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SIMPLE ALGEBRAS OVER A COMLETE LOCAL RING

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Hattori introduced the notion of simple algebras over a commutative ring in [7]. We intend, in this paper, to get further results about such algebras, restricting the coefficient ring to a complete local ring. In [8], Hattori presented some open questions about simple algebras. We shall give an answer to one of these questions about the uniqueness of (relatively) irreducible ideals and division algebras, and shall get several results realized by virtue of this solution; namely, in §2, we shall show that all (relatively) irreducible modules of simple algebras over a complete local ring are mutually isomoplic, and in §3, shall show that an analogy of the classical fundamental theorem of simple algebras holds.

Let R be a commutative ring, and Λ be an R-algebra which is finitely generated as an R-module.(*) Λ is called a left semisimple R-algebra if any finitely generated Λ-module is (Λ, R)-projective in the sense of Hochschild's relative homology. Similarly the right semisimplicity is defined. Left and right semisimple algebras are called *semisimple* algebras. In the case that R is indecomposable, Λ is called a *simple* algebra when Λ is semisimple and has a left module E satisfying the following three conditions;

1) E is finitely generated Λ-projective,
2) E is (Λ, R)-irreducible,
3) E is Λ-completely faithful,

where 2) means that E has no non-trivial Λ-submodule which is an R-direct summand of E, and 3) means that there exist $f_1, \cdots, f_n$ in $\mathrm{Hom}_\Lambda(E, \Lambda)$ and $x_1, \cdots, x_n$ in E such that $\sum f_i(x_i)=1$, or equivalently, E is a generator of the category of left Λ-modules (c.f. Grothendieck [5]). Hattori showed in [7] that we may equally define the simplicity by possession of a right module E satisfying 1), 2), 3).

Hattori defined the simplicity by adopting the R-projectivity of E instead of 1) of us. Since Λ is semisimple, the R-projectivity implies

(*) Throughout this paper we assume that all rings have units, that all modules are unitary, and that algebras are finitely generated as modules.

the Λ-projectivity for any Λ-module, so our condition is slightly weaker. Simple algebras in Hattori's sense are always R-projective algebras. However, in our approach, any Artinean simple ring may be interpreted as a simple algebra over the ring of rational integers Z. So we adopt the Λ-projectivity.

If a semisimple algebra Λ has a Λ-module E which enjoys 1), 2) and,

3′) E is Λ-faithful,

we call that Λ is a *quasi-simple* algebra. Clearly the simplicity implies the quasi-simplicity.

A quasi-simple algebra Λ is called a *strongly simple* algebra if all projective (Λ, R)-irreducible modules are isomorphic to each other. Easily can check that a strongly simple algebra is a simple algebra. In the classical case, these three concepts coincide.

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1. The Morita-equivalence of simple algebras.

Let Λ be a ring and E be a finitely generated projective, completely faithful, left Λ-module. If a ring Γ_1 is ring isomorphic to $\mathrm{Hom}_\Lambda(E, E)$, we call Λ and Γ_1 are Morita-dual, or Λ is *Morita-equivalent* to $\Gamma=\Gamma_1^0$, where Γ_1^0 is the anti-isomorphic copy of Γ_1. We know, in this case, E is a finitely generated projective, completely faithful right Γ-module, and categories $\mathfrak{M}^r(\Lambda)$, $\mathfrak{M}^r(\Gamma)$ ($\mathfrak{M}^l(\Lambda)$, $\mathfrak{M}^l(\Gamma)$) of right (left) Λ, Γ-modules are categorically isomorphic. The categorical isomorphisms

$$\mathfrak{F}: \mathfrak{M}^r(\Lambda) \to \mathfrak{M}^r(\Gamma)$$
$$\mathfrak{G}: \mathfrak{M}^l(\Gamma) \to \mathfrak{M}^l(\Lambda)$$

are given by $\mathfrak{F}: M \rightsquigarrow M\otimes_\Lambda E$, $\mathfrak{F}^{-1}: N \rightsquigarrow N\otimes_\Gamma \mathrm{Hom}_\Lambda(E, \Lambda)$ and $\mathfrak{G}$: $Q \rightsquigarrow E\otimes_\Gamma P$, $\mathfrak{G}^{-1}: P \rightsquigarrow \mathrm{Hom}_\Gamma(E, \Gamma)\otimes_\Lambda P$ for any right Λ-module M, right Γ-module N, left Γ-module Q, and any left Λ-module P. (c.f. Auslander-Goldman [1], Bass [3])

When we deal with simple algebras, we assume the coefficient ring R is always is always indecomposable.

Proposition 1. *Let* Λ *and* Γ *be* R*-algebras which are Morita-equivalent. If* Λ *is simple, then* Γ *is simple. If* Λ *is strongly simple, then* Γ *is also strongly simple.*

Proof. Let E and $\mathfrak{F}$ be as above. Since $\mathfrak{F}$ is a categorical isomorphism, every right Γ-module is isomorphic to $M\underset{\Lambda}{\otimes}E$ for some Λ-module M. While, $M\underset{\Lambda}{\otimes}E$ is (Γ, R)-projective, for,

$$M\underset{\Lambda}{\otimes}E<_{\Gamma}\oplus(M\underset{R}{\otimes}\Lambda)\underset{\Lambda}{\otimes}E\simeq M\underset{R}{\otimes}E<_{\Gamma}\oplus(M\underset{R}{\otimes}\Gamma)\oplus\cdots\oplus(M\underset{R}{\otimes}\Gamma)$$

So, Γ is right semisimple and, by a similar argument, is left semisimple. $\mathfrak{F}$ preserves the finite generation, projectivity, indecomposability and $\mathfrak{F}$ sends a generator to a generator. Therefor Γ is a simple algebra. Assume that Λ is strongly simple. Let $\mathfrak{G}$ be as above. $\mathfrak{G}$ sends two (Γ, R)-irreducible modules to (Λ, R)-irreducible modules which are isomorphic. Since $\mathfrak{G}$ is an isomorphism, $\mathfrak{G}^{-1}$ sends back isomorphically these modules.

A semisimple R-algebra Δ is called a *division* algebra when Δ itself is (Δ, R)-irreducible. Evidently *a division algebra is a simple algebra*. We say that a simple algebra Λ belongs to a division algebra Δ if Λ is Morita-equivalent to Δ. Every simple algebra Λ has a division algebra to which Λ belongs; namely Δ is given by $\Delta^0\simeq\mathrm{Hom}_\Lambda(E, E)$, where E is a Λ-module satisfying 1), 2), 3). Conversely, for a semisimple algebra Λ and a Λ-module E satisfying 1) and 3), if $\mathrm{Hom}_\Lambda(E, E)$ is a division algebra, then E is (Λ, R)-irreducible. Therefore, *the division algebra Δ to which a strongly simple algebra Λ belongs uniquely determined up to isomorphism.*

Theorem 2. *The strongly simple algebra Λ is isomorphic to the full matric algebra with entries in the division algebra Δ to which Λ belongs.*

Proof. By proposition 1, Δ is strong. So every finitely generated Δ-projective module is a free Δ-module. Hence, Λ is anti-isomorphic to the endomorphism ring of a free module over Δ. q.e.d.

To conclude this section, we shall add a some propositions.

Proposition 3. *If a simple algebra Λ belongs to a division algebra Δ, then;*

a) Λ *is central simple if and only if* Δ *is central division.*

b) Λ *is separable if and only if* Δ *is separable divison.*

Proof. a) is clear and b) follows immediately from Theorem 1 of Kanzaki [9].

Proposition 4. *Any finitely generated projective module over a strongly simple algebra is completely faithful.*

Proof. Any fininitely generated projective module can be decomposed into the directsum of relatively irreducible modules which are mutually isomorphic and are completely faithful.

2. Over a complete local ring.

To begin with, we quote some well known facts without proof. (See for example Lemma 18, 1 of Bass [4])

Lemma 5. *a) Let R be a local ring with the maximal ideal* $\mathfrak{m}$, *and Λ be an R-algebra*(*). *Two finitely generated projective Λ-modules M, N are isomorphic if $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are isomorphic as $\Lambda/\mathfrak{m}\Lambda$-modules. b) If R is a complete local ring, then any finitely generated projective $\Lambda/\mathfrak{m}\Lambda$-module is isomorphic to $M/\mathfrak{m}M$ for some finitely generated projective Λ-module M.*

Lemma 6. *Let R, Λ be as in b) of the above lemma. A finitely generated projective Λ-module M is indecomposable if and only if $M/\mathfrak{m}M$ is indecomposable as a $\Lambda/\mathfrak{m}\Lambda$-module. If Λ is semisimple, M is (Λ, R)-irreducible if and only if $M/\mathfrak{m}M$ is $\Lambda/\mathfrak{m}\Lambda$-irreducible.*

Proof. 'if' part: $M \cong M_1 \oplus M_2$ implies $M/\mathfrak{m}M \cong M_1/\mathfrak{m}M_1 \oplus M_2/\mathfrak{m}M_2$ and each of M_i/mM_i is not null by the Nakayama's lemma. 'only if' part: Assume that $M/\mathfrak{m}M$ is decomposable; $M/\mathfrak{m}M = \mathfrak{M}_1 \oplus \mathfrak{M}_2$. Since $\mathfrak{M}_1$, $\mathfrak{M}_2$ are $\Lambda/\mathfrak{m}\Lambda$-projective, there exist projective Λ-modules M_1, M_2 such that $M_1/\mathfrak{m}M_1 \cong \mathfrak{M}_1$ and $M_2/\mathfrak{m}M_2 \cong \mathfrak{M}_2$ by Lemma 6 b). So, we get the isomorphisms, $M_1 \oplus M_2/\mathfrak{m}(M_1 \oplus M_2) \cong M_1/\mathfrak{m}M_1 \oplus M_2/\mathfrak{m}M_2 \cong \mathfrak{M}_1 \oplus \mathfrak{M}_2 = M/\mathfrak{m}M$. Then $M \cong M_1 \oplus M_2$ by Lemma 6, a)

The following lemma is due to S. Endo.

Lemma 7. *Let R be an arbitrary commutative ring and Λ be an R-algebra. And let $\mathfrak{a}$ be an ideal of R. If $\Lambda/\mathfrak{m}\Lambda$ has no non-trivial nil-ideal (the ideal whose elements are all nilpotent), then for any finitely generated faithful Λ-module M, $M/\mathfrak{a}M$ is $\Lambda/\mathfrak{a}\Lambda$-faithful.*

Proof. Assume that $M/\mathfrak{a}M$ is not faithful, then the two sided ideal $A = \{a \in \Lambda \mid aM \subseteq \mathfrak{a}M\}$ properly contains $\mathfrak{a}\Lambda$. $A/\mathfrak{a}\Lambda$ is a two sided non-trivial ideal of $\Lambda/\mathfrak{a}\Lambda$, hence $A/\mathfrak{a}\Lambda$ has a non-nilpotent element; that is, there exists an element α in A such that α^n is not in $\mathfrak{a}\Lambda$ for any natural number n. Since M is R-finitely generated and $\alpha M \subseteq \mathfrak{a}M$, we get $\alpha u_i = \sum_{j=1}^{n} a_{ij} u_j$, where $\{u_i\}$ is a system of R-generators of M and $a_{ij} \in \mathfrak{a}$. $R[\alpha]$ being commutative, the identities:

(*) local ring = Noetherian local ring.

$$\begin{vmatrix} a_{11}-\alpha, & \cdots\cdots, & a_{1n} \\ \vdots & & \\ a_{n1}, & \cdots\cdots, & a_{nn}-\alpha \end{vmatrix} u_j = 0 \qquad (j=1, 2, \cdots, n)$$

hold. Since M is Λ-faithful, this determinant is zero. So $\alpha^n \in \mathfrak{a}\Lambda$; contradiction.

Theorem 8. *Over a complete local ring R with the maximal ideal $\mathfrak{m}$, the three concepts, quasi-simplicity, simplicity and strong simplicity coincide. A semisimple R-algebra Λ is simple if and only if $\Lambda/\mathfrak{m}\Lambda$ is a simple $R/\mathfrak{m}$-algebra, and a semisimple R-algebra Δ is a division algebra if and only if $\Delta/\mathfrak{m}\Delta$ is a division algebra over $R/\mathfrak{m}$*

Proof. Let Λ be a quasi-simple algebra over R and M be a finitely generated Λ-projective, (Λ, R)-irreducible, Λ-faithful module. Since $\Lambda/\mathfrak{m}\Lambda$ is a semisimple $R/\mathfrak{m}$-algebra, $M/\mathfrak{m}M$ is an irreducible faithful $\Lambda/\mathfrak{m}\Lambda$-module from the above lemmas. So, $\Lambda/\mathfrak{m}\Lambda$ is a simple algebra over the field $R/\mathfrak{m}$. By virtue of the classical theory, $\Lambda/\mathfrak{m}\Lambda$-irreducible modules are all isomorphic to each other. Using Lemmas 6, 7, (Λ, R)-irreducible Λ-modules are all isomorphic to each other ; i.e. Λ is strongly simple. The assertion about Δ is proved straightforword.

REMARK. If Λ is semisimale, then $\mathfrak{m}\Lambda$ is the Jocobson radical of Λ. So, the last half part of Theorem 9 means that a semisimple R-algebra $\Lambda(\Delta)$ is a simple (division) algebra if and only if $\Lambda(\Delta)$ is a primary (completely primary) ring.

Corollary 9. *Any simple algebra Λ over a complete local ring is a full matric algebra $(\Delta)_n$ with entries in the division algebra Δ to which Λ belongs. And the division algebra Δ and the matrix degree n are uniquely determined.*

Corollary 10. *For any simple algebra Λ over a complete local ring, every finitely generated Λ-projective module is completely faithful.*

3. The Brauer-equivalence. Commutor.

In this section R will denote a complete local ring with the unique maximal ideal $\mathfrak{m}$. We first give;

Proposition 11. *Let Λ_1, Λ_2 be simple R-algebras, and Δ_1, Δ_2 be division algebras to which Λ_1, Λ_2 belong respectively. Then Δ_1 and Δ_2 are algebra isomorphic if and only if there exist natural numbers n_1, n_2 such that $\Lambda_1 \underset{R}{\otimes} (R)_{n_1}$ and $\Lambda_2 \underset{R}{\otimes} (R)_{n_2}$ are isomorphic.*

Proof. 'only if' part: We put $\Delta = \Delta_1 \simeq \Delta_2$ and assume $\Lambda_1 = (\Delta)_{n_2}$ and $\Lambda_2 = (\Delta)_{n_1}$, then we get $\Lambda \underset{R}{\otimes} (R)_{n_1} \simeq (\Delta)_{n_1 n_2} \simeq \Lambda_2 \underset{R}{\otimes} (R)_{n_2}$

'if' part: We assume $\Lambda_1 \simeq (\Delta_1)_{t_1}$, $\Lambda_2 \simeq (\Delta_2)_{t_2}$. By assumption, there exist n_1, n_2, such that $\Lambda_1 \underset{R}{\otimes} (R)_{n_1} \simeq \Lambda_2 \underset{R}{\otimes} (R)_{n_2}$; that is $(\Delta_1)_{t_1 n_1} \simeq (\Delta_2)_{t_2 n_2}$. So Δ_1, Δ_2 are isomorphic, for simple algebra $(\Delta_1)_{t_1 n_1}$ belongs to both Δ_1 and Δ_2.

We call that R-simple algebras Λ_1, Λ_2 are *Brauer-equivalent* if there exist natural numbers n_1, n_2 such that $\Lambda_1 \underset{R}{\otimes} (R)_{n_1} \simeq \Lambda_2 \underset{R}{\otimes} (R)_{n_2}$(*).

The simple algebra which is Brauer-equivalent to a central separable algebra is also central separable.

Corollary 12. *Let Λ_1, Λ_2 be central separable R-algebras (they are simple by Theorem 4 of [7]) and Δ_1, Δ_2 be division algebras to which Λ_1, Λ_2 belong respectively. Λ_1, Λ_2 give rise to the same element of Brauer group if and only if Δ_1 and Δ_2 are isomorphic.*

Proposition 13. *Two R-simple algebras are Brauer-eqivalent whenever they are Morita-equivalent.*

Proof. Let Λ_1, Λ_2 be simple algebras which are Morita-equivalent and Δ_2 be the division algebra to which Λ_2 belongs. By hypothesis, there exists a finitely generated Λ_2-projective (completely faithful) module E such that $\Lambda_1^0 \simeq \mathrm{Hom}_{\Lambda_2}(E, E)$. E can be decomposed to direct sum of mutually isomorphic (Λ_2, R)-irreducible modules; $E = E_0 \oplus \cdots \oplus E_0$. Hence, we get, $\Lambda \simeq \mathrm{Hom}_{\Lambda_2}(E_0 \oplus \cdots \oplus E_0,\ E_0 \oplus \cdots \oplus E_0) \simeq (\mathrm{Hom}_{\Lambda_2}(E_0, E_0))_n \simeq (\Delta)_n$. So Λ_1 belongs to Δ_2, i.e. Λ_1 is Brauer-equivalent to Λ_2.

In [2] Theorem 6.5, Auslander and Goldman proved that the Brauer group of R is isomorphic to the Brauer group of $R/\mathfrak{m}$. More precisely we have a one-to-one correspondence between central separable algebras over R and central separable (or equivalently central simple) algebras over $R/\mathfrak{m}$.

Proposition 14. *a) The central separable R-algebras Λ_1, Λ_2 are isomorphic if and ooly if $\Lambda_1/\mathfrak{m}\Lambda_1$, $\Lambda_2/\mathfrak{m}\Lambda_2$ are isomorphic as $R/\mathfrak{m}$-algebras. b) For any central separable $R/\mathfrak{m}$-algebra $\mathfrak{A}$, there exists a central separable R-algebra Λ such that $\Lambda/\mathfrak{m}\Lambda \simeq \mathfrak{A}$*

Proof. It is clear that if a simple algebra Λ belongs to a division algebra Δ, then $\Lambda/\mathfrak{m}\Lambda$ belongs to $\Delta/\mathfrak{m}\Delta$. Theorem 6.5 of [2] means;

(*) Auslander and Goldman defined this equivalence using the R-endomorphism ring $\mathrm{Hom}_R(E\,E)$ of R-projective module E. But in our local case, $\mathrm{Hom}_R(E\,E) \simeq (R)_n$, for E is R-free. (c.f. [2])

a′) R-central separable division algebras Δ_1, Δ_2 are isomorphic if $\Delta_1/\mathfrak{m}\Delta_1$, $\Delta_2/\mathfrak{m}\Delta_2$ are isomorphic, and b′) for any division algebra $\mathfrak{D}$ over $R/\mathfrak{m}$, there exists a central separable division algebra Δ over R such that $\Delta/\mathfrak{m}\Delta$, $\mathfrak{D}$ are isomorphic. In fact, we let Δ be the division algebra to which Λ belongs, where Λ is a central separable R-algebra such that $\Lambda/\mathfrak{m}\Lambda$ is Brauer-equivalent to $\mathfrak{D}$.

We put $\Lambda_1 \simeq (\Delta_1)_{n_1}$, $\Lambda_2 \simeq (\Delta_2)_{n_2}$. $\Lambda_1/\mathfrak{m}\Lambda_1 \simeq \Lambda_2/\mathfrak{m}\Lambda_2$ means $(\Delta_1/\mathfrak{m}\Delta_1)_{n_1} \simeq (\Delta_2/\mathfrak{m}\Delta_2)_{n_2}$, so we get $\Delta_1/\mathfrak{m}\Delta_1 \simeq \Delta_2/\mathfrak{m}\Delta_2$ and $n_1 = n_2$. By a′) we get $\Lambda_1 \simeq \Lambda_2$. So a) is proved. We assume $\mathfrak{A}$ belongs to $\mathfrak{D}$, namely, $\mathfrak{A} \simeq (\mathfrak{D})_n$. By b′) there exists an R-central separable division algebra Δ such that $\Delta/\mathfrak{m}\Delta \simeq \mathfrak{D}$. Setting $\Lambda = (\Delta)_n$, we get $\Lambda/\mathfrak{m}\Lambda \simeq \mathfrak{A}$. q.e.d.

Let Γ be an arbitrary ring and Λ be a subring of Γ. We denote the commutor ring of Λ in Γ by $V_\Gamma(\Lambda)$: $V_\Gamma(\Lambda) = \{\gamma \in \Gamma \mid \gamma\lambda = \lambda\gamma$ for all $\lambda \in \Lambda\}$.

The commutor theory of a central separable algebra and its simple subalgebras holds analogusly in our complete local case as the classical theory.

Theorem 16. *Let Γ be a central separable R-algebra and Λ be a simple subalgebra of Γ which is an R-direct summand of Γ. Then,*

a) $V_\Gamma(\Lambda)$ *is a simple algebra*

b) $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$

c) $\Lambda \otimes \Gamma^0$ *is a simple algebra, and its belonging division algebra is anti-isomorphic to the division algebra to which $V_\Gamma(\Lambda)$ belongs.*

Proof. b) is proved in [6], Theorem 3.5. We know $V_\Gamma(\Lambda)$ is naturally isomorphic to $\mathrm{Hom}_{\Lambda\otimes\Gamma_0}(\Gamma, \Gamma)$ and $\Lambda\otimes\Gamma^0$ is semisimple, and if Λ is R-direct summand, then Γ is a finitely generated projective, completely faithful $\Lambda\otimes\Gamma^0$-module (§ 3 of [6]) i.e. $V_\Gamma(\Lambda)$ is Morita-dual to $\Lambda\otimes\Gamma^0$, or $V_\Gamma(\Lambda)^0$, $\Lambda\otimes\Gamma^0$ are Brauer-equivalent by Proposition 14 provided they are simple. So it remains only to prove the simplicity of $\Lambda\otimes\Gamma^0$.

Let $\mathfrak{m}$ be the maximal ideal of R, then

$$\Lambda \otimes \Gamma^0/\mathfrak{m}(\Lambda \otimes \Gamma^0) \simeq \Lambda/\mathfrak{m}\Lambda \underset{R/\mathfrak{m}}{\otimes} (\Gamma/\mathfrak{m}\Gamma)^0 .$$

By Theorem 9, $\Lambda\otimes\Gamma^0$ is simple.

REMARK. For a central separable R-algebra Γ and an arbitrary simple algebra Λ, $\Lambda\otimes\Gamma$ is always a simple algebra. (See, the proof of of theorem above)

4. Over a complete valuation ring.

Over a Dedekind domain, any torsionfree semisimple algebra can be decomposed to the direct sum of simple algebras (§4 of [6] and Theorem 3 of [7])

Let R be a complete, discrete rank one valuation ring, and Γ be a torsionfree semisimple R-algebra.

Set $\Gamma=\Lambda_1\oplus\cdots\oplus\Lambda_2$; where each Λ_i ($i=1, 2, \cdots, r$) is a torsionfree simple algebra, and set $\Lambda_i=L_i\oplus\cdots\oplus L_i$; where L_i is a (Λ_i, R)-irreducible left ideal. So $L_iL_j=0$ if $i\neq j$. Since the (Γ, R)-irreducibility and the (Λ_i, R)-irreducibility coincide, and the decomposition to direct sum of (Γ, R)-irreducible ideals is unique up to isomorphism, the direct sum of all isomorphic (Γ, R)-irreducible components of one decomposition of Γ is a two sided ideal and a simple algebra.

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