ON SEPARABLE ALGEBRAS OVER A COMMUTATIVE RING*

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Introduction. The notion of a separable algebra over a commutative ring was introduced in Auslander-Goldman [2], which coincides with that of a maximally central algebra in Azumaya [3] for a central algebra over a local ring. The basic properties of separable algebras were shown in [2] and [3].

The purpose of this paper is to define the reduced trace and norm of a central separable algebra over a commutative ring and to prove that a separable algebra over a commutative ring is a symmetric algebra.

Let \( \Lambda \) be a central separable algebra over a commutative ring \( R \) and let \( S \) be a commutative \( R \)-algebra such that \( S \otimes_R \Lambda \cong \text{Hom}_S(P, P) \) for some finitely generated, faithful, projective \( S \)-module \( P \). Then \( S \) is called, according to [2], a splitting ring of \( \Lambda \), and especially, if \( R \subseteq S \), it is called a proper splitting ring of \( \Lambda \). It was proved in [2] that a central separable algebra over a Noetherian local ring \( R \) has a proper splitting ring which is a Galois extension of \( R \). However, for a general commutative ring \( R \), it is an open problem whether any central separable \( R \)-algebra has a proper (Galois) splitting ring. Therefore, our method, which will be used to defining the reduced trace and norm of a central separable \( R \)-algebra, is different from the usual one in the classical case (cf. [4]).

In § 1 we shall show that a separable algebra over a general commutative ring is extended from a separable algebra over a Noetherian commutative ring, and, in § 2, we shall prove that, in case \( R \) is a commutative ring included in a semi-local ring, a central separable \( R \)-algebra has a proper splitting ring.

§ 3 is devoted to defining the reduced trace of a central separable \( R \)-algebra \( \Lambda \). If \( \Lambda \) has a proper splitting ring, we can define the reduced characteristic polynomial, trace and norm of \( \Lambda \) by using the characteristic polynomial, trace and norm of a projective module in [7], and we shall also show that there exist the analogous relations to the classical case between these and the characteristic polynomial, trace and norm of an \( R \)-algebra \( \Lambda \). In the general case, we define the reduced trace of \( \Lambda \), by using the above-mentioned result in § 1.

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An algebra $\Lambda$ over a commutative ring $R$, which is a finitely generated, faithful, projective $R$-module, is called, according to [6], a symmetric $R$-algebra, if $\text{Hom}_R(\Lambda, R)$ is $\Lambda^e$-isomorphic to $\Lambda$. In the classical theory, it is well known that any semi-simple algebra over a field is symmetric. However, for a general commutative ring $R$, it is an open problem whether a semi-simple $R$-algebra is symmetric or not.

In § 4 we shall prove, as a partial answer to this, that a separable algebra over a commutative ring is symmetric. This includes the results in Müller [10] and DeMeyer [5]. Throughout this paper a ring means a ring with a unit element, and a (semi-)local ring means a commutative (semi-)local ring which is not always Noetherian.

1. Basic results

First we shall prove, as a generalization of (4.5) and (4.7) in [2],

**Proposition 1.1.** Let $\Lambda$ be an algebra over a (not always Noetherian) commutative ring $R$, which is a finitely generated $R$-module. Then the following conditions are equivalent:

1. $\Lambda$ is a separable $R$-algebra.
2. For any maximal ideal $m$ of $R$, $\Lambda_m$ is a separable $R_m$-algebra.
3. For any maximal ideal $m$ of $R$, $\Lambda/m\Lambda$ is a separable $R/m\Lambda$-algebra.

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are obvious.

(2)$\Rightarrow$(1): We have $\text{w.dim}_{\Lambda^e}\Lambda = \sup_m \text{w.dim}_{\Lambda^e_m}\Lambda_m$ where $m$ runs over all maximal ideals of $R$. If each $\Lambda_m$ is $R_m$-separable, then we have $\text{w.dim}_{\Lambda^e_m}\Lambda_m = 0$ and so $\text{w.dim}_{\Lambda^e}\Lambda = 0$. As $\Lambda$ is $\Lambda^e$-finitely presented, this shows that $\Lambda$ is $\Lambda^e$-projective.

(3)$\Rightarrow$(2): Without loss of generality we may assume that $R$ is a local ring with a maximal ideal $m$. Now suppose that $\Lambda/m\Lambda = R/m\Lambda$ and so $\Lambda$ is $\Lambda^e$-projective if and only if $\Lambda$ is $\Lambda^e$-projective. Hence we may further assume that $R$ is Henselian. Then, for the projective $\Lambda^e/m\Lambda^e$-module $\Lambda/m\Lambda$, there is a finitely generated projective $\Lambda^e$-module $P$ such that $f: P/mP \cong \Lambda/m\Lambda$ as $\Lambda^e$-modules. Since $R$ is local and $P$, $\Lambda^e$ are $\Lambda^e$-projective, there exist $\Lambda^e$-epimorphisms $f: P \rightarrow \Lambda$, which induces $f$ on $P/mP$, and $g: \Lambda^e \rightarrow P$ such that $f \circ g$ is the natural epimorphism of $\Lambda^e$ onto $\Lambda$. The homomorphism $f \circ g$ is $R$-split and so $f$ is also $R$-split. From this it follows directly that $f$ is an isomorphism. Thus $\Lambda$ is $\Lambda^e$-projective, which completes our proof.

It is remarked that, by (1.1), we can omit the assumption that $R$ is Noetherian from almost all of results in [2].
The following proposition will play an important part in § 3.

**Proposition 1.2.** Let $\Lambda$ be a separable $R$-algebra, which is a finitely generated, faithful, projective $R$-module. Then there exist a Noetherian subring $R'$ of $R$ and a separable $R'$-subalgebra $\Lambda'$ of $\Lambda$, which is a finitely generated, faithful, projective $R'$-module, such that $\Lambda = R \otimes_R \Lambda'$.

Proof. Let $\{\lambda_0 = 1, \lambda_1, \ldots, \lambda_t\}$ be a set of generators of $\Lambda$ over $R$. Let $F$ be a free $R$-module with a basis $\{u_0, u_1, \ldots, u_t\}$, and define the $R$-epimorphism $f: F \to \Lambda$ by putting $f(u_i) = \lambda_i$ for each $i$. Since $\Lambda$ is $R$-projective, we have an $R$-homomorphism $g: \Lambda \to F$ such that $f \circ g = 1_\Lambda$. Now we put $g(\lambda_i) = \sum r_i u_j$, $r_{ij} \in R$. Let $R_0$ be the prime ring of $R$ and $x$ the polynomial ring over $R_0$ generated by $\{r, \lambda\}$. Then the module generated by $\lambda_0, \lambda_1, \ldots, \lambda_t$ over $R_0$ is finitely generated and $\Lambda$ is $R_0$-projective. As $\Lambda$ is $R$-separable, defining the $\Lambda^e$-epimorphism $\phi: \Lambda^e \to \Lambda$ by putting $\phi(\lambda_i \otimes R \lambda_0) = \lambda_i \lambda_j$ and $\phi_j(\lambda_i \otimes R \lambda_0) = \lambda_i \lambda_j$, there is a $\Lambda^e$-homomorphism $\psi: \Lambda \to \Lambda^e$ such that $\phi \psi = 1$. Put $\psi(\lambda_i) = \sum s_{ijk}(\lambda_j \otimes R \lambda_k)$, $s_{ijk} \in R$ and $\lambda_i \lambda_j = \sum t_{ijk} \lambda_k$, $t_{ijk} \in R$. Furthermore let $R'$ be the polynomial ring over $R_0$ generated by $\{r_{ij}\}$, $\{s_{ijk}\}$ and $\{t_{ijk}\}$, and denote by $\Lambda'$ the module generated by $\lambda_0, \lambda_1, \ldots, \lambda_t$ over $R'$. Then $R'$ is Noetherian, and $\Lambda'$ is an $R'$-algebra which is a finitely generated, faithful, projective $R'$-module, as $R'$ includes all of $\{r, \lambda\}$ and $\{t_{ijk}\}$. If we define a $\Lambda^e$-epimorphism $\phi': \Lambda^e \to \Lambda'$ by putting $\phi'(\lambda_i \otimes R \lambda_0) = \lambda_i \lambda_j$ and we put $\psi'(\lambda_i) = \sum s_{ijk}(\lambda_j \otimes R \lambda_k)$ for any $i$, then, from the fact that $\Lambda$ is $R$-finitely generated projective, we see easily that $\phi'$ is the well-defined $\Lambda^e$-homomorphism of $\Lambda'$ into $\Lambda^e$ such that $\phi' \circ \phi = 1_{\Lambda'}$. Therefore $\Lambda'$ is a separable $R'$-algebra. Let $\alpha$ be the $R$-algebra epimorphism of $R \otimes R \Lambda'$ onto $\Lambda$ which is defined by $\alpha(r \otimes R \lambda_i) = r \lambda_i$, for any $r \in R$. Let $m$ be a maximal ideal of $R$ and put $\nu = m \cap R'$. Then we have $(R \otimes R \Lambda')_m = R_m \otimes R' \Lambda'_\nu$ and so $\alpha$ induces naturally an $R_m$-algebra epimorphism $\alpha_m: R_m \otimes R' \Lambda'_\nu \to \Lambda_m$. Since $\Lambda'_\nu$ is $R'_\nu$-free, $\alpha_m$ must be an isomorphism. From this it follows immediately that $\alpha$ is an isomorphism. Thus our proof is completed.

2. Central separable algebras with proper splitting rings

Let $\Lambda$ be a central separable $R$-algebra and $S$ a commutative $R$-algebra. If there exists a finitely generated faithful projective $S$-module $P$ such that $S \otimes R \Lambda \cong \text{Hom}_R(P, P)$ as $S$-algebras, then $S$ is called, according to [2], the splitting ring of $\Lambda$. Especially, when $S \supseteq R$, $S$ is called the proper splitting ring of $\Lambda$.

First we give, as a slight generalization of [2], (6.3),

**Proposition 2.1.** Let $R$ be a local ring with a maximal ideal $m$ and $\Lambda$ a central separable $R$-algebra. Then $\Lambda$ has a proper splitting ring $S$ which is a
separable $R$-algebra and a finitely generated free $R$-module. Especially, if $R$ is Henselian, then we can choose as $S$ a local ring with a maximal ideal $mS$.

Proof. By using (1.1) and the Henselization instead of the completion, this can be proved along the same line as in [2], (6.3).

For a central separable algebra over a general commutative ring $R$, we can not assure the existence of the proper splitting ring which is $R$-separable and $R$-finitely generated, projective. In this section, we shall consider only the existence of proper splitting rings. However, we could not prove the existence of a proper splitting ring for a central separable algebra over a general coefficient ring.

**Proposition 2.2.** Let $R$ be a commutative ring which is contained in a semi-local ring. Then any central separable $R$-algebra has a proper splitting ring. Especially, this assumption for $R$ is satisfied by a Noetherian ring or an integral domain.

Proof. It suffices to prove this proposition in case $R$ is itself a semi-local ring. Let $R$ be a semi-local ring with maximal ideals $m_1, m_2, \ldots, m_t$ and put $R' = R_{m_1} \oplus R_{m_2} \oplus \cdots \oplus R_{m_t}$. Then $R \subseteq R'$ and $R' \otimes R = \mathfrak{m}_{m_1} \oplus \mathfrak{m}_{m_2} \oplus \cdots \oplus \mathfrak{m}_{m_t}$. Accordingly to (2.1), there exists a proper splitting ring $S_i$ of $R_{m_i}$ for any $i$. If we put $S = S_1 \oplus S_2 \oplus \cdots \oplus S_t$, then we have $R \subseteq R' \subseteq S$ and $S$ is a proper splitting ring of $R$, as is required.

As another case, which is not included in (2.2), we have

**Proposition 2.3.** Let $R$ be a commutative ring with the total quotient ring $K$ such that any prime ideal of $K$ is maximal. Then any central separable $R$-algebra has a proper splitting ring.

Proof. We may assume $R = K$. If we denote by $\mathfrak{n}$ the nil radical of $R$, then $R/\mathfrak{n}$ is, by our assumption, a regular ring (in the Neumann's sense). Therefore we may further assume that $\Lambda$ is a finitely generated free $R$-module. Let $\{u_1, u_2, \ldots, u_t\}$ be an $R$-basis of $\Lambda$ with $u_1 = 1$, and put $u_i u_j = \sum_{i=1}^{t} r_{ijk} u_k$, $r_{ijk} \in R$. Let $R_0$ be the prime ring of $R$, and put $R' = R_0[[t_{ijk}]]$ and $\Omega' = \{r'_{u_1} + \cdots + r'_{u_t} r_i \in R'\}$. Then $\Omega'$ is a central $R'$-algebra with an $R'$-basis $\{u_1, \ldots, u_t\}$, and we have $R \otimes R' = \Lambda$. Furthermore let $\hat{R}$ be the integral closure of $R'$ in $R$. Since $R/\mathfrak{n}$ is regular, any non-zero divisor of $\hat{R}$ is a unit in $R$, and therefore the total quotient ring $\hat{K}$ of $\hat{R}$ can be regarded as a subring of $R$. From the fact that $\hat{R}$ is integral over $R'$, we see that the total quotient ring $K'$ of $R'$ is included in $R$. Since $R'$ is Noetherian and $\hat{K}/\mathfrak{n} \cap \hat{K}$ is regular, $K'/\mathfrak{n} K'$ is Artinian, and so $K'$ is itself Artinian. If we put $\Lambda' = K' \otimes R'$, then $R' \otimes \Lambda' = \Lambda$ and, as $K'$ is Artinian, we can easily see that $\Lambda$ is a central separable $K'$-algebra. According to (2.1), there
exists a proper splitting ring $F$ of $\Lambda'$ which is a finitely generated projective $K'$-module. Now put $S=F\otimes R$. Then $S\cong F$, $R$ and $S\otimes\Lambda=S\otimes R\otimes\Lambda'=F\otimes R\otimes\Lambda'=(F\otimes R)\otimes F\otimes\Lambda'$. Consequently, $S$ is a proper splitting ring of $\Lambda$, which completes our proof.

3. The trace and norm of a central separable algebra

1. Let $R$ be a commutative ring and $P$ a finitely generated projective $R$-module. Suppose that $P$ has (constant) rank $n$. Then there exists a commutative ring $S\supseteq R$ such that $S\otimes P$ is a free $S$-module of rank $n$. Let $\{u_1, \ldots, u_n\}$ be a $S$-basis of $S\otimes P$. If $f\in\text{Hom}_R(P, P)$, then $f$ can be regarded as an element of $\text{Hom}_S(S\otimes P, S\otimes P)$, and we can put $f(u_i)=\sum_j u_is_{ij}$ for some $s_{ij}\in S$. Now put $Pc_p(f): X=|s_{ij}|-X\delta_{ij}|$, $T_p(f)=\text{traces (}s_{ij}\text{)}$ and $N_p(f)=|s_{ij}|$ where $X$ denotes an indeterminate. It can easily be shown by using the localization at any maximal ideal of $R$ that $Pc_p(f, X)\in R[X]$ and $T_p(f)$, $N_p(f)\in R$ and that these are determined without depending on $S$ and $\{u_1, \ldots, u_n\}$. If $P$ has not constant rank, there is, by [7], § 2, a unique decomposition $R=R_0\oplus\cdots\oplus R_\ell$ such that any $R_i\otimes P$ has rank $n_i$ over $R_i$ where $n_1<\cdots<n_\ell$, and we have $\text{Hom}_R(P, P)=\sum_{i=1}^\ell \oplus \text{Hom}_{R_i}(R_i\otimes P, R_i\otimes P)$. Let $f$ be an element of $\text{Hom}_R(P, P)$ and $f_i$ the $i$-th component of $f$. Then we put $Pc_p(f, X)=\sum_{i=1}^\ell \oplus Pc_p(f_i, X)$, $T_p(f)=\sum_{i=1}^\ell \oplus T_{R_i}(f_i)$ and $N_p(f)=\sum_{i=1}^\ell \oplus N_{R_i}(f_i)$ and we call them the characteristic polynomial, trace and norm of $f$. It can be easily shown that our definitions coincide with those in [7].

If $\Lambda$ is an $R$-algebra which is a finitely generated projective $R$-module, then we use $Pc_{\Lambda/R}(f, X)$, $T_{\Lambda/R}(f)$ and $N_{\Lambda/R}(f)$ instead of $Pc_p(f, X)$, $T_p(f)$ and $N_p(f)$.

2. Now we shall define the reduced characteristic polynomial, trace and norm for a central separable algebra with a proper splitting ring.

Let $\Lambda$ be a central separable $R$-algebra with a proper splitting ring $S$. Then there exists a $S$-algebra isomorphism $h_S: S\otimes\Lambda\cong \text{Hom}_S(P^{(S)}, P^{(S)})$ for some finitely generated projective $S$-module $P^{(S)}$.

**Proposition 3.1** For any element $\lambda$ of $\Lambda$, $Pc_{\Lambda,R}(h_S(\lambda): X)$ is a polynomial of $R[X]$ which does not depend on $S$, $P^{(S)}$ and $h_S$.

**Proof.** First suppose that $R$ is a local ring. Then $\Lambda$ is a projective $R$-module of constant rank, and so $P^{(S)}$ is also a projective $S$-module of constant rank. By replacing $S$ by any extension ring $S'$ of it and by replacing $h_S$ by $1\otimes h'_S$, $Pc_{\Lambda,R}(h_S(\lambda): X)$ is invariant, and therefore we may further assume that
$P^{(S)}$ is $S$-free. Then $h_S$ induces a $S$-algebra isomorphism $k_S: S \otimes \Lambda \cong M_n(S)$ such that $P_{c_{P^{(S)}}(h_S(\lambda))}(X) = |XE_n - k_S(\lambda)|$. On the other hand, according to (2.1), there exists a proper splitting semi-local ring $T$ of $\Lambda$ which is $R$-free. For $T$ we can define, similarly, $h_T, P^{(T)}$ and $k_T$. Since $T$ is $R$-free, we have $R \otimes R = S \otimes R \cap R \otimes T$ in $S \otimes T$, and so we may suppose that there is a commutative ring $U$ containing both $S$ and $T$ and $S \cap T = R$ in $U$. Now the algebra isomorphisms $k_S: S \otimes \Lambda \cong M_n(S)$ and $k_T: T \otimes \Lambda \cong M_n(T)$ can, naturally, be extended to the $U$-algebra isomorphisms $k_S^U, k_T^U: U \otimes \Lambda \cong M_n(U)$. Then $k_S^U \circ k_T^U$ is an $U$-algebra automorphism of $M_n(U)$ and it induces an $U_m$-algebra automorphism of $M_n(U_m)$ for any maximal ideal of $U$. As $U_m$ is a local ring, it is inner, and so we have $|XE_n - k_S^U(\lambda^*)| = |XE_n - k_T^U(\lambda^*)| = 0$ in $U_m[X]$ for any $\lambda^* \in U \otimes \Lambda$. Hence we have $P_{c_{P^{(S)}}}(h_S(\lambda)): X) = |XE_n - k_S(\lambda)| = |XE_n - k_T(\lambda)| = P_{c_{P^{(S)}}}(h_T(\lambda)): X$ in $U[X]$. However, as $P_{c_{P^{(S)}}}(h_S(\lambda)): X) \in S_m[X]$ and $P_{c_{P^{(S)}}}(h_T(\lambda)): X) \in T_m[X]$, we obtain $P_{c_{P^{(S)}}}(h_S(\lambda)): X) = P_{c_{P^{(S)}}}(h_T(\lambda)): X) \in U[X] = S[X] \cap T[X]$. Thus $P_{c_{P^{(S)}}}(h_S(\lambda)): X)$ is a polynomial of $R[X]$. It is obvious from the above proof that this does not depend on $S, P^{(S)}$ and $h_S$, which completes our proof for a local ring $R$.

Let $R$ be a general commutative ring and $m$ a maximal ideal of $R$. Denote by $\lambda_m$ the residue of $\lambda$ in $\Lambda_m$ and by $h_S^m$ the $S_m$-algebra isomorphism: $S_m \otimes \Lambda_m \cong \text{Hom}_{S_m}(P_m, P_m)$ induced by $h_S$. Further let $[P_{c_{P^{(S)}}}(h_S(\lambda)): X]_m$ be the residue of $P_{c_{P^{(S)}}}(h_S(\lambda)): X)$ in $S_m[X]$. Then we see $[P_{c_{P^{(S)}}}(h_S(\lambda)): X]_m = P_{c_{P^{(S)}}}(h_S^m(\lambda_m))$ in $S_m[X]$. Since, by the preceding argument for a local ring, $P_{c_{P^{(S)}}}(h_S^m(\lambda_m)): X) \in R_m[X]$, we have also $[P_{c_{P^{(S)}}}(h_S(\lambda)): X]_m \in R_m[X]$. Consequently we obtain $P_{c_{P^{(S)}}}(h_S(\lambda)): X) \in R[X]$. It is also evident in this case that $P_{c_{P^{(S)}}}(h_S(\lambda)): X)$ does not depend on $S, P^{(S)}$ and $h_S$.

Now we denote $P_{c_{P^{(S)}}}(h_S(\lambda)): X)$ by $P_{crd_{\Lambda/R}}(\lambda): X)$ and we call it the reduced characteristic polynomial of $\lambda$. Furthermore, if we put $\text{Trd}_{\Lambda/R}(\lambda) = T_{P^{(S)}}(h_S(\lambda))$ and $\text{Nrd}_{\Lambda/R}(\lambda) = N_{P^{(S)}}(h_S(\lambda))$, then they are elements of $R$ which do not depend on $S, P^{(S)}$ and $h_S$ and we call them the reduced trace and norm of $\lambda$, respectively.

From our definitions it follows immediately

**Proposition 3.2.** For any $\lambda, \lambda_1, \lambda_2 \in \Lambda$ and any $r \in R$, we have

$$\text{Trd}_{\Lambda/R}(\lambda_1 + \lambda_2) = \text{Trd}_{\Lambda/R}(\lambda_1) + \text{Trd}_{\Lambda/R}(\lambda_2),$$
$$\text{Trd}_{\Lambda/R}(r\lambda) = r \text{Trd}_{\Lambda/R}(\lambda),$$
$$\text{Trd}_{\Lambda/R}(\lambda_1 \lambda_2) = \text{Trd}_{\Lambda/R}(\lambda_1 \lambda_2),$$
$$\text{Nrd}_{\Lambda/R}(\lambda_1 \lambda_2) = \text{Nrd}_{\Lambda/R}(\lambda_1) \text{Nrd}_{\Lambda/R}(\lambda_2).$$

Especially, if $\Lambda$ has rank $n^2$ over $R$, then we have
Nrd_{\Lambda/R}(r\lambda) = r^n Nrd_{\Lambda/R}(\lambda)

From this proposition, it follows that Trd_{\Lambda/R} is an R-homorphism of \Lambda into R and Nrd_{\Lambda/R} is a semi-group homomorphism of \Lambda into R as the multiplicative semi-groups.

For any maximal ideal \( m \) of \( R \), let \([\text{Prd}_{\Lambda/R}(\lambda; X)]_m\) be the residue of \( \text{Prd}_{\Lambda/R}(\lambda; X) \) in \((R/m)[X]\) and denote by \( \bar{\lambda}_m \) the residue of \( \lambda \) in \( \Lambda/m\Lambda \). Now we can show \([\text{Prd}_{\Lambda/R}(\lambda; X)]_m = \text{Prd}_{\Lambda/m\Lambda/R/m}(\bar{\lambda}_m; X)\). In fact, it suffices to prove this in case \( R \) is a Henselian local ring with a maximal ideal \( m \). However, in this case, there is, by (2.1), a proper splitting local ring \( S \) of \( \Lambda \) such that \( mS \) is a maximal ideal of \( S \) and \( S \) is a finitely generated free \( R \)-module. Then \( S/mS \) becomes the splitting field of the classical central separable \( R/m \)-algebra \( \Lambda/m\Lambda \), from which our result follows immediately. Accordingly, \( \text{Trd}_{\Lambda/R} \) and \( \text{Nrd}_{\Lambda/R} \) induce, naturally, \( \text{Trd}_{\Lambda/m\Lambda/R/m} \) and \( \text{Nrd}_{\Lambda/m\Lambda/R/m} \), respectively, which coincide with those in the classical sense. By summarizing these, we obtain

**Proposition 3.3.** For any maximal ideal \( m \) of \( R \), the residue of \( \text{Prd}_{\Lambda/R} \) in \((R/m)[X]\) coincides with \( \text{Prd}_{\Lambda/m\Lambda/R/m} \). Especially, the residues of \( \text{Trd}_{\Lambda/R} \) and \( \text{Nrd}_{\Lambda/R} \) in \( R/m \) coincide with \( \text{Trd}_{\Lambda/m\Lambda/R/m} \) and \( \text{Nrd}_{\Lambda/m\Lambda/R/m} \), respectively.

3. Here we shall determine the relations between the trace (norm) and reduced trace (reduced norm) of a central separable algebra, which are given in the same form as in the classical one (cf. [4]).

Assume that \( \Lambda \) is a projective \( R \)-module of the constant rank \( m \). Then we may suppose \( S \otimes \Lambda \simeq M_n(S) \), where \( m = n^2 \). From our definitions, it follows directly that \( \text{Trd}_{\Lambda/R}(1) = n, \text{Trd}_{\Lambda/R}(\lambda) = n \text{Trd}_{\Lambda/R}(\lambda) \) and \( \text{Nrd}_{\Lambda/R}(\lambda) = [\text{Nrd}_{\Lambda/R}(\lambda)]^n \).

In the general case, let \( R = R_1 \oplus \cdots \oplus R_t \) be the unique decomposition of \( R \) such that \( R_i \otimes \Lambda \) has rank \( m_i \) over \( R_i \) where \( m_1 < m_2 < \cdots < m_t \). Then we can put \( m_i = n_i^2 \) for any \( i \). Let \( e_i \) be a unit element of \( R_i \) and \( \lambda_i \) the \( i \)-th component of \( \lambda \). Then we obtain

**Proposition 3.4.** \( \text{Trd}_{R_i \otimes \Lambda/R_i}(e_i) = n_i e_i \) for each \( i \),

\[
\text{T}_{\Lambda/R}(\lambda) = \text{Trd}_{\Lambda/R}(1) \text{Trd}_{\Lambda/R}(\lambda) = \sum_{i=1}^t n_i \text{Trd}_{R_i \otimes \Lambda/R_i}(\lambda_i)
\]

\[
\text{N}_{\Lambda/R}(\lambda) = \sum_{i=1}^t [\text{Nrd}_{R_i \otimes \Lambda/R_i}(\lambda_i)]^n
\]

The following result will be used in § 4.

**Proposition 3.5.** \( \text{Trd}_{\Lambda/R} \) is an \( R \)-epimorphism of \( \Lambda \) onto \( R \).

Proof. By the remark after (3.2), it suffices to prove that \( \text{Trd}_{\Lambda/R} \) is an epimorphism. By virtue of the classical result, for any maximal ideal \( m \) of \( R \), \( \text{Trd}_{\Lambda/m\Lambda/R/m} \) is an epimorphism of \( \Lambda/m\Lambda \) onto \( R/m \). According to (2.3), then, \( \text{Trd}_{\Lambda/R} \) must be an epimorphism of \( \Lambda \) onto \( R \),
Corollary 3.6. The complete image $T_{A/R}(\Lambda)$ of $T_{A/R}$ is a principal ideal of $R$ generated by $\text{Trd}_{A/R}(1)$. Especially, $\Lambda$ is strongly separable if and only if $\text{Trd}_{A/R}(1)$ is a unit of $R$.

Proof. This is an immediate consequence of (3.4) and (3.5).

4. As is remarked in § 2, we could not succeed in proving the existence of a proper splitting ring for a central separable algebra in the general case. Hence we cannot define the reduced characteristic polynomial for a central separable algebra in the case where we cannot show the existence of a proper splitting ring. However we can define, by using (1.2), the reduced trace for any central separable $R$-algebra $\Lambda$. In fact, by virtue of (1.2), there exist a Noetherian subring $R'$ of $R$ and a central separable $R'$-algebra $\Lambda'$ such that $\Lambda=R\otimes_{R'}\Lambda'$. Since $\Lambda'$ has a proper splitting ring by (2.2), there exists, according to 2, the reduced trace $\text{Trd}_{\Lambda'/R'}: \Lambda'\to R'$. Now we define the reduced trace $\text{Trd}_{\Lambda/R}: \Lambda\to R$, by putting $\text{Trd}_{\Lambda/R}(r\otimes\lambda')=r\text{Trd}_{\Lambda'/R'}(\lambda')$ for any $r\in R$ and for any $\lambda'\in\Lambda'$. It can be easily shown that, for any maximal ideal $m$ of $R$, the $R_m$-homomorphism $(\text{Trd}_{\Lambda/R})_m: \Lambda_m\to R_m$, which is induced on $\Lambda_m$ by $\text{Trd}_{\Lambda/R}$, coincides with the reduced trace $\text{Trd}_{\Lambda_m/R_m}$ of $\Lambda_m$ defined by using the proper splitting ring of $\Lambda_m$. Especially, if $\Lambda$ has a proper splitting ring, $\text{Trd}_{\Lambda/R}$ coincides with that defined in 2. Furthermore we can also prove (3.2)-(3.6) in this case.

4. The symmetricity of a separable algebra

Let $\Lambda$ be an $R$-algebra, which is a finitely generated, faithful, projective $R$-module. We shall consider $\Lambda^*=\text{Hom}_R(\Lambda, R)$ as a left $\Lambda^*$-module through the operations $(\lambda \cdot f)(\mu)=f(\mu\lambda), (f \cdot \lambda)(\mu)=f(\lambda\mu)$ where $f\in\Lambda^*, \lambda, \mu\in\Lambda$. Following [6], we call $\Lambda$ a Frobenius $R$-algebra if $\Lambda^*$ is isomorphic to $\Lambda$ as left (or equivalently right) $\Lambda$-modules, and, furthermore, is called a symmetric $R$-algebra if $\Lambda^*$ is $\Lambda^*$-isomorphic to $\Lambda$. From our definitions it follows that any symmetric $R$-algebra is Frobenius.

We begin with

Lemma 4.1. Let $S$ be a symmetric, commutative $R$-algebra and $\Lambda$ a symmetric $S$-algebra. Then $\Lambda$ is a symmetric $R$-algebra.

Proof. By our assumptions we have $\Lambda\simeq\text{Hom}_S(\Lambda, S)$ as two-sided $\Lambda$-modules and $S\simeq\text{Hom}_R(S, R)$ as $S$-modules. So we obtain $\text{Hom}_S(\Lambda, S)\simeq\text{Hom}_S(\Lambda, \text{Hom}_R(S, R))\simeq\text{Hom}_R(\Lambda\otimes_S R, R)\simeq\text{Hom}_R(\Lambda, R)$ as two-sided $\Lambda$-modules. This shows that $\Lambda$ is a symmetric $R$-algebra.

It is well known, in the classical theory, that a semi-simple algebra over a field is symmetric. However, for any commutative ring $R$, it is an open question whether a semi-simple $R$-algebra is symmetric or not,
Now we give, as a partial answer to this question,

**Theorem 4.2.** A separable R-algebra Λ, which is a finitely generated, faithful, projective R-module, is a symmetric R-algebra.

Proof. Let C be the center of Λ. According to [2] (2.1), Λ is a finitely generated projective C-module. By our assumption, Λ is R-finitely generated projective, and so C is also a finitely generated projective R-module, as C is a C-direct summand of Λ. Since, by [2], A.4, a commutative separable R-algebra, which is a finitely generated, faithful, projective R-module, is symmetric, C must be a symmetric R-algebra. Therefore, by (4.1), it suffices to prove our theorem in case \( R=C \).

Let Λ be a central separable R-algebra and denote by \( \text{Trd}_{\Lambda/R} \) the reduced trace of Λ, defined in § 3. Then \( \text{Trd}_{\Lambda/R} \) is a symmetric R-homomorphism of Λ into \( R \): i.e., we have \( \text{Trd}_{\Lambda/R}(\lambda \mu) = \text{Trd}_{\Lambda/R}(\mu \lambda) \) for any \( \lambda, \mu \in \Lambda \). Hence, putting \( \Phi(\lambda)(\mu) = \text{Trd}_{\Lambda/R}(\lambda \mu) \) for any \( \lambda, \mu \in \Lambda \), \( \Phi \) is a \( \Lambda^* \)-homomorphism of Λ into \( \Lambda^* \). By (3.3), for any maximal ideal \( m \) of \( R \), \( \text{Trd}_{\Lambda/R} \) induces naturally the reduced trace \( \text{Trd}_{\Lambda/m\Lambda/R/m} \) in the classical sense on \( \Lambda/m\Lambda \), and therefore \( \Phi \) induces, naturally, the \( \Lambda^*/m\Lambda^* \)-homomorphism \( \Phi_m : \Lambda/m\Lambda \rightarrow \Lambda^*/m\Lambda^* \cong (\Lambda/m\Lambda)^* \) such that \( \Phi_m(\lambda)(\mu) = \text{Trd}_{\Lambda/m\Lambda/R/m}(\lambda \mu) \) for any \( \lambda, \mu \in \Lambda/m\Lambda \). From the classical result it follows that \( \Phi_m \) is a \( \Lambda^*/m\Lambda^* \)-isomorphism. As both Λ and \( \Lambda^* \) are finitely generated projective R-modules, we can easily see from this that \( \Phi \) itself is an isomorphism of Λ onto \( \Lambda^* \). This completes our proof.

We remark that (4.2) was known in some special cases (cf. [2], [5] and [10]). Finally we give, as an additional remark,

**Proposition 4.3.** Let \( \Lambda \) be a central R-algebra which is a finitely generated projective R-module. Then the following statements are equivalent:

1. \( \Lambda \) is a separable R-algebra.
2. The R-module \( \Lambda/[\Lambda, \Lambda] \) is isomorphic to \( R \), and, for any maximal ideal \( m \) of \( R \), \( \Lambda/m\Lambda \) is a semi-simple \( R/m \)-algebra.

Here we denote by \([\Lambda, \Lambda]\) the R-module generated by all elements of \( \Lambda \) in the form \( \lambda \mu - \mu \lambda \), \( \lambda, \mu \in \Lambda \).

Proof. (1)⇒(2): Suppose that \( \Lambda \) is a separable R-algebra. Then the second assertion of (2) follows from [2], (1.6) and so it suffices to prove \( \Lambda/[\Lambda, \Lambda] \cong R \). Let \( \text{Trd}_{\Lambda/R} \) be the reduced trace of Λ. Then \( \text{Trd}_{\Lambda/R} \) is a symmetric R-epimorphism of Λ onto \( R \), and therefore, putting Ker \( \text{Trd}_{\Lambda/R} = K \), we have an \( R \)-exact sequence:

\[
0 \rightarrow K \rightarrow \Lambda \xrightarrow{\text{Trd}_{\Lambda/R}} R \rightarrow 0
\]

and \( K \supseteq [\Lambda, \Lambda] \). Hence we have only to show \( K = [\Lambda, \Lambda] \). As is shown in § 3,
Trd\textsubscript{Λ/R} induces naturally the reduced trace Trd\textsubscript{Λ/mΛ/R/m} of Λ/mΛ for any maximal ideal m of R, and we have Ker Trd\textsubscript{Λ/mΛ/R/m} = K/mK. However, it is well known, in the classical theory, that the kernel of the reduced trace of a central separable R/m-algebra Λ/mΛ coincides with [Λ/mΛ, Λ/mΛ]. Consequently we must have K/mK = [Λ/mΛ, Λ/mΛ] for any maximal ideal m of R. From this we easily see K = [Λ, Λ], as K is R-finitely generated. Thus the implication (1)⇒(2) is proved. (2)⇒(1). Conversely suppose (2). By (1.1) it suffices to prove that Λ/mΛ has R/m as its center. By our assumption we have an R-exact sequence:

\[ 0 \to [Λ, Λ] \to Λ \overset{α}{\longrightarrow} R \to 0. \]

This induces an R/m-exact sequence:

\[ 0 \to [Λ, Λ]/m[Λ, Λ] \to Λ/mΛ \overset{α}{\longrightarrow} R/mR \to 0. \]

and so we have [Λ, Λ]/m[Λ, Λ] ≅ [Λ/mΛ, Λ/mΛ]. Therefore we have Λ/mΛ ≅ [Λ/mΛ, Λ/mΛ]⊕R/m. On the other hand, since Λ/mΛ is R/m-semisimple, Λ/mΛ is separable over its center C, and then we have Λ/mΛ ≅ [Λ/mΛ, Λ/mΛ]⊕C. As C ⊆ R/m, we see from these that C coincides with R/m. This completes our proof.

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**References**


