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ON SEPARABLE ALGEBRAS OVER A COMMUTATIVE RING*

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Introduction. The notion of a separable algebra over a commutative ring was introduced in Auslander-Goldman [2], which coincides with that of a maximally central algebra in Azumaya [3] for a central algebra over a local ring. The basic properties of separable algebras were shown in [2] and [3].

The purpose of this paper is to define the reduced trace and norm of a central separable algebra over a commutative ring and to prove that a separable algebra over a commutative ring is a symmetric algebra.

Let Λ be a central separable algebra over a commutative ring R and let S be a commutative R-algebra such that $S \underset{R}{\otimes} \Lambda \cong \mathrm{Hom}_S(P, P)$ for some finitely generated, faithful, projective S-module P. Then S is called, according to [2], a *splitting ring* of Λ, and especially, if $R \subseteq S$, it is called a *proper splitting ring* of Λ. It was proved in [2] that a central separable algebra over a Noetherian local ring R has a proper splitting ring which is a Galois extension of R. However, for a general commutative ring R, it is an open problem whether any central separable R-algebra has a proper (Galois) splitting ring. Therefore, our method, which will be used to defining the reduced trace and norm of a central separable R-algebra, is different from the usual one in the classical case (cf. [4]).

In § 1 we shall show that a separable algebra over a general commutative ring is extended from a separable algebra over a Noetherian commutative ring, and, in § 2, we shall prove that, in case R is a commutative ring included in a semi-local ring, a central separable R-algebra has a proper splitting ring.

§ 3 is devoted to defining the reduced trace of a central separable R-algebra Λ. If Λ has a proper splitting ring, we can define the reduced characteristic polynomial, trace and norm of Λ by using the characteristic polynomial, trace and norm of a projective module in [7], and we shall also show that there exist the analogous relations to the classical case between these and the characteristic polynomial, trace and norm of an R-algebra Λ. In the general case, we define the reduced trace of Λ, by using the above-mentioned result in § 1.

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An algebra Λ over a commutative ring R, which is a finitely generated, faithful, projective R-module, is called, according to [6], a *symmetric R-algebra*, if $\mathrm{Hom}_R(\Lambda, R)$ is Λ^e-isomorphic to Λ. In the classical theory, it is well known that any semi-simple algebra over a field is symmetric. However, for a general commutative ring R, it is an open problem whether a semi-simple R-algebra is symmetric or not.

In § 4 we shall prove, as a partial answer to this, that a separable algebra over a commutative ring is symmetric. This includes the results in Müller [10] and DeMeyer [5].

Throughout this paper a ring means a ring with a unit element, and a (semi-) local ring means a commutative (semi-) local ring which is not always Noetherian.

1. Basic results

First we shall prove, as a generalization of (4.5) and (4.7) in [2],

Proposition 1.1. *Let Λ be an algebra over a (not always Noetherian) commutative ring R, which is a finitely generated R-module. Then the following conditions are equivalent:*

(1) *Λ is a separable R-algebra.*

(2) *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda_{\mathfrak{m}}$ is a separable $R_{\mathfrak{m}}$-algebra.*

(3) *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a separable $R/\mathfrak{m}$-algebra.*

Proof. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are obvious.

(2)$\Rightarrow$(1): We have $\mathrm{w.dim}_{\Lambda^e}\Lambda=\sup_{\mathfrak{m}} \mathrm{w.dim}_{\Lambda^e_{\mathfrak{m}}}\Lambda_{\mathfrak{m}}$ where $\mathfrak{m}$ runs over all maximal ideals of R. If each $\Lambda_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-separable, then we have $\mathrm{w.dim}_{\Lambda^e_{\mathfrak{m}}}\Lambda_{\mathfrak{m}}=0$ and so $\mathrm{w.dim}_{\Lambda^e}\Lambda=0$. As Λ is Λ^e-finitely presented, this shows that Λ is Λ^e-projective.

(3)$\Rightarrow$(2): Without loss of generality we may assume that R is a local ring with a maximal ideal $\mathfrak{m}$. Now suppose that $\Lambda/\mathfrak{m}\Lambda$ is $R/\mathfrak{m}$-separable. Let $\hat{R}$ be the Henselization of R and put $\hat{\Lambda}=\hat{R}\otimes_R\Lambda$. Then we have $\hat{R}/\mathfrak{m}\hat{R}=R/\mathfrak{m}$ and $\hat{\Lambda}/\mathfrak{m}\hat{\Lambda}=\Lambda/\mathfrak{m}\Lambda$. Since $\hat{R}$ is R-faithfully flat, we have $\mathrm{w.dim}_{\Lambda^e}\Lambda=\mathrm{w.dim}_{\hat{\Lambda}^e}\hat{\Lambda}$ and so Λ is Λ^e-projective if and only if $\hat{\Lambda}$ is $\hat{\Lambda}^e$-projective. Hence we may further assume that R is Henselian. Then, for the projective $\Lambda^e/\mathfrak{m}\Lambda^e$-module $\Lambda/\mathfrak{m}\Lambda$, there is a finitely generated projective Λ^e-module P such that $\bar{f}$: $P/\mathfrak{m}P\cong\Lambda/\mathfrak{m}\Lambda$ as Λ^e-modules. Since R is local and P, Λ^e are Λ^e-projective, there exist Λ^e-epimorphisms f: $P\to\Lambda$, which induces $\bar{f}$ on $P/\mathfrak{m}P$, and g: $\Lambda^e\to P$ such that $f\circ g$ is the natural epimorphism of Λ^e onto Λ. The homomorphism $f\circ g$ is R-split and so f is also R-split. From this it follows directly that f is an isomorphism. Thus Λ is Λ^e-projective, which completes our proof.

It is remarked that, by (1.1), we can omit the assumption that R is Noetherian from almost all of results in [2].

The following proposition will play an important part in § 3.

Proposition 1.2. *Let Λ be a separable R-algebra, which is a finitely generated, faithful, projective R-module. Then there exist a Noetherian subring R' of R and a separable R'-subalgebra Λ' of Λ, which is a finitely generated, faithful, projective R'-module, such that $\Lambda = R \otimes_{R'} \Lambda'$.*

Proof. Let $\{\lambda_0 = 1, \lambda_1, \cdots, \lambda_t\}$ be a set of generators of Λ over R. Let F be a free R-module with a basis $\{u_0, u_1, \cdots, u_t\}$, and define the R-epimorphism $f\colon F \to \Lambda$ by putting $f(u_i) = \lambda_i$ for each i. Since Λ is R-projective, we have an R-homomorphism $g\colon \Lambda \to F$ such that $f \circ g = 1_\Lambda$. Now we put $g(\lambda_i) = \sum_{j=1}^{t} r_{ij} u_j$, $r_{ij} \in R$. Let R_0 be the prime ring of R and R_1 the polynomial ring over R generated by $\{r_{ij}\}$. Then the module generated by $\lambda_0, \lambda_1, \cdots, \lambda_t$ over R_1 is R_1-projective. As Λ is R-separable, defining the Λ^e-epimorphism $\varphi\colon \Lambda^e \to \Lambda$ by putting $\varphi(\lambda_i \otimes_R \lambda_j^0) = \lambda_i \lambda_j$, there is a Λ^e-homomorphism $\psi\colon \Lambda \to \Lambda^e$ such that $\varphi\psi = 1$. Put $\psi(\lambda_i) = \sum_{j,k} s_{ijk}(\lambda_j \otimes_R \lambda_k^0)$, $s_{ijk} \in R$ and $\lambda_i \lambda_j = \sum_k t_{ijk} \lambda_k$, $t_{ijk} \in R$. Furthermore let R' be the polynomial ring over R_0 generated by $\{r_{ij}\}$, $\{s_{ijk}\}$ and $\{t_{ijk}\}$, and denote by Λ' the module over R' generated by $\lambda_0, \lambda_1, \cdots, \lambda_t$. Then R' is Noetherian, and Λ' is an R'-algebra which is a finitely generated, faithful, projective R'-module, as R' includes all of $\{r_{ij}\}$ and $\{t_{ijk}\}$. If we define a Λ'^e-epimorphism $\varphi'\colon \Lambda'^e \to \Lambda'$ by putting $\varphi'(\lambda_i \otimes_{R'} \lambda_j^0) = \lambda_i \lambda_j$ and we put $\psi'(\lambda_i) = \sum_{jk} s_{ijk}(\lambda_j \otimes_{R'} \lambda_k)$ for any i, then, from the fact that Λ is R-finitely generated projective, we see easily that ψ' is the well-defined Λ'^e-homomorphism of Λ' into Λ'^e such that $\varphi' \circ \psi' = 1_{\Lambda'}$. Therefore Λ' is a separable R'-algebra. Let α be the R-algebra epimorphism of $R \otimes_{R'} \Lambda'$ onto Λ which is defined by $\alpha(r \otimes_{R'} \lambda_i) = r\lambda_i$, for any $r \in R$. Let $\mathfrak{m}$ be a maximal ideal of R and put $\mathfrak{p}' = \mathfrak{m} \cap R'$. Then we have $(R \otimes_{R'} \Lambda')_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_{R'_{\mathfrak{p}'}} \Lambda'_{\mathfrak{p}'}$ and so α induces naturally an $R_{\mathfrak{m}}$-algebra epimorphism $\alpha_{\mathfrak{m}}\colon R_{\mathfrak{m}} \otimes_{R'_{\mathfrak{p}'}} \Lambda'_{\mathfrak{p}'} \to \Lambda_{\mathfrak{m}}$. Since $\Lambda'_{\mathfrak{p}'}$ is $R'_{\mathfrak{p}'}$-free, $\alpha_{\mathfrak{m}}$ must be an isomorphism. From this it follows immediately that α is an isomorphism. Thus our proof is completed.

2. Central separable algebras with proper splitting rings

Let Λ be a central separable R-algebra and S a commutative R-algebra. If there exists a finitely generated faithful projective S-module P such that $S \otimes_R \Lambda \cong \mathrm{Hom}_S(P, P)$ as S-algebras, then S is called, according to [2], the *splitting ring* of Λ. Especially, when $S \supseteq R$, S is called the *proper splitting ring* of Λ.

First we give, as a slight generalization of [2], (6.3),

Proposition 2.1. *Let R be a local ring with a maximal ideal $\mathfrak{m}$ and Λ a central separable R-algebra. Then Λ has a proper splittting ring S which is a* *separable R-algebra and a finitely generated free R-module. Especially, if R is Henselian, then we can choose as S a local ring with a maximal ideal* $\mathfrak{m}S$.

Proof. By using (1.1) and the Henselization instead of the completion, this can be proved along the same line as in [2], (6.3).

For a central separable algebra over a general commutative ring R, we can not assure the existence of the proper splitting ring which is R-separable and R-finitely generated, projective. In this section, we shall consider only the existence of proper splitting rings. However, we could not prove the existence of a proper splitting ring for a central separable algebra over a general coefficient ring.

Proposition 2.2. *Let R be a commutative ring which is contained in a semi-local ring. Then any central separable R-algebra has a proper splitting ring. Especially, this assumption for R is satisfied by a Noetherian ring or an integral domain.*

Proof. It suffices to prove this proposition in case R is itself a semi-local ring. Let R be a semi-local ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ and put $R'=R_{\mathfrak{m}_1}\oplus R_{\mathfrak{m}_2}\oplus\cdots\oplus R_{\mathfrak{m}_t}$. Then $R\subseteq R'$ and $R'\underset{R}{\otimes}\Lambda=\Lambda_{\mathfrak{m}_1}\oplus\Lambda_{\mathfrak{m}_2}\oplus\cdots\oplus\Lambda_{\mathfrak{m}_t}$. Accordingly to (2.1), there exists a proper splitting ring S_i of $\Lambda_{\mathfrak{m}_i}$ for any i. If we put $S=S_1\oplus S_2\cdots\oplus S_t$, then we have $R\subseteq R'\subseteq S$ and S is a proper splitting ring of Λ, as is required.

As another case, which is not included in (2.2), we have

Proposition 2.3. *Let R be a commutative ring with the total quotient ring K such that any prime ideal of K is maximal. Then any central separable R-algebra has a proper splitting ring.*

Proof. We may assume $R=K$. If we denote by $\mathfrak{n}$ the nil radical of R, then $R/\mathfrak{n}$ is, by our assumption, a regular ring (in the Neumann's sense). Therefore we may further assume that Λ is a finitely generated free R-module. Let $\{u_1, u_2, \cdots, u_t\}$ be an R-basis of Λ with $u_1=1$, and put $u_iu_j=\sum_{k=1}^{t} r_{ijk}u_k$, $r_{ijk}\in R$. Let R_0 be the prime ring of R, and put $R'=R_0[\{r_{ijk}\}]$ and $\Omega'=\{r'_1u_1+\cdots+r'_tu_t\,|\,r'_i\in R'\}$. Then Ω' is a central R'-algebra with an R'-basis $\{u_1, \cdots, u_t\}$, and we have $R\underset{R'}{\otimes}\Omega'=\Lambda$. Furthermore let $\tilde{R}$ be the integral closure of R' in R. Since $R/\mathfrak{n}$ is regular, any non-zero divisor of $\tilde{R}$ is a unit in R, and therefore the total quotient ring $\tilde{K}$ of $\tilde{R}$ can be regarded as a subring of R. From the fact that $\tilde{R}$ is integral over R', we see that the total quotient ring K' of R' is included in R. Since R' is Noetherian and $\tilde{K}/\mathfrak{n}\cap\tilde{K}$ is regular, $K'/\mathfrak{n}K'$ is Artinian, and so K' is itself Artinian. If we put $\Lambda'=K'\underset{R'}{\otimes}\Omega'$, then $R'\underset{K'}{\otimes}\Lambda'=\Lambda$ and, as K' is Artinian, we can easily see that Λ is a central separable K'-algebra. According to (2.1), there exists a proper splitting ring F of Λ' which is a finitely generated projective K'-module. Now put $S=F\underset{K'}{\otimes}R$. Then $S\supseteq F$, R and $S\underset{R}{\otimes}\Lambda=S\underset{R}{\otimes}R\underset{K'}{\otimes}\Lambda'=F\underset{K'}{\otimes}R\underset{K'}{\otimes}\Lambda'=(F\underset{K'}{\otimes}R)\underset{F}{\otimes}F\underset{K'}{\otimes}\Lambda'$. Consequently, S is a proper splitting ring of Λ, which completes our proof.

3. The trace and norm of a central separable algebra

1. Let R be a commutative ring and P a finitely generated projective R-module. First suppose that P has (constant) rank n. Then there exists a commutative ring $S\supseteq R$ such that $S\underset{R}{\otimes}P$ is a free S-module of rank n. Let $\{u_1, \cdots, u_n\}$ be a S-basis of $S\underset{R}{\otimes}P$. If $f\in\mathrm{Hom}_R(P, P)$, then f can be regarded as an element of $\mathrm{Hom}_S(S\underset{R}{\otimes}P, S\underset{R}{\otimes}P)$, and we can put $f(u_j)=\sum_{i=1}^{n}u_i s_{ij}^{(f)}$ for some $s_{ij}^{(f)}\in S$. Now put $\mathrm{Pc}_P(f\colon X)=|s_{ij}^{(f)}-X\delta_{ij}|$, $\mathrm{T}_P(f)=\text{traces }(s_{ij}^{(f)})$ and $\mathrm{N}_P(f)=|s_{ij}^{(f)}|$ where X denotes an indeterminate. It can easily be shown by using the localization at any maximal ideal of R that $\mathrm{Pc}_P(f, X)\in R[X]$ and $\mathrm{T}_P(f)$, $\mathrm{N}_P(f)\in R$ and that these are determined without depending on S and $\{u_1, \cdots, u_n\}$. If P has not constant rank, there is, by [7], § 2, a unique decomposition $R=R_1\oplus\cdots\oplus R_t$ such that any $R_i\underset{R}{\otimes}P$ has rank n_i over R_i where $n_1<n_2\cdots<n_t$, and we have $\mathrm{Hom}_R(P, P)=\sum_{i=1}^{t}\oplus\mathrm{Hom}_{R_i}(R_i\underset{R}{\otimes}P, R_i\underset{R}{\otimes}P)$. Let f be an element of $\mathrm{Hom}_R(P, P)$ and f_i the i-th component of f. Then we put $\mathrm{Pc}_P(f\colon X)=\sum_{i=1}^{t}\oplus\mathrm{Pc}_{R_i\otimes P}(f_i\colon X)$, $\mathrm{T}_P(f)=\sum_{i=1}^{t}\oplus\mathrm{T}_{R_i\underset{R}{\otimes}P}(f_i)$ and $\mathrm{N}_P(f)=\sum_{i=1}^{t}\oplus\mathrm{N}_{R_i\otimes P}(f_i)$ and we call them the characteristic polynomial, trace and norm of f. It can be easily shown that our definitions coincide with those in [7].

If Λ is an R-algebra which is a finitely generated projective R-module, then we use $\mathrm{Pc}_{\Lambda/R}(f\colon X)$, $\mathrm{T}_{\Lambda/R}(f)$ and $\mathrm{N}_{\Lambda/R}(f)$ instead of $\mathrm{Pc}_\Lambda(f\colon X)$, $\mathrm{T}_\Lambda(f)$ and $\mathrm{N}_\Lambda(f)$.

2. Now we shall define the reduced characteristic polynomial, trace and norm for a central separable algebra with a proper splitting ring.

Let Λ be a central separable R-algebra with a proper splitting ring S. Then there exists a S-algebra isomorphism $h_S\colon S\underset{R}{\otimes}\Lambda\cong\mathrm{Hom}_S(P^{(S)}, P^{(S)})$ for some finitely generated projective S-module $P^{(S)}$.

Proposition 3.1 *For any element λ of Λ, $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)$ is a polynomial of $R[X]$ which does not depend on S, $P^{(S)}$ and h_S.*

Proof. First suppose that R is a local ring. Then Λ is a projective R-module of constant rank, and so $P^{(S)}$ is also a projective S-module of constant rank. By replacing S by any extension ring S' of it and by replacing h_S by $1\underset{S}{\otimes}h_S$, $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)$ is invariant, and therefore we may further assume that

$P^{(S)}$ is S-free. Then h_S induces a S-algebra isomorphism k_S: $S\underset{R}{\otimes}\Lambda\cong \mathrm{M}_n(S)$ such that $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)=|XE_n-k_S(\lambda)|$. On the other hand, according to (2.1), there exists a proper splitting semi-olcal ring T of Λ which is R-free. For T we can define, similarly, h_T, $P^{(T)}$ and k_T. Since T is R-free, we have $R\underset{R}{\otimes}R=S\underset{R}{\otimes}R\cap R\underset{R}{\otimes}T$ in $S\underset{R}{\otimes}T$, and so we may suppose that there is a commutative ring U containing both S and T and $S\cap T=R$ in U. Now the algebra isomorphisms k_S: $S\underset{R}{\otimes}\Lambda\cong\mathrm{M}_n(S)$ and k_T: $T\underset{R}{\otimes}\Lambda\cong\mathrm{M}_n(T)$ can, naturally, be extended to the U-algebra isomorphisms k_S^*, k_T^*: $U\underset{R}{\otimes}\Lambda\cong\mathrm{M}_n(U)$. Then $k_S^*\circ k_T^*$ is an U-algebra automorphism of $\mathrm{M}_n(U)$ and it induces an $U_{\mathfrak{m}}$-algebra automorphism of $\mathrm{M}_n(U_{\mathfrak{m}})$ for any maximal ideal of U. As $U_{\mathfrak{m}}$ is a local ring, it is inner, and so we have $|XE_n-k_S^*(\lambda^*)|=|XE_n-k_T^*(\lambda^*)|$ in $U_{\mathfrak{m}}[X]$ for any $\lambda^*\in U\underset{R}{\otimes}\Lambda$. Hence we have $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)=|XE_n-k_S(\lambda)|=|XE_n-k_S^*(\lambda)|=|XE_n-k_T^*(\lambda)|=\mathrm{Pc}_{P^{(T)}}(h_T(\lambda)\colon X)$ in $U[X]$. However, as $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)\in S[X]$ and $\mathrm{Pc}_{P^{(T)}}(h_T(\lambda)\colon X)\in T[X]$, we obtain $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)=\mathrm{Pc}_{P^{(T)}}(h_T(\lambda)\colon X)\in R[X]=S[X]\cap T[X]$. Thus $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)$ is a polynomial of $R[X]$. It is obvious from the above proof that this does not depend on S, $P^{(S)}$ and h_S, which completes our proof for a local ring R.

Let R be a general commutative ring and $\mathfrak{m}$ a maximal ideal of R. Denote by $\lambda_{\mathfrak{m}}$ the residue of λ in $\Lambda_{\mathfrak{m}}$ and by $h_{S_{\mathfrak{m}}}$ the $S_{\mathfrak{m}}$-algebra isomorphism: $S_{\mathfrak{m}}\underset{R_{\mathfrak{m}}}{\otimes}\Lambda_{\mathfrak{m}}\cong \mathrm{Hom}_{S_{\mathfrak{m}}}(P_{\mathfrak{m}}^{(s)}, P_{\mathfrak{m}}^{(s)})$ induced by h_S. Further let $[\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)]_{\mathfrak{m}}$ be the residue of $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X])$ in $S_{\mathfrak{m}}[X]$. Then we see $[\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)]_{\mathfrak{m}}=\mathrm{Pc}_{P_{\mathfrak{m}}^{(s)}}(h_{S_{\mathfrak{m}}}(\lambda_{\mathfrak{m}})\colon X)$. Since, by the preceding argument for a local ring, $\mathrm{Pc}_{P_{\mathfrak{m}}}(h_{S_{\mathfrak{m}}}(\lambda)\colon X)\in R_{\mathfrak{m}}[X]$, we have also $[\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)]_{\mathfrak{m}}\in R_{\mathfrak{m}}[X]$. Consequently we obtain $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)\in R[X]$. It is also evident in this case that $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)$ does not depend on S, $P^{(S)}$ and h_S.

Now we denote $\mathrm{Pc}_{P^{(S)}}(h_S(\lambda)\colon X)$ by $\mathrm{Pcrd}_{\Lambda/R}(\lambda\colon X)$ and we call it the reduced characteristic polynomial of λ. Furthermore, if we put $\mathrm{Trd}_{\Lambda/R}(\lambda)=\mathrm{T}_{P^{(S)}}(h_S(\lambda))$ and $\mathrm{Nrd}_{\Lambda/R}(\lambda)=\mathrm{N}_{P^{(S)}}(h_S(\lambda))$, then they are elements of R which do not depend on S, $P^{(S)}$ and h_S and we call them the reduced trace and norm of λ, respectively.

From our definitions it follows immediately

Proposition 3.2. *For any $\lambda, \lambda_1, \lambda_2\in\Lambda$ and any $r\in R$, we have*

$$\begin{aligned}\mathrm{Trd}_{\Lambda/R}(\lambda_1+\lambda_2) &= \mathrm{Trd}_{\Lambda/R}(\lambda_1)+\mathrm{Trd}_{\Lambda/R}(\lambda_2)\,,\\ \mathrm{Trd}_{\Lambda/R}(r\lambda) &= r\,\mathrm{Trd}_{\Lambda/R}(\lambda)\,,\\ \mathrm{Trd}_{\Lambda/R}(\lambda_1\lambda_2) &= \mathrm{Trd}_{\Lambda/R}(\lambda_2\lambda_1)\,,\\ \mathrm{Nrd}_{\Lambda/R}(\lambda_1\lambda_2) &= \mathrm{Nrd}_{\Lambda/R}(\lambda_1)\,\mathrm{Nrd}_{\Lambda/R}(\lambda_2)\end{aligned}$$

Especially, if Λ has rank n^2 over R, then we have

$$\mathrm{Nrd}_{\Lambda/R}(r\lambda) = r^n\,\mathrm{Nrd}_{\Lambda/R}(\lambda)$$

From this proposition, it follows that $\mathrm{Trd}_{\Lambda/R}$ is an R-homorphism of Λ into R and $\mathrm{Nrd}_{\Lambda/R}$ is a semi-group homomorphism of Λ into R as the multiplicative semi-groups.

For any maximal ideal $\mathfrak{m}$ of R, let $\overline{[\mathrm{Prd}_{\Lambda/R}(\lambda\colon X)]_{\mathfrak{m}}}$ be the residue of $\mathrm{Prd}_{\Lambda/R}(\lambda\colon X)$ in $(R/\mathfrak{m})[X]$ and denote by $\bar{\lambda}_{\mathfrak{m}}$ the residue of Λ in $\Lambda/\mathfrak{m}\Lambda$. Now we can show $\overline{[\mathrm{Prd}_{\Lambda/R}(\lambda\colon X)]_{\mathfrak{m}}}=\mathrm{Prd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}(\bar{\lambda}_{\mathfrak{m}}\colon X)$. In fact, it suffices to prove this in case R is a Henselian local ring with a maximal ideal $\mathfrak{m}$. However, in this case, there is, by (2.1), a proper splitting local ring S of Λ such that $\mathfrak{m}S$ is a maximal ideal of S and S is a finitely generated free R-module. Then $S/\mathfrak{m}S$ becomes the splitting field of the classical central separable $R/\mathfrak{m}$-algebra $\Lambda/\mathfrak{m}\Lambda$, from which our result follows immediately. Accordingly, $\mathrm{Trd}_{\Lambda/R}$ and $\mathrm{Nrd}_{\Lambda/R}$ induce, naturally, $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$ and $\mathrm{Nrd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$, respectively, which coincide with those in the classical sense. By summarizing these, we obtain

Proposition 3.3. *For any maximal ideal* $\mathfrak{m}$ *of* R, *the residue of* $\mathrm{Prd}_{\Lambda/R}$ *in* $(R/\mathfrak{m})[X]$ *coincides with* $\mathrm{Prd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$. *Especially, the residues of* $\mathrm{Trd}_{\Lambda/R}$ *and* $\mathrm{Nrd}_{\Lambda/R}$ *in* $R/\mathfrak{m}$ *coincide with* $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$ *and* $\mathrm{Nrd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$, *respectively.*

3. Here we shall determine the relations between the trace (norm) and reduced trace (reduced norm) of a central separable algebra, which are given in the same form as in the classical one (cf. [4]).

Assume that Λ is a projective R-module of the constant rank m. Then we may suppose $S\underset{R}{\otimes}\Lambda\cong M_n(S)$, where $m=n^2$. From our definitions, it follows directly that $\mathrm{Trd}_{\Lambda/R}(1)=n$, $\mathrm{T}_{\Lambda/R}(\lambda)=n\,\mathrm{Trd}_{\Lambda/R}(\lambda)$ and $\mathrm{N}_{\Lambda/R}(\lambda)=[\mathrm{Nrd}_{\Lambda/R}(\lambda)]^n$. In the general case, let $R=R_1\oplus\cdots\oplus R_t$ be the unique decomposition of R such that $R_i\underset{R}{\otimes}\Lambda$ has rank m_i over R_i where $m_1<m_2<\cdots<m_t$. Then we can put $m_i=n_i^2$ for any i. Let e_i be a unit element of R_i and λ_i the i-th component of λ. Then we obtain

Proposition 3.4. $\mathrm{Trd}_{R_i\otimes\Lambda/R_i}(e_i)=n_ie_i$ *for each* i,

$$\mathrm{T}_{\Lambda/R}(\lambda) = \mathrm{Trd}_{\Lambda/R}(1)\,\mathrm{Trd}_{\Lambda/R}(\lambda) = \sum_{i=1}^{t} n_i\,\mathrm{Trd}_{R_i\otimes\Lambda/R_i}(\lambda_i)$$
$$\mathrm{N}_{\Lambda/R}(\lambda) = \sum_{i=1}^{t}[\mathrm{Nrd}_{R\otimes\Lambda_i/R_i}(\lambda_i)]^{n_i}$$

The following result will be used in § 4.

Proposition 3.5. $\mathrm{Trd}_{\Lambda/R}$ *is an R-epimorphism of* Λ *onto* R.

Proof. By the remark after (3.2), it suffices to prove that $\mathrm{Trd}_{\Lambda/R}$ is an epimorphism. By virtue of the classical result, for any maximal ideal $\mathfrak{m}$ of R, $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$ is an epimorphism of $\Lambda/\mathfrak{m}\Lambda$ onto $R/\mathfrak{m}$. According to (2.3), then, $\mathrm{Trd}_{\Lambda/R}$ must be an epimorphism of Λ onto R,

Corollary 3.6. *The complete image* $T_{\Lambda/R}(\Lambda)$ *of* $T_{\Lambda/R}$ *is a principal ideal of* R *generated by* $\mathrm{Trd}_{\Lambda/R}$ (1). *Especially,* Λ *is strongly separable if and only if* $\mathrm{Trd}_{\Lambda/R}$ (1) *is a unit of* R.

Proof. This is an immediate consequence of (3.4) and (3.5).

4. As is remarked in § 2, we could not succeed in proving the existence of a proper splitting ring for a central separable algebra in the general case. Hence we can not define the reduced characteristic polynomial for a central separable algebra in the case where we can not show the existence of a proper splitting ring. However we can define, by using (1.2), the reduced trace for any central separable R-algebra Λ. In fact, by virtue of (1.2), there exist a Noetherian subring R' of R and a central separable R'-algebra Λ' such that $\Lambda=R\otimes_{R'}\Lambda'$. Since Λ' has a proper splitting ring by (2.2), there exists, according to 2, the reduced trace $\mathrm{Trd}_{\Lambda'/R'}\colon \Lambda'\to R'$. Now we define the reduced trace $\mathrm{Trd}_{\Lambda/R}\colon \Lambda\to R$, by putting $\mathrm{Trd}_{\Lambda/R}(r\otimes_R\lambda')=r\,\mathrm{Trd}_{\Lambda'/R'}(\lambda')$ for any $r\in R$ and for any $\lambda'\in\Lambda'$. It can be easily shown that, for any maximal ideal $\mathfrak{m}$ of R, the $R_{\mathfrak{m}}$-homomorphism $(\mathrm{Trd}_{\Lambda/R})_{\mathfrak{m}}\colon \Lambda_{\mathfrak{m}}\to R_{\mathfrak{m}}$, which is induced on $\Lambda_{\mathfrak{m}}$ by $\mathrm{Trd}_{\Lambda/R}$, coincides with the reduced trace $\mathrm{Trd}_{\Lambda_{\mathfrak{m}}/R_{\mathfrak{m}}}$ of $\Lambda_{\mathfrak{m}}$ defined by using the proper splitting ring of $\Lambda_{\mathfrak{m}}$. Especially, if Λ has a proper splitting ring, $\mathrm{Trd}_{\Lambda/R}$ coincides with that defined in 2. Furthermore we can also prove (3.2)～(3.6) in this case.

4. The symmetricity of a separable algebra

Let Λ be an R-algebra, which is a finitely generated, faithful, projective R-module. We shall consider $\Lambda^*=\mathrm{Hom}_R(\Lambda, R)$ as a left Λ^e-module through the operations $(\lambda\cdot f)(\mu)=f(\mu\lambda)$, $(f\cdot\lambda)(\mu)=f(\lambda\mu)$ where $f\in\Lambda^*$, $\lambda, \mu\in\Lambda$. Following [6], we call Λ a *Frobenius R-algebra* if Λ^* is isomorphic to Λ as left (or equivalently right) Λ-modules, and, furthermore, is called a *symmetric R-algebra* if Λ^* is Λ^e-isomorphic to Λ. From our definitions it follows that any symmetric R-algebra is Frobenius.

We begin with

Lemma 4.1. *Let* S *be a symmetric, commutative R-algebra and* Λ *a symmetric S-algebra. Then* Λ *is a symmetric R-algebra.*

Proof. By our assumptions we have $\Lambda\cong\mathrm{Hom}_S(\Lambda, S)$ as two-sided Λ-modules and $S\cong\mathrm{Hom}_R(S, R)$ as S-modules. So we obtain $\mathrm{Hom}_S(\Lambda, S)\cong\mathrm{Hom}_S(\Lambda, \mathrm{Hom}_R(S, R))\cong\mathrm{Hom}_R(\Lambda\otimes_S S, R)\cong\mathrm{Hom}_R(\Lambda, R)$ as two-sided Λ-modules. This shows that Λ is a symmetric R-algebra.

It is well known, in the classical theory, that a semi-simple algebra over a field is symmetric. However, for any commutative ring R, it is an open question whether a semi-simple R-algebra is symmetric or not, Now we give, as a partial answer to this question,

Theorem 4.2. *A separable R-algebra Λ, which is a finitely generated, faithful, projective R-module, is a symmetric R-algebra.*

Proof. Let C be the center of Λ. According to [2] (2.1), Λ is a finitely generated projective C-module. By our assumption, Λ is R-finitely generated projective, and so C is also a finitely generated projective R-module, as C is a C-direct summand of Λ. Since, by [2], A.4, a commutative separable R-algebra, which is a finitely generated, faithful, projective R-module, is symmetric, C must be a symmetric R-algebra. Therefore, by (4.1), it suffices to prove our theorem in case $R=C$.

Let Λ be a central separable R-algebra and denote by $\mathrm{Trd}_{\Lambda/R}$ the reduced trace of Λ, defined in § 3. Then $\mathrm{Trd}_{\Lambda/R}$ is a symmetric R-homomorphism of Λ into R: i.e., we have $\mathrm{Trd}_{\Lambda/R}(\lambda\mu)=\mathrm{Trd}_{\Lambda/R}(\mu\lambda)$ for any λ, $\mu\in\Lambda$. Hence, putting $\Phi(\lambda)(\mu)=\mathrm{Trd}_{\Lambda/R}(\lambda\mu)$ for any λ, $\mu\in\Lambda$, Φ is a Λ^e-homomorphism of Λ into Λ^*. By (3.3), for any maximal ideal $\mathfrak{m}$ of R, $\mathrm{Trd}_{\Lambda/R}$ induces naturally the reduced trace $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$ in the classical sense on $\Lambda/\mathfrak{m}\Lambda$, and therefore Φ induces, naturally, the $\Lambda^e/\mathfrak{m}\Lambda^e$-homomorphism $\overline{\Phi}_{\mathfrak{m}}$: $\Lambda/\mathfrak{m}\Lambda\to\Lambda^*/\mathfrak{m}\Lambda^*\cong(\Lambda/\mathfrak{m}\Lambda^*)$ such that $\overline{\Phi}_{\mathfrak{m}}(\bar{\lambda})(\bar{\mu})=\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}(\bar{\lambda}\bar{\mu})$ for any $\bar{\lambda}$, $\bar{\mu}\in\Lambda/\mathfrak{m}\Lambda$. From the classical result it follows that $\overline{\Phi}_{\mathfrak{m}}$ is a $\Lambda^e/\mathfrak{m}\Lambda^e$-isomorphism. As both Λ and Λ^* are finitely generated projective R-modules, we can easily see from this that Φ itself is an isomorphism of Λ onto Λ^*. This completes our proof.

We remark that (4.2) was known in some special cases (cf. [2], [5] and [10]).

Finally we give, as an additional remark,

Proposition 4.3. *Let Λ be a central R-algebra which is a finitely generated projective R-module. Then the following statements are equivalent:*

(1) *Λ is a separable R-algebra.*

(2) *The R-module $\Lambda/[\Lambda, \Lambda]$ is isomorphic to R, and, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a semi-simple $R/\mathfrak{m}$-algebra.*

Here we denote by $[\Lambda, \Lambda]$ the R-module generated by all elements of Λ in the form $\lambda\mu-\mu\lambda$, λ, $\mu\in\Lambda$.

Proof. (1)⇒(2): Suppose that Λ is a separable R-algebra. Then the second assertion of (2) follows from [2], (1.6) and so it suffices to prove $\Lambda/[\Lambda, \Lambda]\cong R$. Let $\mathrm{Trd}_{\Lambda/R}$ be the reduced trace of Λ. Then $\mathrm{Trd}_{\Lambda/R}$ is a symmetric R-epimorphism of Λ onto R, and therefore, putting $\mathrm{Ker}\ \mathrm{Trd}_{\Lambda/R}=K$, we have an R-exact sequence:

$$0\to K\to\Lambda\xrightarrow{\mathrm{Trd}_{\Lambda/R}} R\to 0$$

and $K\supseteq[\Lambda, \Lambda]$. Hence we have only to show $K=[\Lambda, \Lambda]$. As is shown in § 3,

$\mathrm{Trd}_{\Lambda/R}$ induces naturally the reduced trace $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}$ of $\Lambda/\mathfrak{m}\Lambda$ for any maximal ideal $\mathfrak{m}$ of R, and we have Ker $\mathrm{Trd}_{\Lambda/\mathfrak{m}\Lambda/R/\mathfrak{m}}=K/\mathfrak{m}K$. However, it is well known, in the classical theory, that the kernel of the reduced trace of a central separable $R/\mathfrak{m}$-algebra $\Lambda/\mathfrak{m}\Lambda$ coincides with $[\Lambda/\mathfrak{m}\Lambda, \Lambda/\mathfrak{m}\Lambda]$. Consequently we must have $K/\mathfrak{m}K=[\Lambda/\mathfrak{m}\Lambda, \Lambda/\mathfrak{m}\Lambda]$ for any maximal ideal $\mathfrak{m}$ of R. From this we easily see $K=[\Lambda, \Lambda]$, as K is R-finitely generated. Thus the implication (1)⇒ (2) is proved. (2)⇒(1). Conversely suppose (2). By (1.1) it suffices to prove that $\Lambda/\mathfrak{m}\Lambda$ has $R/\mathfrak{m}$ as its center. By our assumption we have an R-exact sequence:

$$0 \to [\Lambda, \Lambda] \to \Lambda \xrightarrow{\alpha} \mathrm{R} \to 0\,.$$

This induces an $R/\mathfrak{m}$-exact sequence:

$$0 \to [\Lambda, \Lambda]/\mathfrak{m}[\Lambda, \Lambda] \to \Lambda/\mathfrak{m}\Lambda \xrightarrow{\bar{\alpha}} R/\mathfrak{m}R \to 0\,.$$

and so we have $[\Lambda, \Lambda]/\mathfrak{m}[\Lambda, \Lambda]\cong[\Lambda/\mathfrak{m}\Lambda, \Lambda/\mathfrak{m}\Lambda]$. Therefore we have $\Lambda/\mathfrak{m}\Lambda\cong[\Lambda/\mathfrak{m}\Lambda, \Lambda/\mathfrak{m}\Lambda]\oplus R/\mathfrak{m}$. On the other hand, since $\Lambda/\mathfrak{m}\Lambda$ is $R/\mathfrak{m}$-semisimple, $\Lambda/\mathfrak{m}\Lambda$ is separable over its center $\bar{C}$, and then we have $\Lambda/\mathfrak{m}\Lambda\cong[\Lambda/\mathfrak{m}\Lambda, \Lambda/\mathfrak{m}\Lambda]\oplus\bar{C}$. As $\bar{C}\supseteq R/\mathfrak{m}$, we see from these that $\bar{C}$ coincides with $R/\mathfrak{m}$. This completes our proof.

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