{\rho,p-1}(u)\,.
\end{aligned}\tag{3.11}$$

From (3.5)$\sim$(3.11) and $qp^{-1}\leqq e^{q-p}$ it follows that

$$\begin{aligned}
N_{\rho,q}(u)&\leqq C_{10}e(1+q^{-1})\Big[\sqrt{2\rho}\{2m+1+2^{\tau-1}m\Gamma(\tau+1)N\}\rho N_0(2^\tau\rho N)^q\\
&\quad+\sqrt{2\rho}\,mKN_0q(\rho N)^q+\Gamma(\tau+1)^{-1}N_{\rho,q-1}(u)\\
&\quad+\tau M_0\textstyle\sum_{p=0}^{q-1}(\rho M)^{q-p}N_{\rho,p}(u)+\Gamma(\tau)^{-1}M_0\sum_{p=0}^{q-1}(e\rho M)^{q-p}N_{\rho,p-1}(u)\\
&\quad+M_0q(\rho M)^q\Big(\int_{-\rho}^{\rho}\|u(t)\|_0^2\,dt\Big)^{1/2}\Big]\,.
\end{aligned}\tag{3.12}$$

We want to show that there exist constants H_0, $H \geqq 0$ such that for any non-negative integer q

$$N_{\rho,q}(u) \leqq H_0 H^q . \tag{3.13}$$

With the aid of (3.12) we can proceed by induction without difficulty to verify (3.13) provided that H_0 and H are so large that

$$12\, C_{10} e\sqrt{2\rho}\{2m+1+2^{\tau-1} m\Gamma(\tau+1)N\}\rho N_0 \leqq H_0 ,$$

$$12\, C_{10} e\sqrt{2\rho}\; mKN_0 \leqq H_0 ,\ \ 12\, C_{10} e M_0\left(\int_{-\rho}^{\rho} ||u(t)||_0^2\, dt\right)^{1/2} \leqq H_0 ,$$

$$\max(2^\tau \rho N,\ e\rho N,\ 2e\rho M) \ \leqq H,\ \ 12\, C_{10} e\Gamma(\tau+1)^{-1} \leqq H,$$

$$24\, C_{10} e\tau M_0 \rho M \leqq H,\ \ 24\, \Gamma(\tau)^{-1} C_{10} e^2 M_0 \rho M \leqq H^2 .$$

The proof of the theorem has been completed.

Next we show that u belongs to the same class in the space $H_{4m}(\Omega)$ under the following more restrictive assumptions:

(V′) the derivatives in x of the coefficients of A of order up to $2m$ and those in x of the coefficients of B_j of order up to $4m-m_j$, $j=1, \cdots, m$, all belong to Gevrey's class $G(\tau)$ as functions of t uniformly;

(VI′) f and g_j, $j=1, \cdots, m$, belong to Gevrey's class $G(\tau)$ when considered as functions of t with values in $H_{2m}(\Omega)$ and $H_{4m-m_j}(\Omega)$ respectively.

Theorem 3.2. *Under the assumptions* (I)$\sim$(IV), (V′), (VI′) *any solution of* (0.1)-(0.2) *belongs to Gevrey's class* $G(\tau)$ *when considered as a function of t with values in* $H_{4m}(\Omega)$.

Proof. By assumption and Theorem 3.1 there exist constants $\bar{M}_0$, $\bar{M}$, $\bar{N}_0$, $\bar{N}$ and L_0, L such that for all $q=1, 2, \cdots$

$$\sup|D_t^q D_x^\kappa a_\alpha(x, t)| \leqq \bar{M}_0 \bar{M}^q \Gamma(\tau q+1),\ \ |\kappa| \leqq 2m , \tag{3.14}$$

$$\sup|D_t^q D_x^\kappa b_{j,\beta}(x, t)| \leqq \bar{M}_0 \bar{M}^q \Gamma(\tau q+1),\ \ |\kappa| \leqq 4m-m_j,\ j=1, \cdots, m , \tag{3.15}$$

$$||f^{(q)}(t)||_{2m} \leqq \bar{N}_0 \bar{N}^q \Gamma(\tau q+1) , \tag{3.16}$$

$$||g^{(q)}(t)||_{4m-m_j} \leqq \bar{N}_0 \bar{N}^q \Gamma(\tau q+1),\ j=1, \cdots, m , \tag{3.17}$$

$$||D_t^q u(t)||_{2m} \leqq L_0 L^q \Gamma(\tau q+1) . \tag{3.18}$$

We want to show that there exist constants $\bar{L}_0$ and $\bar{L}$ such that

$$||D_t^q u(t)||_{4m} \leqq \bar{L}_0 \bar{L}^q \Gamma(\tau q+1) \tag{3.19}$$

for all integers $q \geqq 0$. Supposing that (3.19) is true for $q=0, 1, \cdots, l-1$, let us prove that the same is true for $q=l$. In view of the Agmon-

Douglis-Nirenberg inequality concerning the system $(A(x, t, D_x), \{B_j(x, t, D_x)\}, \Omega)$

$$\begin{aligned} \|D_t^l u(t)\|_{4m} \leqq C_{11}\{&\|A(x, t, D_x)D_t^l u(t)\|_{2m} \\ &+\textstyle\sum_{j=1}^m \langle B_j(x, t, D_x)D_t^l u(t)\rangle_{4m-m_j} + \|D_t^l u(t)\|_0\} . \end{aligned} \tag{3.20}$$

Differentiating both sides of (0.1)–(0.2) we get

$$\begin{aligned} A(x, t, D_x)D_t^l u(x, t) &= -D_t^{l+1}u(x, t) + D_t^l f(x, t) \\ &-\textstyle\sum_{k=0}^{l-1}\binom{l}{k}A^{(l-k)}(x, t, D_x)D_t^k u(x, t),\ x\in\Omega , \end{aligned} \tag{3.21}$$

$$\begin{aligned} B_j(x, t, D_x)D_t^l u(x, t) &= D_t^l g_j(x, t) \\ &-\textstyle\sum_{k=0}^{l-1}\binom{l}{k}B^{(l-k)}(x, t, D_x)D_t^k u(x, t),\ x\in\partial\Omega,\ j=1, \cdots , m , \end{aligned} \tag{3.22}$$

where $A^{(l-k)}$ and $B_j^{(l-k)}$ are differential operators obtained by differentiating the corresponding coefficients of A and B_j $l-k$ times with respect to t respectively. In view of (3.14) and an elementary calculation we get

$$\|A^{(l-k)}(x, t, D_x)D_t^k u(t)\|_{2m} \leqq C_{12}\bar{M}_0\bar{M}^{l-k}\Gamma(\tau(l-k)+1)\|D_t^k u(t)\|_{4m} ,$$

and hence with the aid of (3.16), (3.18), (3.21) and the induction hypothesis we obtain

$$\begin{aligned} \|A(x, t, D_x)D_t^l u(t)\|_{2m} &\leqq L_0 L^{l+1}\Gamma(\tau(l+1)+1) \\ &+\bar{N}_0\bar{N}^l\Gamma(\tau l+1) + C_{13}\bar{M}_0\bar{L}\textstyle\sum_{k=0}^{l-1}\bar{M}^{l-k}\bar{L}^k . \end{aligned} \tag{3.23}$$

Estimating $\langle B_j(x, t, D_x)D_t^l u(t)\rangle_{4m-m_j}$ in a similar manner and using (2.23) we can show without difficulty that (3.19) holds provided that $\bar{L}_0$ and $\bar{L}$ are sufficiently large.

4. Estimates for derivatives in all variables. In addition to the assumptions in section 1 we assume in this section that

(VII) all the functions a_α, $|\alpha|\leqq 2m$, $b_{j,\beta}$, $|\beta|\leqq m_j$, f and g_j, $j=1, \cdots, m$, belong to Gevrey's class $G(\sigma, \tau)$;

(VIII) $\tau=2m\sigma$;

(IV′) Ω is a bounded domain of the class $G(\sigma)$ in the sense that each point of $\partial\Omega$ is contained in some open subset of $\partial\Omega$ which can be mapped onto a subset of a hyperplane by means of a one-to-one mapping of Gevrey's class $G(\sigma)$.

Under the assumptions above we show that any solution of (0.1)–(0.2) belongs to the class $G(\sigma, \tau)=G(\sigma, 2m\sigma)$. In this section we denote by $C_{13}, C_{14}, \cdots$ constants depending only on the assumptions stated so far. By (IV′) we may suppose that the origin is located on a part of $\partial\Omega$ which is contained in the hyperplane $x_n=0$. First we prove that the Cauchy data of u are in Gevrey's class $G(\sigma, \tau)$ near the origin.

We shall employ the following semi-norms and norms:

$$|v|_i^2 = |v|_{i,\Omega}^2 = \sum_{|\kappa|=i}\int_\Omega |D_x^\kappa v(x)|^2 dx\,, \tag{4.1}$$

$$|v|_{i,r}^2 = \sum_{|\kappa|=i}\int_{|x|<r, x_n>0} |D_x^\kappa v(x)|^2 dx\,, \tag{4.2}$$

$$||v||_{k,r}^2 = \textstyle\sum_{i=0}^k |v|_{i,r}^2\,. \tag{4.3}$$

We may choose constants c_0 and c_1 in such a manner that

$$|v|_i \leqq c_0 |v|_j^{i/j} |v|_0^{(j-i)/j} + c_1 |v|_0\,, \tag{4.4}$$

$$|v|_{i,r} \leqq c_0 |v|_{j,r}^{i/j} |v|_{0,r}^{(j-i)/j} + c_1 r^{-i} |v|_{0,r}\,. \tag{4.5}$$

for $0<i<j<2m$ and $0<r$. From now on we shall distinguish the normal variable x_n from tangential space variables $x'=(x_1, \cdots, x_{n-1})$ and by ∇^p we denote any derivative of order p in x'. We denote by $A^\#$ and $B_j^\#$ the principal parts of A and B_j respectively:

$$A^\#(x, t, D_x) = \textstyle\sum_{|\alpha|=2m} a_\alpha(x, t) D_x^\alpha\,, \tag{4.6}$$

$$B_j^\#(x, t, D_x) = \textstyle\sum_{|\beta|=m_j} b_{j,\beta}(x, t) D_x^\beta\,,\ j=1, \cdots, m\,. \tag{4.7}$$

Let $\rho_0(<1)$ be a positive number such that

$$\{(x', x_n):\ |x|<\rho_0,\ x_n>0\} \subset \Omega\,. \tag{4.8}$$

Let η_1 be a smooth function such that $\eta_1(t)\equiv 1$ for $|t|\leqq 1$, $\eta_1(t)=0$ for $|t|\geqq 2$. For $p, q=0, 1, 2, \cdots$ we denote by $M_{p,q}$ constants such that for all α, β, κ, j with $|\alpha|\leqq 2m$, $|\beta|\leqq m_j$, $|\kappa|\leqq 2m-m_j$, $j=1, \cdots, m$,

$$|D_t^q \nabla^p a_\alpha(x, t)| \leqq M_{p,q}\,, \tag{4.9}$$

$$|D_x^\kappa D_t^q \nabla^p b_{j,\beta}(x, t)| \leqq M_{p,q}\,, \tag{4.10}$$

$$\int_{-\infty}^\infty |(\eta_1 D_t^q \nabla^p b_{j,\beta})^\wedge(x, \lambda)|\, d\lambda \leqq M_{p,q}\,, \tag{4.11}$$

$$\int_{-\infty}^\infty |\lambda|^{(2m-m_j)/2m} |(\eta_1 D_t^q \nabla^p b_{j,\beta})^\wedge(x, \lambda)|\, d\lambda \leqq M_{p,q} \tag{4.12}$$

in $\Omega\times(-\infty, \infty)$ or Ω. We shall use the following notations:

$$d_{p,q}(u, r) = \max\left\{\left(\int_{-r}^r |D_t^{q+1}\nabla^p u(t)|_{0,r}^2 dt\right)^{1/2} + \left(\int_{-r}^r |D_t^q \nabla^p u(t)|_{2m,r}^2 dt\right)^{1/2}\right\},$$

$$e_{p,q}(f, r) = \max\left(\int_{-r}^r |D_t^q \nabla^p f(t)|_{0,r}^2 dt\right)^{1/2},$$

for $p, q=0, 1, 2, \cdots$, $0<r<\rho_0$, with the maximum taken over all derivatives ∇^p of order p. Let φ be a function stated in the preceding section. Then as in [7, pp. 181–187] we get

Lemma 4.1. *If ρ_1 is sufficiently small, then for any $\delta>0$, $r>0$ such that $r+\delta<\rho_1$*

$$\begin{aligned}d_{p,q}(u, r)\leqq C_{13}\Big[& e_{p,q}(f, r+\delta)\\ & +\textstyle\sum_{j=1}^m\Big(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|(\varphi D_t^q\nabla^p g_j)^\wedge(\lambda)|_{0,r+\delta})^2 d\lambda\Big)^{1/2}\\ & +\textstyle\sum_{j=1}^m\Big(\int_{-r-\delta}^{r-\delta}|D_t^q\nabla^p g_j(t)|_{2m-m_j,r+\delta}^2\,dt\Big)^{1/2}\\ & +\textstyle\sum_{j=1}^m\delta^{m_j-2m}\Big(\int_{-r-\delta}^{r-\delta}|D_t^q\nabla^p g_j(t)|_{0,r+\delta}^2 dt\Big)^{1/2}\\ & +(r+\delta+\varepsilon)d_{p,q}(u, r+\delta)+\varepsilon^{1-2m}\delta^{-2m}\Big(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{0,r+\delta}^2\,dt\Big)^{1/2}\\ & +\textstyle\sum'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}d_{p',q'}(u, r+\delta)\\ & +\delta^{-2m}\textstyle\sum'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}\Big(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2 dt\Big)^{1/2}\Big], \end{aligned}\tag{4.13}$$

where $\sum'$ means that the summation extends over all (p', q') satisfying $0\leqq p'\leqq p$, $0\leqq q'\leqq q$ except $(p, q)=(p', q')$, and ε is an arbitrary positive number.

By assumption there exist constants N_0, N, M_0 and M such that for any pair of integers $p, q\geqq 0$

$$\sup|D_t^q\nabla^p f(x, t)|\leqq N_0N^{p+q}\Gamma(\sigma p+\tau q+1), \tag{4.14}$$

$$\sup|D_x^\kappa D_t^q\nabla^p g_j(x, t)|\leqq N_0N^{p+q}\Gamma(\sigma p+\tau q+1), \tag{4.15}$$

$$|\kappa|\leqq 2m-m_j,\ j=1, \cdots, m,$$

$$M_{p,q}\leqq M_0M^{p+q}\Gamma(\sigma p+1)\Gamma(\tau q+1). \tag{4.16}$$

Then as in section 3 we get

$$e_{p,q}(f, r+\delta)\leqq C_{14}N_0N^{p+q}\Gamma(\sigma p+\tau q+1), \tag{4.17}$$

$$\begin{aligned}&\Big(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|(\varphi D_t^q\nabla^p g_j)^\wedge(\lambda)|_{0,r+\delta})^2 d\lambda\Big)^{1/2}\\ &\leqq C_{15}\Big\{N_0N^{p+q+1}\Gamma(\sigma p+\tau q+\tau+1)\\ &\quad+(1+K\delta^{-1})N_0N^{p+q}\Gamma(\sigma p+\tau q+1)\Big\},\end{aligned}\tag{4.18}$$

$$\Big(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p g_j(t)|_{2m-m_j,r+\delta}^2\,dt\Big)^{1/2}\leqq C_{16}N_0N^{p+q}\Gamma(\sigma p+\tau q+1), \tag{4.19}$$

$$\left(\int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p g_j(t)|_{0,r+\delta}^2 dt\right)^{1/2} \leq C_{17} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1). \tag{4.20}$$

With the aid of (4.13), (4.16)$\sim$(4.20) and (2.23) we get

Lemma 4.2. *Under the assumptions of the preceding lemma the following inequality holds for any pair of integers $p \geqq 2m$ and $q \geqq 0$:*

$$\begin{aligned} d_{p,q}(u, r) \leqq C_{18} \Big[& N_0 N^{p+q+1} \Gamma(\sigma p + \tau q + \tau + 1) \\ & + \delta^{-2m} N_0 N^{p+q} \Gamma(\sigma p + \tau q + 1) + (r + \delta + \varepsilon) d_{p,q}(u, r + \delta) \\ & + \delta^{-2m} \varepsilon^{1-2m} d_{p-2m,q}(u, r + \delta) \\ & \quad + \tau\sigma M_0 \sum{}' \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} d_{p',q'}(u, r + \delta) \\ & \quad + \frac{\tau\sigma M_0}{\delta^{2m}} \sum_{p' \geqq 2m}{}' \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} d_{p'-2m,q'}(u, r + \delta) \\ & \quad + \frac{\tau\sigma M_0}{\delta^{2m}} \sum_{p' < 2m}{}' \frac{\Gamma(\tau q + 1)}{\Gamma(\tau q' + 1)} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} M^{p-p'+q-q'} \\ & \qquad \times \left(\int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0,r+\delta}^2 dt\right)^{1/2} \Big]. \end{aligned} \tag{4.21}$$

We introduce the notation

$$N_{\rho,p,q}(u) = \Gamma(\sigma p + \tau q + 1)^{-1} \sup_{\rho/2 \leqq r < \rho} d_{p,q}(u, r)(\rho - r)^{p+q+2m} \tag{4.22}$$

for $p, q = 0, 1, 2, \cdots$.

Lemma 4.3. *If $\rho(<\rho_1)$ is sufficiently small, there exist constants H_0 and H such that*

$$N_{\rho,p,q}(u) \leqq H_0 H^{p+q} \tag{4.23}$$

for any pair of integers $p, q \geqq 0$.

Proof. Suppose $p > 2m$ and $\delta = (\rho - r)/(1 + \sigma p + \tau q)$. Let us multiply both sides of (4.21) by $\Gamma(\sigma p + \tau q + 1)^{-1}(\rho - r)^{p+q+2m}$. Noting $\delta = (\rho - r - \delta)/(\sigma p + \tau q)$, $\rho - r = \{1 + (\sigma p + \tau q)^{-1}\}(\rho - r - \delta)$ and $\{1 + (\sigma p + \tau q)^{-1}\}^{p+q+2m} \leqq e^2$ we get

$$\begin{aligned} & \Gamma(\sigma p + \tau q + 1)^{-1} d_{p,q}(u, r)(\rho - r)^{p+q+2m} \\ & \quad \leqq C_{19}(I + II + III + IV + V + VI + VII), \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} I &= \Gamma(\sigma p + \tau q + 1)^{-1} N_0 N^{p+q+1} \Gamma(\sigma p + \tau q + \tau + 1)(\rho - r)^{p+q+2m}, \\ II &\leqq e^2 (\sigma p + \tau q)^{2m} N_0 N^{p+q} (\rho - r - \delta)^{p+q}, \\ III &= e^2 (\rho + \varepsilon) \Gamma(\sigma p + \tau q + 1)^{-1} d_{p,q}(u, r + \delta)(\rho - r - \delta)^{p+q+2m}, \end{aligned}$$

$$IV = e^2\varepsilon^{1-2m}(\sigma p+\tau q)^{2m}\Gamma(\sigma p+\tau q+1)^{-1}d_{p-2m,q}(u, r+\delta)(\rho-r-\delta)^{p+q},$$

$$V = e^2\tau\sigma M_0{\sum}' \frac{\Gamma(\tau q+1)}{\Gamma(\tau q'+1)}\frac{\Gamma(\sigma p+1)}{\Gamma(\sigma p'+1)}\frac{M^{p-p'+q-q'}}{\Gamma(\sigma p+\tau q+1)} \times d_{p',q'}(u, r+\delta)(\rho-r-\delta)^{p+q+2m},$$

$$VI = e^2\tau\sigma M_0(\sigma p+\tau q)^{2m}\sum_{p'\geqq 2m}{}' \frac{\Gamma(\tau q+1)}{\Gamma(\tau q'+1)}\frac{\Gamma(\sigma p+1)}{\Gamma(\sigma p'+1)} \times\frac{M^{p-p'+q-q'}}{\Gamma(\sigma p+\tau q+1)}d_{p'-2m,q'}(u, r+\delta)(\rho-r-\delta)^{p+q},$$

$$VII = e^2\tau\sigma M_0(\sigma p+\tau q)^{2m}\sum_{p'<2m} \frac{\Gamma(\tau q+1)}{\Gamma(\tau q'+1)}\frac{\Gamma(\sigma p+1)}{\Gamma(\sigma p'+1)} \times\frac{M^{p-p'+q-q'}}{\Gamma(\sigma p+\tau q+1)}\left(\int_{-\rho}^{\rho}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2\,dt\right)^{1/2}(\rho-r-\delta)^{p+q}.$$

Since by Lemma 2.5

$$\Gamma(\sigma p+\tau q+\tau+1)\leqq 2^{\sigma p+\tau q+\tau-1}\Gamma(\tau+1)\Gamma(\sigma+1)\Gamma(\sigma p+\tau q+1), \tag{4.25}$$

we get

$$I\leqq 2^{\tau-1}\Gamma(\tau+1)\rho^{2m}N_0N(2^\sigma\rho N)^p(2^\tau\rho N)^q. \tag{4.26}$$

It is easy to show

$$II\leqq e^2N_0(\sigma p+\tau q)^{2m}(\rho N)^{p+q}, \tag{4.27}$$

$$III\leqq(\rho+\varepsilon)e^2N_{\rho,p,q}(u). \tag{4.28}$$

It follows from Lemma 2:5 that

$$\Gamma(\sigma(p-2m)+\tau q+1)\leqq\Gamma(2m\sigma-2m+1)^{-1}\Gamma(\sigma p+\tau q+1-2m) \tag{4.29}$$

for $p\geqq 2m$, and hence

$$\begin{aligned}&(\sigma p+\tau q)^{2m}\Gamma(\sigma(p-2m)+\tau q+1)\Gamma(\sigma p+\tau q+1)^{-1}\\ &\leqq(\sigma p+\tau q)^{2m}\Gamma(2m\sigma-2m+1)^{-1}\Gamma(\sigma p+\tau q+1-2m)\Gamma(\sigma p+\tau q+1)^{-1}\\ &=(\sigma p+\tau q)^{2m}\Gamma(2m\sigma-2m+1)^{-1}\{(\sigma p+\tau q)(\sigma p+\tau q-1)\cdots\\ &\cdots(\sigma p+\tau q+1-2m)\}^{-1}\leqq(2m)^{2m}\Gamma(2m\sigma-2m+1)^{-1}.\end{aligned} \tag{4.30}$$

From (4.22) and (4.30) it follows that

$$IV\leqq e^2(2m)^{2m}\varepsilon^{1-2m}\Gamma(2m\sigma-2m+1)^{-1}N_{\rho,p-2m,q}(u). \tag{4.31}$$

With the aid of

$$\frac{\Gamma(\tau q+1)}{\Gamma(\tau q'+1)}\frac{\Gamma(\sigma p+1)}{\Gamma(\sigma p'+1)}\frac{\Gamma(\sigma p'+\tau q'+1)}{\Gamma(\sigma p+\tau q+1)}\leqq e^{\sigma(p-p')+\tau(q-q')} \tag{4.32}$$

which follows from Lemma 2.5, we get

$$V \leqq e^2 \tau\sigma M_0 \sum{}' (e^\sigma \rho M)^{p-p'} (e^\tau \rho M)^{q-q'} N_{\rho,p',q'}(u) . \tag{4.33}$$

(4.29) with p, q replaced by p', q' implies

$$\begin{aligned} \Gamma(\sigma p' + \tau q' + 1) &= (\sigma p' + \tau q')(\sigma p' + \tau q' - 1) \cdots \\ &\cdots (\sigma p' + \tau q' + 1 - 2m) \Gamma(\sigma p' + \tau q' + 1 - 2m) \\ &\geqq (\sigma p' + \tau q' + 1 - 2m)^{2m} \Gamma(2m\sigma - 2m + 1) \Gamma(\sigma(p' - 2m) + \tau q' + 1) \end{aligned} \tag{4.34}$$

for $p' \geqq 2m$. If $p \geqq p' \geqq 2m$ and $q \geqq q'$

$$\begin{aligned} \frac{\sigma p + \tau q}{\sigma p' + \tau q' + 1 - 2m} &= 1 + \frac{\sigma(p - p') + \tau(q - q') + 2m - 1}{\sigma p' + \tau q' + 1 - 2m} \\ &\leqq 1 + \sigma(p - p') + \tau(q - q') + 2m - 1 \\ &\leqq \exp(\sigma(p - p') + \tau(q - q') + 2m - 1) . \end{aligned} \tag{4.35}$$

With the aid of (4.34) and (4.35)

$$\begin{aligned} VI \leqq \sigma\tau M_0 e^{2m(2m-1)+2} \Gamma(2m\sigma - 2m + 1)^{-1} \\ \times \sum_{p' \geqq 2m}{}' (e^{(2m+1)\sigma} \rho M)^{p-p'} (e^{(2m+1)\tau} \rho M)^{q-q'} N_{\rho,p'-2m,q'}(u) . \end{aligned} \tag{4.36}$$

By Theorem 3.2 there exist constants R_0 and R such that

$$||D_t^q u(t)||_{4m,\Omega} \leqq R_0 R^q \Gamma(\tau q + 1) \tag{4.37}$$

for any integer $q \geqq 0$. We may assume $R \geqq 2M$. Hence

$$\left(\int_{-\rho}^{\rho} |D_t^{q'} \nabla^{p'} u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \leqq \sqrt{2\rho}\, R_0 R^{q'} \Gamma(\tau q' + 1) . \tag{4.38}$$

Noting

$$(\sigma p + \tau q)^{2m} \frac{\Gamma(\sigma p + 1)}{\Gamma(\sigma p' + 1)} \frac{\Gamma(\tau q + 1)}{\Gamma(\sigma p + \tau q + 1)} \leqq \frac{(\sigma p + \tau q + 1)^{2m}}{\Gamma(\sigma p' + 1)}$$

which is also a simple consequence of Lemma 2.5, we easily obtain

$$\begin{aligned} VII \leqq 2\sqrt{2\rho}\, e^2 \tau\sigma M_0 R_0 \sum_{p'=0}^{2m-1} \Gamma(\sigma p' + 1)^{-1} M^{-p'} \\ \times (\sigma p + \tau q + 1)^{2m} \rho^{p+q} M^p R^q . \end{aligned} \tag{4.39}$$

Using (4.26), (4.27), (4.28), (4.31), (4.33), (4.36), (4.39) and then choosing ρ and ε sufficiently small, we obtain

$$\begin{aligned} N_{\rho,p,q}(u) \leqq C_{20} [& N_0 N (2^\sigma \rho N)^p (2^\tau \rho N)^q + N_0 (\sigma p + \tau q)^{2m} (\rho N)^{p+q} \\ & + N_{\rho,p-2m,q}(u) + M_0 \sum{}' (e^\sigma \rho M)^{p-p'} (e^\tau \rho M)^{q-q'} N_{\rho,p',q'}(u) \\ & + M_0 \sum_{p' \geqq 2m}{}' (e^{(2m+1)\sigma} \rho M)^{p-p'} (e^{(2m+1)\tau} \rho M)^{q-q'} N_{\rho,p'-2m,q'}(u) \\ & + M_0 R_0 (\sigma p + \tau q + 1)^{2m} \rho^{p+q} M^p R^q] . \end{aligned} \tag{4.40}$$

(4.37) implies that (4.23) is true for $0\leqq p\leqq 2m$, $q=0, 1, 2, \cdots$ with some constants H_0 and H. If H_0 and H are so large that

$$\begin{aligned}
&6C_{20}N_0\leqq H_0, \quad 6C_{20}N_0N\leqq H_0, \quad 6C_{20}M_0R_0\leqq H_0,\\
&e^{\tau}\rho N\leqq H, \quad 6C_{20}\leqq H^{2m}, \quad 2e^{(2m+1)\tau}\rho M\leqq H,\\
&12C_{20}M_0\rho M(2e^{\tau}+e^{\sigma})\leqq H,\\
&12C_{20}M_0\rho M(2e^{(2m+1)\tau}+e^{(2m+1)\sigma})\leqq H^{2m+1},\\
&(\sigma p+\tau q)^{2m}\leqq(\rho^{-1}N^{-1}H)^{p+q},\\
&(\sigma p+\tau q)^{2m}\leqq(\rho^{-1}R^{-1}H)^{p+q}
\end{aligned}$$

for all p and q, then with the aid of (4.40) we can first verify that (4.23) is true for $q=0$, $p=0, 1, 2, \cdots$ and then that the same is valid for all p and q by means of the induction argument concerning $p+q$. Thus the proof of Lemma 4.3 is completed.

So far we have not used $\tau=2m\sigma$. Especially if $\tau=\sigma=1$, (4.23) implies the analyticity of the Cauchy data of u, and hence with the aid of Holmgren's theorem and Cauch-Kowalevskii theorem it follows that u is analytic near the origin ([7]).

In what follows we denote the normal variable by y (i.e $y=x_n$), and introduce the notation

$$\bar{N}_{p,k,q}(u)=\max\left(\int_{-\rho/2}^{\rho/2}|D_t^q\nabla^p D_y^k u(t)|_{0,\rho/2}^2 dt\right)^{1/2} \tag{4.41}$$

for $p, k, q=0, 1, 2, \cdots$ with the maximum taken over all derivatives ∇_p of order p.

Lemma 4.4. *There exist constants $\bar{L}_0$, $\bar{L}$ and $\theta\leqq 1/2$ such that*

$$\bar{N}_{p,q,k}(u)\leqq\bar{L}_0\bar{L}^{\sigma p+\sigma k+\tau q}\theta^{\sigma p+\tau q}\Gamma(\sigma p+\tau k+\tau q+1) \tag{4.42}$$

for all p, q, $k\geqq 0$. $\bar{L}_0$, $\bar{L}$ and θ may depend on ρ, but are independent of p, q, k.

Proof. From (4.23) it follows that there exist constants L_0 and L such that for $p\geqq 0$, $q\geqq 0$, $0\leqq k\leqq 2m$

$$\left(\int_{-\rho/2}^{\rho/2}|D_t^q\nabla^p u(t)|_{k,\rho/2}^2 dt\right)^{1/2}\leqq L_0L^{\sigma p+\sigma k+\tau q}\Gamma(\sigma p+\sigma k+\tau q+1),$$

L_0 and L being allowed to depend on ρ. Hence

$$\bar{N}_{p,k,q}(u)\leqq L_0L^{\sigma p+\sigma k+\tau q}\Gamma(\sigma p+\sigma k+\tau q+1) \tag{4.43}$$

for $p\geqq 0$, $q\geqq 0$ and $0\leqq k\leqq 2m$. Due to the ellipticity of A we can solve

(0.1) with respect to $D_y^{2m}u$ near the origin to obtain

$$D_y^{2m}u=\sum_{k=0}^{2m-1}\sum_{|\beta|=2m-k}c_{\beta,k}D_{x'}^{\beta}D_y^k u$$
$$+\sum_{k=0}^{2m-1}\sum_{|\beta|\leqq 2m-k-1}c_{\beta,k}D_{x'}^{\beta}D_y^k u+cD_t u+af. \tag{4.44}$$

By assumption there exist constants $\bar{M}_0$ and $\bar{M}$ such that if h stands for any of the functions $c_{\beta,k}$, c, a, f, then

$$\sup|D_t^q D_x^\gamma h|\leqq \bar{M}_0\bar{M}^{\sigma|\kappa|+\tau q}\Gamma(\sigma|\gamma|+1)\Gamma(\tau q+1) \tag{4.45}$$

for any q and γ. Hence with some constants $\bar{R}_0$ and $\bar{R}$

$$\left(\int_{-\rho/2}^{\rho/2}|D_t^q\nabla^p D_y^l(af)(t)|_{0,\rho/2}^2 dt\right)^{1/2}$$
$$\leqq \bar{R}_0\bar{R}^{\sigma p+\tau q+\sigma l}\Gamma(\sigma p+\tau q+\sigma l+\tau+1) \tag{4.46}$$

for any $q, p, l=0, 1, 2, \cdots$. (4.42) is valid for $0\leqq k\leqq 2m$ if

$$L_0\leqq \bar{L}_0, \quad L\leqq \bar{L}\theta. \tag{4.47}$$

We show by induction that (4.41) is valid for all p, q, k if $\bar{L}_0$ and $\bar{L}$ are so large and θ is so small that (4.47) as well as the following inequalities are all true:

$$\bar{L}\theta\geqq 2\bar{M}e^2, \tag{4.48}$$

$$64\sigma\tau(\tau+1)C_0\bar{M}_0\theta^\sigma\leqq 1, \tag{4.49}$$

$$32\sigma\tau(\tau+1)\bar{M}_0\theta^\tau\leqq 1, \tag{4.50}$$

$$32\sigma\tau(\tau+1)\bar{M}_0\sum_{k=0}^{2m-1}\sum_{|\beta|\leqq 2m-k-1}\Gamma(\sigma(2m-k-|\beta|)+1)^{-1}\leqq \bar{L}^\sigma, \tag{4.51}$$

$$4\bar{R}_0\leqq \bar{L}_0, \quad \bar{R}\leqq \bar{L}\theta, \tag{4.52}$$

where C_0 is the number of β with $|\beta|\leqq 2m$. To see this we first differentiate both sides of (4.44) to obtain

$$D_t^q\nabla^p D_y^{2m+l}u=\sum_{k=0}^{2m-1}\sum_{|\beta|=2m-k}\binom{q}{q'}\binom{p}{p'}\binom{l}{l'}$$
$$\times D_t^{q-q'}\nabla^{p-p'}D_y^{l-l'}c_{\beta,k}\cdot D_t^{q'}D_{x'}^{\beta}\nabla^{p'}D_y^{k+l'}u+\cdots. \tag{4.53}$$

Suppose (4.42) is true for $0\leqq k\leqq 2m+l-1$. When we estimate the right side of (4.53), we use (2.23) for $\binom{q}{q'}$ and $\binom{p}{p'}$, and for $\binom{l}{l'}$ use

$$\binom{l}{l'}\leqq\frac{(\tau+1)\Gamma(\sigma l+\tau+1)}{\Gamma(\sigma l'+\tau+1)\Gamma(\sigma(l-l')+1)} \tag{4.54}$$

which also follows from Lemma 2.5. Hence with the aid of (4.45), the induction hypothesis and the inequalities

$$\frac{\Gamma(\tau q+1)}{\Gamma(\tau q'+1)}\frac{\Gamma(\sigma p+1)}{\Gamma(\sigma p'+1)}\frac{\Gamma(\sigma l+\tau+1)}{\Gamma(\sigma l'+\tau+1)}$$
$$\leqq \exp(2\sigma(p-p')+2\tau(q-q')+\sigma(l-l')),$$
$$\Gamma(\sigma(p'+|\beta|)+\sigma(l'+k)+\tau q'+1)$$
$$\leqq \Gamma(\sigma p'+\sigma l'+\tau q'+\tau+1)\Gamma(\sigma(2m-k-|\beta|)+1)^{-1}$$

which are consequences of Lemma 2.5, we get

$$\begin{aligned}\bar{N}_{p,q,l+2m}(u) &\leqq \Gamma(\sigma p+\tau q+\sigma l+\tau+1)\\ &\times\Big\{\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0 e^{2\sigma p+2\tau q+\sigma l}\bar{M}^{\sigma p+\sigma l+\tau q}\bar{L}^{\tau}\\ &\times\sum_{k=0}^{2m-1}\sum_{|\beta|=2m-k}\sum_{p'=0}^{p}\sum_{q'=0}^{q}\sum_{l'=0}^{l}\left(\frac{\bar{L}\theta}{e^2\bar{M}}\right)^{\sigma p'+\tau q'}\left(\frac{\bar{L}}{e\bar{M}}\right)^{\sigma l'}\theta^{\sigma(2m-k)}\\ &+\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0 e^{2\sigma p+2\tau q+\sigma l}\bar{M}^{\sigma p+\sigma l+\tau q}\\ &\times\sum_{k=0}^{2m-1}\sum_{|\beta|\leqq 2m-k-1}\sum_{p'=0}^{p}\sum_{q'=0}^{q}\sum_{j'=0}^{l}\left(\frac{\bar{L}\theta}{e^2\bar{M}}\right)^{\sigma p'+\tau q'}\left(\frac{\bar{L}}{e\bar{M}}\right)^{\sigma l'}\\ &\times\frac{\bar{L}^{\sigma(|\beta|+k)}}{\Gamma(\sigma(2m-k-|\beta|)+1)}+\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0 e^{2\sigma p+2\tau q+\sigma l}\bar{M}^{\sigma p+\sigma l+\tau q}\bar{L}^{\tau}\theta^{\tau}\\ &\times\sum_{p'=0}^{p}\sum_{q'=0}^{q}\sum_{l'=0}^{l}\left(\frac{\bar{L}\theta}{e^2\bar{M}}\right)^{\sigma p'+\tau q'}\left(\frac{\bar{L}}{e\bar{M}}\right)^{\sigma l'}+\bar{R}_0\bar{R}^{\sigma p+\tau q+\sigma l}\Big\}.\end{aligned}$$

If (4.48) is true, we easily get

$$\begin{aligned}\bar{N}_{p,q,l+2m}(u) &\leqq \Gamma(\sigma p+\tau q+\sigma l+\tau+1)\\ &\times\{16C_0\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0\bar{L}^{\sigma p+\sigma l+\tau q+\tau}\theta^{\sigma p+\tau q+\sigma}\\ &+8\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0\textstyle\sum_{k=0}^{2m-1}\sum_{|\beta|\leqq 2m-k-1}\Gamma(\sigma(2m-k-|\beta|+1)^{-1}\\ &\times\bar{L}^{\sigma p+\sigma l+\tau q+\tau-\sigma}\theta^{\sigma p+\tau q}\\ &+8\sigma\tau(\tau+1)\bar{M}_0\bar{L}_0\bar{L}^{\sigma p+\tau q+\sigma l+\tau}\theta^{\sigma p+\tau q+\tau}+\bar{R}_0\bar{R}^{\sigma p+\tau q+\sigma l}\}.\end{aligned}\tag{4.55}$$

Thus if (4.48)∼(4.52) are all true, it is immediately seen that the right side of (4.55) is dominated by

$$\bar{L}_0\bar{L}^{\sigma p+\sigma(l+2m)+q}\theta^{\sigma p+\tau q}\Gamma(\sigma p+\sigma(l+2m)+\tau q+1).$$

Thus the proof of Lemma 4.4 is completed.

The interior estimates of the derivatives of the solution is easier to be obtained, and hence we conclude

Theorem 4.1. *Under the assumptions* (I), (II), (III), (VII), (VIII) *and* (IV′) *any solution of* (0.1)-(0.2) *belongs to Gevrey's class* $G(\sigma, \tau)$.

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