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SELF-INJECTIVE QUOTIENT RINGS AND INJECTIVE QUOTIENT MODULES*

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1. Introduction

Until further notice, we assume that R is a ring (with unity) and S is a multiplicatively closed set of regular elements of R such that R satisfies the multiplicity condition with respect to S (for every a, s in R, s in S, there exist a_1, s_1 in R, s_1 in S such that $a\,s_1=s\,a_1$). Let Q denote the (Asano's) quotient ring R_S of R. If 1 denotes the identity of R, $1'$ the identity of Q, then $1.1'=1.s.s^{-1}=(1.s)\,s^{-1}=s.s^{-1}=1'$. Also $1.1'=1$, because $1'$ is the identity of a bigger ring Q. So that the identities of the two rings coincide.

Let M be a (unital) right R-module. M is said to be S-free if $m\,s=0$, $m\in M$, $s\in S$ implies $m=0$. M is said to be S-divisible iff $Ms=M$ $\forall s\in S$. If M is both S-free and S-divisible R-module, then the module composition $M\times R\to M$ can be extended to $M\times Q\to M$ in one and only one way such that M becomes a Q-module, by defining $m(a.s^{-1})=m'$ where m' is such that $m's=m.a$ (note that m' exists because of S-divisibility and is unique because of S-freeness). This composition is well defined, because if $a.s^{-1}=a_1.s_1^{-1}$, suppose $m'.s=m.a$, $m''.s_1=m.a_1$. Now there exist s_2, s_3, $s_2\in S$ such that $s.s_2=s_1.s_3$. Then $a\,s_2=a_1s_3$. $m.a.s_2=m.a_1.s_3$. $m'.s.s_2=m''.s_1.s_3$ implies $m'=m''$. It is not very difficult to check that M is Q module with this composition.

However if M is S-free R-module, then there exists a Q-module M' such that $M_R\subset M'_R$ and $M'=MQ=\{m.s^{-1};\ m\in M,\ s\in S\}$. This module M' is unique upto isomorphism over M. There are various construction for this module M' available in the literature.

Asano's construction. In $M\times Q$ define $(m, q)\sim(m', q')$ if there exists $s\in S$ such that $q.s\in R$, $q's\in R$ and $m(q\,s)=m'(q'\,s)$. The relation $\sim$ can be verified to be an equivalence relation. In M'=the set of equivalence classes of $M\times Q$ define '+' and '·' as follows: $\overline{(m, q)}+\overline{(m', q')}=\overline{(m(q\,s)+m(q\,s), s^{-1})}$ where $s\in S$ is such that $q\,s\in R$ and $q's\in R$. $\overline{(m, q)}.q'=\overline{(m, q\,q')}$. It can be verified that these compositions are well defined and M' is Q module. The mapping

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$\sigma: m \to \overline{(m, 1)}$ is a R-isomorphism of M_R into M'_R so that identifying σM with M, we find that

$$\overline{(m, q)} = \overline{(m, a s^{-1})} = \overline{(m, a)} s^{-1} = \overline{(ma, 1)} s^{-1} = (m, a) s^{-1} ,$$

so that

$$M' = MQ = \{m s^{-1}: m \in M, s \in S\} .$$

Another construction of M′. Let $M' = M \otimes_R Q$. M' is a Q-module. The mapping $\sigma: m \to m \otimes 1$ embeds M_R into M'_R and

$$\sum_{i=1}^{n} (m_i \otimes q_i) = \sum_{i=1}^{n} (m_i \otimes a_i s_i^{-1}) = \sum_{i=1}^{n} (m_i \otimes a_i s_i^{-1}) s s^{-1} ,$$

where s is such that $a_i s_i^{-1} s \in R$ $i=1, 2, \cdots, n$, so that

$$\sum_{i=1}^{n} (m_i \otimes q_i) = (\sum_{i=1}^{n} m_i (a_i s_i^{-1} s)) \otimes s^{-1} = (m \otimes 1) s^{-1}$$
$$= (\sigma m) s^{-1} , \quad \text{where} \quad m = \sum_{i=1}^{n} m_i (a_i s_i^{-1} s) .$$

Note that if M is S-free R module, then $MQ = M$ iff M is S-divisible.

The starting point of this paper is the following result: If M is a Q-module, the M_Q is injective iff M_R is injective. This result is used to prove that if M is a Q module then injective dimension of M_Q=injective dimension of M_R. The following corollary follows:

The right Global dim. of $Q \leqslant$ right Global dim. of R.

If M is any Q-module, then the injective hulls $E(M_Q)$ and $E(M_R)$ are seen to coincide. The following result is also proved: Every S-free S-divisible module over R is injective if and only if Q is semi-simple Artinian ring.

Similar results about quasi-injective modules have also been proved.

If M be an S-free R module, then the Q module M' (mentioned above) when regarded as a right R moudle is an essential extension of M. Necessary and sufficient condition on M such that M' becomes the injective hull of M are obtained. This result is applied to characterize rings whose classical quotient rings are qusi-Frobenius rings.

These results have also above been applied to obtain necessary conditions and sufficient conditions on a ring under which the Utumi's ring of quotients of the ring is an Asano's quotient ring. Necessary and sufficient conditions, when $R_r^{\Delta}=0$, follow.

Hereditary orders in semi-simple Artinian rings have been characterized. These rings are found to 'resemble' Dedekind domains, which are precisely hereditary orders in commutative fields. A principal right ideal semi-prime ring is found to be a hereditary ring.

It is proved that if R is a semi-prime Goldie ring, and Q its semi-simple

Artinian classical right quotient ring, then Q_R is never projective except when $Q=R$.

Returning back to arbitrary R, with an Asano's quotient ring $Q=R_S$, if M be a projective R module, then trivially M is S-free. It is proved that $M'(=MQ)$ is a projective Q-module. This result is used to rededuce our earlier result: right Global dim. $Q \leqslant$ right Global dim. R.

Finally, necessary and sufficient conditions on R such that $Q(=R_S)$ becomes a hereditary ring are obtained. An immediate corollary thereof being: if R is right hereditary, then Q is also right hereditary.

2. Modules over Asano's quotient ring

2.1. Theorem. *Let R be a ring and S be a multiplicatively closed set of regular elements of R such that R satisfies the multiplicity condition with respect to S. Let Q denote the Asano's quotient ring R_S of R with respect to S. Let M be a module over Q. Then M is clearly a module over R, M is an injective Q-module if and only if M is an injective R-module.*

Proof. Let M be an injective Q module. Let $f\colon I_R \to M_R$, I a right ideal of R. We know that $IQ=\{a s^{-1}\colon a \in I,\ s \in S\}$ Define $f'\colon IQ \to M$ as follows:

$$f'(a s^{-1}) = f(a) s^{-1}$$

f' is well defined for if

$a s^{-1} = a_1 s_1^{-1}$, a, $a_1 \in I$, s, $s_1 \in S$, then there exist $s_2 \in S$ such that (see Asano [1]) $s^{-1} s_2 \in R$ and $s_1^{-1} s_2 \in R$. Clearly $a(s^{-1}s_2) = a_1(s_1^{-1}s_2)$, therefore $f(a)(s^{-1}s_2) = f(a_1)(s_1^{-1}s_2)$.

Post multiplying by s_2^{-1} we get $f(a)s^{-1} = f(a_1)s_1^{-1}$.

We check that f' is a Q-homomorphism of IQ into M.

$$f'(a s^{-1} + a' s'^{-1}) = f'((a s^{-1} + a' s'^{-1}) s_1 s_1^{-1})$$

where s_1 is an element of S such that $s^{-1} s_1$, $s'^{-1} s_1 \in R$

$$\begin{aligned} &= (f(a s^{-1} s_1 + a' s'^{-1} s_1)) s_1^{-1} \\ &= (f(a) s^{-1} s_1 + f(a') s'^{-1} s_1) s_1^{-1} \\ &= f(a) s^{-1} + f(a') s'^{-1} \\ &= f'(a s^{-1}) + f'(a' s'^{-1}) \end{aligned}$$

$$\begin{aligned} f'(a s^{-1} \cdot r s_1^{-1}) &= f'(a r_1 s_2^{-1} s_1^{-1}) \qquad \text{where } s^{-1} r = r_1 s_2^{-1} \\ &= f(a r_1)(s_1 s_2)^{-1} \\ &= f(a) r_1 s_2^{-1} s_1^{-1} \\ &= f(a) s^{-1} r s_1^{-1} \\ &= f'(a s^{-1}) \cdot r s_1^{-1} \end{aligned}$$

M being an injective Q-module there exists $m \in M$ such that

$$f'(x) = mx \ \forall x \in IQ .$$

But $f'(x) = f(x) \ \forall x \in I$. Therefore $f(x) = mx \ \forall x \in I$.

Conversely suppose M_R is injective. Let $f\colon I \to M$, be Q-homomorphism, where I is a right ideal of Q. Then let $J = I \cap R$. J is a right ideal of R and $JQ = I$. Let f' denote the restriction of f to J. Then f' is clearly a R-homomorphism of J into M. As M is an injective R-module, there exists $m \in M$ such that $f'(x) = mx \ \forall x \in J$. Any element of I is of the form xs^{-1}, $x \in J$, $s \in S$.

$$f(xs^{-1}) = f'(x)s^{-1} = m(xs^{-1})$$

Hence M is an injective Q-module.

2.2. Theorem. *Let M be a module over Q, then M_Q is quasi-injective iff M_R is quasi-injective.*

Proof. Assume that M is a quasi-injective module over Q. Let $f\colon N \to M$ be a R-homomorphism where N is a R-submodule of M. We know that $NQ = \{ns^{-1}\colon n \in N, s \in S\}$

Define a mapping $f'\colon NQ \to M$

$$f'(ns^{-1}) = f(n)s^{-1} .$$

It can be checked that f' is well defined and f' is a Q-homomorphism of NQ into M. Also f' coincides with f on N. As M is Q-quasi-injective, therefore there exists an extension f'' of f' such that $f'' \in \mathrm{Hom}_Q(M, M)$. Clearly $f'' \in \mathrm{Hom}_R(M, M)$ and f'' is an extension of f.

Conversely suppose that M_R is quasi-injective. Let $f\colon N \to M$ be a Q-homonorphism, where N is Q-submodule of M. N is also R-submodule of M. There exists $g \in \mathrm{Hom}_R(M, M)$ such that g coincides with f on N. We prove that g is infact a Q-homomorphism.

$$g(m.rs^{-1})s = g(mrs^{-1}s) = g(mr) = g(m)r .$$

Therefore

$$g(mrs^{-1}) = g(m)rs^{-1} \ \forall r \in R,\ s \in S,\ m \in M .$$

2.3. Theorem. *Every S-divisible S-free module over R is injective iff Q $(=R_S)$ is a semi-simple Artinian ring.*

Proof. Let every S-free S-divisible module over R be injective. Then in order to prove that Q is semi-simple Artinian, we shall prove every module over Q is injective. Let M be any Q module, then M is S-free and S-divisible R-module ($ms=0$ implies $mss^{-1}=0$ $m=0$, $Ms \supset (Ms^{-1})s = M \supset Ms$, $M = Ms \ \forall s$

$\in S$). Therefore M is an injective R-module. By theorem 2.1 M is an injective Q module. Hence Q is semi-simple Artinian, see Cartan-Eilenberg [3, page 11, Theorem 4.2].

Conversely let Q be a semi-simple Artinian ring, then if M be a S-free and S-divisible R-module, M can be regarded as a Q-module, the module composition $M\times Q\rightarrow M$ being such that it extends the original module composition $M\times R\rightarrow M$, in one and only one way (see introduction). Q being semi-simple Artinian, any module over Q is injective, therefore M is an injective Q-module. Consequently M is an injective R-module by theorem 2.1.

2.4. DEFINITION. Let M be a module over R. M is said to be a torsion free module if $mx=0$, $m\in M$, x regular in R implies $m=0$. M is said to be a divisible module if $Mx=M$ $\forall$regular element x in M.

2.5. Corollary. (Levy, 1963) *If R be a ring having a classical right quotient ring Q, then every torsion free divisible R-module over R is injective if and only if Q is semi-simple Artinian.*

2.6. Theorem. *Let R be a ring and $Q(=R_S)$ be an Asano's quotient ring of R with respect to a set S of regular elements of R. Then every S-free S-divisible module over R is quasi-injective iff Q is semi-simple Artinian.*

Proof. In the proof we use the following result of Faith and Utumi: [6, page 169, Cor. 2.4]. A ring Q is semi-simple Artinian if and only if every module over Q is quasi-injective.

Assume that every S-free S-divisible R-module is quasi-injective. Let M be any module over Q. Then M is S-free, S-divisible module over R. Therefore M is a quasi-injective R-module. Therefore by theorem 2.2 M is a quasi-injective Q module. Hence Q is semi-simple Artinian.

Converse is proved in the previous theorem 2.3.

2.7. Corollary. *If R is a ring with a right classical quotient ring Q, then every torsion-free divisible R-module is quasi-injective if and only if Q is semi-simple Artinian.*

2.8. Theorem. *Let Q $(=R_S)$ be an Asano's quotient ring of R. Let M be a Q-module, then if E is the injective hull of M_Q then E_R=the injective hull of M_R.*

Proof. E_Q being an injective module, E_R is an injective module by 2.1. E_R is an essential extension of M_R, because if $0\neq m\in E$, then $mQ\cap M\neq 0$, $0\neq mrs^{-1}\in M$ for some r in R, s in S, $0\neq mr\in M$. Therefore $mR\cap M\neq 0$ $\forall m\neq 0$ in E. Hence E_R is an injective hull of M_R.

2.9. Quasi-injective hull. The concept of quasi-injective hull of a module was introduced by Johnson and Wong [15] who proved that if M is any module, then $M'=\Lambda M$, where $\Lambda=\mathrm{Hom}_R(E, E)$, E being the injective hull of M is the unique minimal quasi-injective essential extension of M. Faith and Utumi [6] proved that M is infact a unique minimal quasi-injective extension by observing that complement (closed) submodules of a quasi-injective module are quasi-injective, see Faith and Utumi [6, Corollary 2.2].

2.10. Theorem. *If Q $(=R_S)$ be an Asano's quotient ring of R and M be a Q-module, then quasi-injective hull of M_Q=quasi-injective hull of M_R.*

Proof. Let E_Q be an injective hull of M_Q. Then E_R is an injective hull of M_R by 2.8. Now note that $\mathrm{Hom}_Q(E, E)=\mathrm{Hom}_R(E, E)$, because if $f\in\mathrm{Hom}_R(E, E)$, then $f(mrs^{-1})s=f(mr)=f(m)r$ $\forall r\in R$, $s\in S$, therefore $f(mrs^{-1})=f(m)rs^{-1}$. Let $\Lambda=\mathrm{Hom}_Q(E, E)=\mathrm{Hom}_R(E, E)$. The quasi-injective hull of M_Q is $\Lambda M=$ the quasi-injective hull of M_R.

2.11. DEFINITION. Injective resolution of a module.
Injective dimension of a module.

Let R be an arbitrary ring. Let M_R be a module over R. An exact sequence

$$0\to M\xrightarrow{\cdots} M_0\xrightarrow{d_0} M_1\xrightarrow{d_1} M_2\xrightarrow{d_2} M_3\to\cdots\to M_{n-1}\xrightarrow{d_{n-1}} M_n\to\cdots$$

where each M_1 is injective is called an injective resolution of M. The least integer n such that kernel d_n is injective is called the injective dimension of M. If no such integer n exists, then the injective dimension of M is defined to be ∞.

It has to be noticed that the injective dimension of a module is independent of the injective resolution.

2.12. Theorem. *Let Q $(=R_S)$ be an Asano's quotient ring of a ring R. If M be a module over Q, then injective dimension (M_Q)=injective dimension (M_R).*

Proof. Let

$$0\to M\xrightarrow{\cdots} M_0\xrightarrow{d_0} M_1\xrightarrow{d_1} M_2\to\cdots\to M_{n-1}\xrightarrow{d_{n-1}} M_n\to\cdots$$

be an injective resolution of the module M_Q. Then by 2.1 this is also an injective resolution of the module M_R. If ker. d_n is never Q injective, then the ker. d_n is never R-injective by 2.1, so that if dim. $M_Q=\infty$, then dim. $M_R=\infty$. If however dim. $M=n$, then ker. d_n is Q injective and therefore it is R-injective. Ker. d_m, $m<n$ cannot be R-injective, because if it is R-injective, then it will be Q injective by 2.1, which is a contradiction. Hence injective dimension $M_Q=$ injective dimension M_R.

2.13. DEFINITION. If R is any ring, then right Global dimension of $R=$ supremum of the injective dimension of all R modules.

2.14. Theorem. *If Q $(=R_S)$ be an Asano's quotient ring of R, then right Global dimension of $Q \leqslant$ right Global dimension of R.*

Proof. Immediate from 2.12.

3. Let R be a ring and Q $(=R_S)$ be the Asano's quotient ring of R with respect to a set S of regular elements of R. If M be an S-free module over R, then two known constructions for the 'quotient module' M' $(=MQ)$ have been outlined in the introduction. A still new construction for this module M, which seems to be more natural is given below. But before we give this construction we observe one lemma which in essence is due to Levy [18].

3.1. Lemma. *If S be a multiplicatively closed set of regular elements of a ring R, then R_S exists (i.e. R satisfies the multiplicity condition with respect to S) if and only if for every R-module M, $T(M)=\{m \in M,\ ms=0$ for some $s \in S\}$ is a submodule of M.*

Proof. This result can be proved exactly as in, Levy [18, theorem 1.4, page 134].

3.2. Lemma. *Every injective module is divisible.*

Proof. Let M be an injective R-module. Let a be a regular element of R. Let $m \in M$. The mapping f: $aR \rightarrow M$ such that $f(ar)=mr\ \forall r \in R$ can be realized by an element m' of M. $m'ar=mr\ \forall r \in R$. Hence $m'a=m$. Hence $Ma=M$.

We are now ready to give another construction of the quotient module MQ.

3.3. Let R be a ring and Q $(=R_S)$ be the Asano's right quotient ring of R with respect to S. Let M be a S-free right R module. Let E denote the injective hull of M_R. E is S-divisible by 3.2. E is S-free, because $T(E)=\{m \in E\colon ms=0$ for some $s \in S\}$ is a submodule of E by 3.1. But $T(E) \cap M=0$ because M is S-free. Therefore $T(E)=0$. Therefore E is an S-free S-divisible module, therefore E becomes a Q-module (see introduction) Let $M'=MQ \subset E$. Then M' is the required module over Q such that $M_R \subset M'_R$ and $M'=MQ$.

In view of this construction of MQ, a natural question arises when is $MQ=E(M)$? The following theorem answers this question.

3.4. Theorem. *Let R be a ring and Q $(=R_S)$ be an Asano's quotient ring of R. Let M be an S-free module over R. Then the 'quotient' module M' $(=MQ)$ is injective Q-module (or injective R-module see 2.1) iff every R-homomorphism $f\colon I \rightarrow M$, I a right ideal of R can be extended to a R-homomorphism $g\colon J \rightarrow M$, where J is a right ideal of R containing I and containing an element of S.*

Proof. Let M_R satisfy the condition. We prove that $M'=MQ$ is an injective R-module. Let there be a R-homomorphism $f\colon I\to MQ$, where I is a right ideal of R.

Set $I'=\{x\in I\colon f(x)\in M\}$. I' is a right ideal of R, $I'\subset I$. We claim that for each $r\in I$, there exists $s\in S$ such that $rs\in I'$. Let $r\in I$. Then $f(r)\in MQ$. Therefore $f(r)=ms^{-1}$, where $m\in M$ and $s\in S$. Then $f(rs)=f(r)s=ms^{-1}s=m\in M$. $rs\in I'$. Let f' denote the restriction of f to I. There exists by hypothesis $g\colon J\to M$, where J is a right ideal of R containing I and containing an element of S and $g(x)=f'(x)\ \forall x\in I'$. As J contains an element of S, $JQ=Q$. There exists an extension $g'\colon JQ\to MQ$ of g defined as follows:

$$g'(js^{-1})=g(j)s^{-1}\,.$$

The mapping g' is a Q-homomorphism. Let $g'(1)=m$. Let $r\in I$. There exists $s\in S$ such that $rs\in I'$.

$$f(r)s=f(rs)=f'(rs)=g(rs)=g'(rs)=g'(1).rs=(m.r.).s$$

Therefore $f(r)=mr\ \forall r\in I$.

Conversely, suppose MQ be an injective R-module (injective Q-module). Let there be a R-homomorphism $f\colon I\to M$, where I is a right ideal of R. As MQ is R-injective and $M\subset MQ$, therefore there exists an element $m.s^{-1}$ of MQ such that

$$ms^{-1}x=f(x)\ \forall x\in I\,.$$

Let $J=sR+I$. Define $g\colon J\to M$ as follows:

$$g(sr+x)=ms^{-1}(sr+x)=mr+ms^{-1}x\ \forall r\in R,\ x\in I\,.$$

Surely g is a R-homomorphism of J into M. J contains I and an element s of S. Also $g(x)=ms^{-1}x=f(x)\ \forall x\in I$.

Self injective quotient rings

3.5. Corollary. *A quotient ring Q $(=R_S)$ of R is self injective (Q_R is injective) if and only if for every R-homomorphism $f\colon I\to R$, I a right ideal of R, there exists a R-homomorphism $g\colon J\to R$, where J is a right ideal of R containing I and an element of S and g is such that*

$$g(x)=f(x)\ \forall x\in I\,.$$

3.6. Corollary. *Let R be a ring having a classical right quotient ring Q. The ring Q is self injective (Q_R is injective or Q_R is the injective hull of R_R) iff every R-homomorphism*

$$f\colon I\to R\,,$$

where I is a right ideal of R can be extended to a R-homomorphism

$$g: J \rightarrow R$$

where J is a right ideal of R containing I and a regular element of R.

3.7. REMARK. Semi-prime right Goldie rings form a class of rings which possess self injective classical quotient rings. One is tempted to verify whether this condition is really satisfied by the class of rings.

If $f: I \rightarrow R$, then let J be a complement of I in R, so that $I \oplus J$ is an essential right ideal of R. $I \oplus J$ contains a regular element of R. Goldie [9, Theorem 3.9]. One can trivially extend f to $f': (I+J) \rightarrow R$ by defining $f(i+j)=f(i)$ $\forall i \in I$, $j \in J$.

If Q is a classical quotient of a ring R, we say that R is an order in Q.

In view of the corollary 3.6 and the characterization of orders in perfect rings, orders in semi-primary rings, orders in Artinian rings and Noetherian orders in Artinian rings given in [10] and [14] we have the following four results:

Orders in self-injective perfect rings

3.8. Theorem. *A ring R is an order in a (left) perfect self injective ring iff*
(*i*) *$N(R)$, the upper nil radical of R is left T-nilpotent.*
(*ii*) *$R/N(R)$ is a right Goldie ring.*
(*iii*) *$a_\alpha R_\alpha$ is an essential right ideal of R_α, for every right regular element a_α in R_α and for every ordinal number α, where $R_\alpha = R/T_\alpha$, $a_\alpha = a + T_\alpha(R)$, $T_\alpha(R)$, an ideal of R defined as follows:*

$$T_0(R) = 0,\ T_{\alpha+1}(R) = \{x: x \in N(R),\ xN(R) \subset T_\alpha(R)\},$$

for an ordinal number of the type $\alpha+1$.

$$T_\alpha(R) = \bigcup_{\beta<\alpha} T_\beta(R),$$

for a limit ordinal α.
(*iv*) *If $a+N(R)$ is regular in $R/N(R)$ then a is regular in R.*
(*v*) *Every R-homomorphism $f: I \rightarrow R$, I a right ideal of R can be extended to a R-homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and a regular element of R.*

Orders in self-injective semi-primary rings

3.9. Theorem. *A ring R is an order in a self-inejctive semi-primary ring if and only if*
(*i*) *$N(R)$, the upper nil radical of R is nilpotent.*
(*ii*) *$R/N(R)$ is a right Goldie ring.*
(*iii*) *$a_i R_i$ is essential right ideal of R_i for every right regular element a_i in R_i and for every integer $i \geqslant 0$, where $R_i = R/T_i$, $a_i = a + T_i$, T_i being a two sided ideal of R defined as follows: $T_0 = (0)$, $T_i = l(N(R)^i) \cap N(R)$ for $i \geqslant 1$.*

(iv) If $a+N(R)$ is regular in $R/N(R)$, then a is regular in R.
(v) Every R-homomorphism $f: I \to R$, I a right ideal of R, can be extended to a R-homomorphism $g: J \to R$, where J is a right ideal of R containing I and a regular element of R.

DEFINITION. A ring R such that $I^{lr}=I$ $\forall$ right ideal I and $L^{lr}=L$ $\forall$left ideal of L and R satisfies the d.c.c. on left and right ideals is called a quasi-Frobenius ring.

It is well known that a ring R is quasi-Frobenius iff R is Artinian and right self-injective see Faith [5], Eilenberg-Nakayama [4].

Orders in qusai-Frobenius rings

3.10. Theorem. *A ring R is an order in a quasi-Frobenius ring iff*
(i) $N(R)$, the upper nil radical of R is nilpotent.
(ii) $R/N(R)$ is a right Goldie ring.
(iii) R/T_i has no infinite direct sum of right ideals for every $i \geqslant 0$, where T_i is defined as follows:

$$T_0 = (0),\ T_i = l(N(R)^i) \cap N(R) \qquad i \geqslant 1\,.$$

(iv) If $a+N(R)$ is regular in R, then a is regular in R.
(v) Every R-homomorphism $f: I \to R$, I a right ideal of R, can be extended to $g: J \to R$, where J is a right ideal of R containing I and a regular element of R.

3.11. Theorem. *A Noetherian ring R is an order in a quasi-Frobenius ring if and only if*
(i) If $a+N(R)$ is regular in $R/N(R)$, then a is regular in R.
(ii) Every R-homomorphism $f: I \to R$, I a right ideal of R, can be extended to a R-homomorphism $g: J \to R$, where J is a right ideal of R containing I and a regular element of R.

(In connection with Theorem 3.10, the author wishes to point out that when this manuscript was ready, the author received an unpublished paper entitled 'Orders in quasi-Frobenius Rings' from Professor J.P. Jans wherein the following result is proved:

A ring R is an order in a quasi-Frobenius ring iff
(i) R has no infinite direct sum of right ideals.
(ii) $A_r(E(R), R)$ satisfies a.c.c. where $E(R)$ is the injective hull of R_R, $A_r(E(R), R)=\{S^\perp: S \subset E(R)\}$, $S^\perp=\{x \in R, Sx=0\}$ for subsets S of $E(R)$.
(iii) $T(E(R)/R)=E(R)/R$, where for any R-module M, $T(M)=\{m \in M: ma=0$ for some regular element a in $R\}$.
(iv) If M is a finitely generated (cyclic) R-module such that $T(M)=(0)$, then $E(M)$, the injective hull of $M \subset (E(R))^n$.)

Semi-prime right Goldie rings belong to the class of rings R for which

$Q(R)$, the classical right quotient ring of R exists, $Q_R=E_R$, the injective hull of R_R and E_R is Σ-injective, see Faith [5, page 189, Corollary 3]. (An injective module M_R is said to be Σ-injective iff the direct sum of arbitrarily many copies of M_R is an injective R-module). The following theorem precisely determines this class of rings:

3.12. Theorem. *For a ring R, $Q(R)$ exists, $Q_R=E_R$, the injective hull of R_R, E_R is Σ-injective iff R is an order in a quasi-Frobenius ring.*

Proof. If R is an order in a quasi-Frobenius ring, then let $Q=Q(R)$, Q is quasi-Frobenious. Q_Q is injective. Therefore Q_R is injective by 2.1. $Q_R=E_R$, the injective hull of R_R. Now Q_Q is Σ-injective, because Q is quasi-Frobenius (Q is quasi-Frobenious iff every projective module over Q is an injective Q module, see Faith [5]). Therefore Q_R is Σ-injective by 2.1. Hence E_R is Σ-injective. Conversely let us assume $Q(R)$ exists, $Q_R=E_R$ and E_R is Σ-injective, then Q_R is injective and Q_R is Σ-injective. Then Q is self injective and Q_Q is Σ-injective by 2.1. Hence any free module over Q is injective. Therefore any projective Q module is Q injective. Hence Q is quasi-Frobenius.

3.13. Theorem. *If R has a classical right quotient ring Q, then Q is quasi-Frobenius iff every projective Q module is R-injective.*

Proof. Let Q be quasi-Frobenius. Let M be a projective module over Q, then by Faith [5], M is an injective Q module. Therefore by 2.1 M is an injective R module. Conversely suppose that any projective module over Q is an injective R module, then any projective module over Q is an injective Q module. Hence Q is quasi-Frobenius, Faith [5].

Faith and Walker [8] proved that a ring Q is quasi-Frobenius iff every injective module over Q is projective. From this and with the help of 2.1 we obtain the following theorem:

3.14. Theorem. *Let R be a ring having a classical right quotient ring Q. Then Q is quasi-Frobenius iff every torsion free injective R-module is a projective Q-module.*

Utumi's quotient ring and Johnson's quotient ring

3.15. Theorem. *Let R be a ring such that R has a multiplicatively closed set S of regular elements with respect to which R satisfies the multiplicity condition and every $f: I\to R$, I a right ideal of R, can be extended to $g: J\to R$, where J is a right ideal of R containing I and an element of S, then Q $(=R_S)$ is the Utumi's quotient ring of R.*

Proof. Lambek [16] proved that Q the Utumi's ring of quotients of R has the following characterization. $Q=$Bicommutant of E_R where E_R is the injective hull of R_R. In our case by 3.5 Q_Q is injective. Therefore by 2.1, Q_R is injective. Also $R_R \subset' Q_R$. Therefore Q_R is the injective hull of R_R. Now $\mathrm{Hom}_R(Q_R, Q_R) \cong Q$. Also $\mathrm{Hom}_Q({}_QQ, {}_QQ) \cong Q$. Therefore Q is the Utumi's quotient ring of R.

Conversely we have the following result:

3.16. Theorem. *If R is such that Q_R, where Q is the Utumi's quotient ring of R, is the injective hull of R_R (see, Lambek [17, page 95]) and Q is an Asano's quotient ring of R, then there exists a multiplicatively closed set S of regular elements of R such that R satisfies the multiplicity condition with respect to S and every R-homomorphism $f: I \to R$, I a right ideal of R, can be extended to $g: J \to R$, where J is a right ideal of R containing I and an element of S.*

Proof. Let $S = \{x \in R: x \text{ invertibe in } Q\}$. Then $Q = R_S$ and the rest follows, because Q_R is injective by 3.5.

Combining the previous two theorems and noting that the Utumi's quotient ring of a ring R coincides with $\hat{R}$, the Johnson's maximal right quotient ring of R, if $R_r^{\Delta} = 0$, and that $\hat{R}_R$, where $\hat{R}$ denotes the Johnson's maximal quotient ring of R, is the injective hull of R_R, we notice the following result:

3.17. Theorem. *If R is a ring with $R_r^{\Delta} = 0$, then $\hat{R}_r$, the Johnson's quotient ring of R is an Asano's quotient ring of R iff there exists a multiplicatively closed set S regular elements of R such that (i) R satisfies the multiplicity condition with respect to S and (ii) every R-homomorphism $f: I \to R$, I a right ideal of R, can be extended to a R-homomorphism $g: J \to R$, where J is a right ideal of R containing I and an element of S. In this case $\hat{R}_r = R_S$.*

3.18. Corollary. *If R is a ring such that $R_r^{\Delta} = 0$, and $Q(R)$ the classical right quotient ring of R exists, then $\hat{R} = Q(R)$ iff every $f: I \to R$, I a right ideal of R, can be extended to $g: J \to R$, where J contains I and a regular element of R.*

4. If R is a commutative integral domain and Q its field of quotients, then Q_R is projective implies $Q = R$, see Tsi-Che-Te [20, page 174]. The aim of the present section is to generalize this result from commutative integral domain to an arbitrary semi-prime right Goldie ring.

4.1. Theorem. *Let R be a semi-prime ring with the Goldie conditions on right ideals and let Q be its semi-simple Artinian classical right quotient ring. If Q_R is projective, then $R = Q$.*

Lemma. *Let R be a prime ring with right Goldie conditions and Q its classical right quotient ring. If there exists $0 \neq q$ in Q such that $q \cdot Q \subset R$, then $R = Q$.*

Proof. By a theorem of Faith and Utumi [7, Theorem 3, page 56], there exists a complete set $M = \{e_{ij}: i, j = 1, 2, \cdots, n\}$ of matrix units in Q with the following property: If D is the centralizer of M in Q, then D is a division ring and

$$Q=\sum_{i,j=1}^{n} De_{ij}\supset R\supset \sum_{i,j=1}^{n} Fe_{ij}$$

where F is a right Ore-domain contained in $R\cap D$ and D is the right quotient field of F. (It is to be noted that F in general does not contain identity, see example given by Faith and Utumi [7, (C) page 60]) Let

$$q=\sum_{i,j=1}^{n}\alpha_{ij}e_{ij},\ \alpha_{ij}\in D.$$

Suppose $\alpha_{kl}\neq 0$. Fix a non-zero element a of F. $ae_{rk}\in R$ for every $r=1, 2, \cdots, n$. Also because $qQ\subset R$,

$$\Big(\sum_{i,j=1}^{n}\alpha_{ij}e_{ij}\Big)(a\cdot\alpha_{kl})^{-1}de_{ls}\in R$$

for every $s=1, \cdots, n$ and every $d\in D$. Therefore

$$ae_{rk}\Big(\sum_{i,j=1}^{n}\alpha_{ij}e_{ij}\Big)(a\alpha_{kl})^{-1}de_{ls}\in R$$

for every r, $s=1, 2, \cdots, n$ and every $d\in D$. But

$$ae_{rk}\Big(\sum_{i,j=1}^{n}\alpha_{ij}e_{ij}\Big)(a\alpha_{kl})^{-1}de_{ls}=de_{rs}.$$

Therefore $\sum_{i,j=1}^{n}De_{rs}=Q\subset R$. Hence $Q=R$.

4.2. Lemma. *If Q be an Asano's right quotient ring of R and $f\in Hom_R(Q, R)$ then $f(q)=f(1).q$ for every $q\in Q$.*

Proof. Let $q=ab^{-1}$, a, b in R, b regular. Then

$$f(q)b=f(ab^{-1})b=f(ab^{-1}b)=f(a)=f(1)a.$$

Therefore $f(q)=f(1)ab^{-1}=f(1)q$.

Proof of the Theorem. A module M_R is projective iff there exist subsets $\{m_i\}_{i\in I}$ of M and $\{f_i\}_{i\in I}$ of $\mathrm{Hom}_R(M, R)$, such that for each $m\in M$ $f_i(m)=0$ for almost all i and $m=\sum_{i\in I}m_if_i(m)$, see Bass [2, (4.8) page 477].

As Q_R is projective there exists subsets $\{b_i\}_{i\in I}$ of Q and $\{f_i\}_{i\in I}$ of $\mathrm{Hom}_R(Q, R)$ with the above properties. Now

$$l=b_1f_1(1)+b_2f_2(1)+\cdots\cdots+b_nf_n(1)$$

for finite subset $(1, 2, \cdots, n)$ of I. By the above lemma 4.2 $f_k(1)\,Q\subset R$ for each $k=1, 2, \cdots, n$. Let $e_1, e_2, \cdots, e_m$ the central idempotents in Q such that e_1Q, e_2Q, $\cdots$, e_mQ are the simple components of the semi-simple Artinian ring Q. Clearly for each $i=1, \cdots, m$, there exists $k(i)$, $1\leqslant k(i)\leqslant n$ such that $f_{k(i)}(1)e_i\neq 0$. But

$$(\underset{f(i)}{f(1)}e_i)e_i Q \subset e_i R \cap R .$$

But $e_i Q$ is the classical quotient ring of $e_i R$ R and $e_i Q$ is a simple Artinian ring, therefore by the lemma $e_i Q = e_i R \subset R$. Therefore $e_i Q \cap R$ for every i. Therefore $Q = R$.

5. Hereditary orders in semi-simple Artinian rings

In this section right hereditary semi-prime right Goldie-rings have been characterized.

5.1. Theorem. *If R be a semi-prime right Goldie ring, and Q be its classical right quotient ring, then R is hereditary if and only if for every essential right ideal I of R, there exist* $b_1, b_2, \cdots, b_n$ *in I and* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *in Q such that* $\sum_{i=1}^{n} b_i \alpha_i = 1$ *and* $\alpha_i I \subset R$ *for every* $i = 1, 2, \cdots, n$.

Proof. Assume R is hereditary. Let I be an essential right ideal of R. I is a projective right R module. By the characterization of projective modules mentioned in the proof of Theorem 4.1, there exist subsets $\{b_j\}_{j\in J}$ of I and $\{f_j\}_{j\in J}$ of $\mathrm{Hom}_R(I, R)$ such that for every $b \in I$, $f_j(b) = 0$ for almost all values of j and $b = \sum_{j\in J} b_j f_j(b)$. Now I being an essential right ideal of R, I contains a regular element, Goldie [9, Theorem 3.9]. Therefore $IQ = Q$. Each f_j has a unique extension f_j' in $\mathrm{Hom}_Q(Q, Q)$. Let $f_j'(1) = \alpha_j$. Then $f_j'(q) = f_j'(1.q) = f_j'(1)q = \alpha_j(q)$ for every q in Q. Therefore $\alpha_j b = f_j(b) = f_j(b)$ for every $b \in I$. Consequently $\alpha_j I \subset R$. Now let a be a regular element of I.

$$a = \sum_{j=1}^{n} b_j f_j(a) = (\sum_{j=1}^{n} b_j \alpha_j) a$$

for some finite subset $(1, 2, \cdots, n)$ of J. Therefore $1 = \sum_{i=1}^{n} b_i \alpha_i$. The sets $(b_1, b_2, \cdots, b_n)$ and $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ are the desired sets.

Conversely since every right ideal is a direct summand of an essential right ideal and a direct summand of a projective module is projective, it is sufficient to show that every essential right ideal is projective. Let I be an essential right ideal of R. There exist $b_1, b_2, \cdots, b_n$ in I and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in Q such that $\alpha_i I \subset R$ for every $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} b_i \alpha_i = 1$. Define $f_1, f_2, \cdots, f_n$ such that $f_i(b) = \alpha_i b$, $\forall b \in I$, $f_i \in \mathrm{Hom}_R(I, R)$. Also $b = (\sum_{i=1}^{n} b_i \alpha_i) b = \sum_{i=1}^{n} b_i (\alpha_i b) = \sum_{i=1}^{n} b_i f_i(b)$ $\forall b \in I$. Hence I is projective by the characterization of a projective module mentioned in the proof of Theorem 4.1. (The 'only if' part of this result is due to Levy [18]).

5.2. Corollary. (Levy, 1963) *A hereditary semiprime right Goldie ring is a right Noetherian ring.*

Proof. Since every right ideal is a direct summand of an essential right ideal, it is sufficient to show that every essential right ideal is finitely generated. Let I be an essential right ideal. There exist $b_1, b_2, \cdots, b_n$ in I and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in Q such that $\sum_{i=1}^{n} b_i\alpha_i=1$ and $\alpha_i I\subset R$ $i=1, 2, \cdots, n$. Therefore

$$I=\sum_{i=1}^{n} b_i(\alpha_i I)\subset\sum_{i=1}^{n} b_i R\subset I .$$

Hence

$$I=\sum_{i=1}^{n} b_i R .$$

5.3. Corollary. *A commutative integral domain R is hereditary if and only if it is a Dedekind domain.*

Proof. It is easy to make the following observations (a) An ideal I of R is invertible in the semi-group of fractionary ideals if and only if there exist $b_1, b_2, \cdots, b_n$ in I and $\alpha_1, \alpha_2, \cdots, \alpha_n$, in Q, the field of quotients of R such that $\sum_{i=1}^{n} b_i\alpha_i=1$ and $\alpha_i I\subset R$ for every $i=1, 2, \cdots, n$.

(b) Every ideal of R is an essential ideal. These two observations along with the theorem 6.1 proves the corollary.

5.4. Theorem. *A semi-prime principal right ideal ring is a hereditary ring.*

Proof. A semi-prime principal right ideal ring is clearly a semi-prime Goldie ring. By a remark made in the previous theorem 5.1 it is sufficient to prove that every essential right ideal of R is a projective R-module. Let I be an essential right ideal of R, then $I=aR$ for some a in R. The element a is regular, see Processi and Small [19, page 81]. Therefore $R_R\cong(aR)_R$ under the mapping $r\leftrightarrow ar$. Hence $I\,(=aR)$ is a projective R-module.

6. We return to the study of modules over a ring R, which has an Asano's quotient ring $Q\,(=R_S)$. If M_R is a projective R-module, then $M'\,(=MQ)$ is proved to be a projective Q-module. This result is used to prove that if M is an S-free R-module then projective dimension $(MQ)_Q\leqslant$ projective dimension M_R.

6.1. Category of S-free modules. Let $\mathfrak{F}$ denote the category of all S-free R-modules (with R-homomorphisms as the maps).

Let $\mathfrak{H}$ denote the category of Q-modules (with Q-homomorphisms as maps).

If $M\in\mathfrak{F}$, then there exists M' (unique upto isomorphism over M) in $\mathfrak{H}$ such that $M_R\subset M'_R$ and $M'=MQ$. If $M_1, M_2\in\mathfrak{F}$ and $f\colon M_1\to M_2$, then there exists unique Q-homomorphism $f'\colon M_1Q\to M_2Q$, which extends f. This map f' is defined as follows:

$$f'(m_1 s^{-1})=f(m_1)s^{-1}\ \forall m_1\in M_1,\ s\in S .$$

In fact the rule T:

$$T(M) = MQ$$
$$T(f) = f'$$

is an additive covariant functor from the category $\mathfrak{F}$ into the category $\mathfrak{H}$. T is seen below to be an exact functor:
Let

$$0 \to M_1 \xrightarrow{j} M \xrightarrow{\pi} M_2 \to 0$$

be an exact sequence in $\mathfrak{F}$. Then the sequence

$$0 \to M_1 Q \xrightarrow{j'} MQ \xrightarrow{\pi'} M_2 Q \to 0$$

is also an exact sequence.

Exactness at $M_1 Q$: $j'(m_1 s^{-1}) = 0$ implies $j(m_1) s^{-1} = 0$,
$j(m_1) = 0$, $m_1 = 0$, $m_1 s^{-1} = 0$.

Exactness at MQ: $\pi' j'(m_1 s^{-1}) = \pi'(j'(m_1 s^{-1})) = \pi'(j(m_1) s^{-1})$
$= (\pi j(m_1)) s^{-1} = 0$.

If $\pi'(m s^{-1}) = 0$, then $\pi(m) = 0$, $m = j(m_1)$ for some m_1 in M_1.
$j'(m_1 s^{-1}) = j(m_1) s^{-1} = m s^{-1}$.

Exactness at $M_2 Q$: If $m_2 s^{-1}$ be any element of $M_2 Q$, then there exists $m \in M$ such that $\pi(m) = m_2$, $\pi'(m s^{-1}) = m_2 s^{-1}$.

6.2. We observe in passing that a projective R-module is torsion free.

6.3. Theorem. *Let R be a ring and Q $(=R_S)$ be an Asano's right quotient ring of R. If M is a projective R-module, then MQ is a projective Q-module.*

Proof. Let

$$\begin{array}{ccccc} & & (MQ)_Q & & \\ & & \downarrow f & & \\ A_Q & \xrightarrow[\pi]{} & B_Q & \longrightarrow & 0 \end{array}$$

be a diagram of Q-modules with exact row. Let $f_1 : M \to B$ denote the restriction of f to M. f_1 is R-homomorphism of M into B. We therefore have the diagram

$$\begin{array}{ccccc} & & M_R & & \\ & & \downarrow f_1 & & \\ A_R & \xrightarrow[\pi]{} & B_R & \longrightarrow & 0 \end{array}$$

of R-modules with exact row. Since M_R is projective there exists $g: M \to A$ such that $\pi g = f_1$. Let g': $MQ \to AQ$ $(=A)$ denote the extension of g: $M \to A$. Then

$$\pi g'(m s^{-1}) = \pi(g(m) s^{-1}) = \pi g(m) s^{-1} = f_1(m) s^{-1} = f(m.s^{-1}) .$$

6.4. Corollary. *If M is a Q-module such that M_R is projective, then M_Q is also projective.*

6.5. DEFINITION. Projective dimension of a module
Global dimension of a ring

Let R be an arbitrary ring and M be a module over R. An exact sequence

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \to \cdots\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\cdots} M \to 0$$

where each M_i is projective is called a projective resolution of M. The smallest positive integer n such that kernel d_n is projective is called the projective dimension of the module M. If no such integer n exists, then the projective dimension of M is infinity. (It is known that the projective dimension of M is independent of the projective resolution). Right Global dimension of a ring R=supremum of the projective dimensions of all R-modules (see sec. 2).

It is well known that for any ring R, supremum of the injective dimensions of all R-modules=supremum of all the projective dimensions of all R-modules. Therefore the same term Right Global dimension is used for both the supremums.

6.6. Theorem. *Let R be a ring and $Q(=R_S)$ be an Asano's right quotient ring of R. If M is an S-free R-module, then projective dimension $(MQ)_Q \leqslant$ projective dimension M_R.*

Proof. Let

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \to \cdots\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\cdots} M \to 0 \qquad \text{(i)}$$

be a projective resolution of M_R. If projective dimension $M_R = \infty$, we have nothing to prove. Let the projective dimension $M = n(<\infty)$. Then K_n, the kernel of d_n is projective R-module.

From the projective resolution (i) of M_R we get a projective resolution of $(MQ)_Q$

$$\cdots \to M_n Q \xrightarrow{d'_n} M_{n-1} Q \to \cdots\cdots \to M_1 Q \xrightarrow{d_1'} M_0 Q \xrightarrow{\cdots} MQ \to 0$$

(We note that the sequence is exact because we have noticed that the functor T is an exact functor. Also each $M_i Q$ is a projective Q-module by theorem 6.3).

The kernel of d_n=(kernel d_n)Q which is a projective Q-module by theorem 6.3. Hence projective dimension $(MQ)_Q \leqslant$ projective dimension M_R .

6.7. Corollary. *If M is a Q module, then projective dimension $M_Q \leqslant$ projective dimension M_R.*

6.8. Corollary. *Right Global dimension $Q \leqslant$ right Global dimension R.*

7. Hereditary quotient rings

Let R be a ring and S be a multiplicatively closed set of regular elements of R such that R_S exists. Denote R_S by Q. In this section we obtain necessary and sufficient conditions on R such that Q becomes a hereditary ring.

7.1. Theorem. *A quotient ring Q of R is hereditary if and only if for every exact sequence of R modules*

$$M \xrightarrow{\pi} N \longrightarrow 0$$

where M is S-free and injective and N is S-free, N is injective.

Proof. Let Q be hereditary. Let

$$M \xrightarrow{\pi} N \longrightarrow 0$$

be an exact sequence of R-modules, where M is S-free and injective and N is S-free. Since every injective module is divisible (see 3.2), M is S-divisible and therefore M is Q-module. Also because M is S-divisible, therefore N is also S-divisible, therefore N is also a Q-module. The map π is Q-homomorphism (see 6.1, $\pi' = \pi$ in this case). Now since M_R is injective, therefore M_Q is injective. As Q is hereditary N_Q is injective and therefore N_R is injective.

Conversely we prove that Q is hereditary. It is sufficient to show that for every exact sequence of Q-modules

$$M \xrightarrow{\pi} N \longrightarrow 0$$

where M is injective, N is injective. Now regarding the above sequence as a sequence of R-modules we note that M_R is S-free, M_R is injective and N is S-free. Therefore N_R is injective. Hence N_Q is injective.

7.2. Corollary. *If R is hereditary, then a quotient ring Q $(=R_S)$ of R is also hereditary.*

Proof. Since R is hereditary, for any exact sequence of R-modules

$$M \longrightarrow N \longrightarrow 0$$

where M is injective, N is injective. Therefore by the theorem 7.1 Q is hereditary.

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