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ON CERTAIN COHOMOLOGY GROUPS ATTACHED TO HERMITIAN SYMMETRIC SPACES (II)

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Introduction

Let $X=G/K$ be a bounded symmetric domain in $\boldsymbol{C}^N$, where G is a semisimple Lie group with finite center and K is a maximal compact subgroup of G. An automorphic factor j on X is a C^∞-mapping j: $G\times X\to GL(S)$, S being a finite dimensional complex vector space, which satisfies the conditions:

1) $j(s, x)$ is holomorphic in $x\in X$ for each $s\in G$;
2) $j(ss', x)=j(s, s'x)j(s', x)$ for $x\in X$ and $s, s'\in G$.

Let x_0 be the point of $X=G/K$ represented by the coset K. An automorphic factor j defines a representation τ of the group K by the formula $\tau(t)=j(t, x_0)$ for $t\in K$ and we say that j is a prolongation of the representation τ of K. We know that, given a representation τ of K in a complex vector space S, there exists an automorphic factor J_τ: $G\times X\to GL(S)$ which is a prolongation of τ and which we call the canonical automorphic factor of type τ [4, 6, Part II]. Moreover, if τ is an irreducible representation of K, then the automorphic factors which are prolongations of τ are equivalent to each other [6, Appendix].

Let j be an automorphic factor on X. Then G acts on $X\times S$ as a group of holomorphic transformations if we define the action of $s\in G$ by putting

$$s(x, u) = (sx, j(s, x)u)$$

for $(x, u)\in X\times S$.

Now let Γ be a discrete subgroup of G. Then Γ acts on X properly and discontinuously. In the following we assume that the quotient space $\Gamma\backslash X$ is compact and that Γ acts freely on X and let $M=\Gamma\backslash X$. Then M is a compact complex Kähler manifold. Moreover, Γ acts on $X\times S$ and let $E(j)$ be the quotient of $X\times S$ by the action of Γ: $E(j)=\Gamma\backslash(X\times S)$. Then $E(j)$ is a holomorphic vector bundle over M with typical fibre S. In this paper we consider exclusively the case where j is a canonical automorphic factor J_τ. In this case the vector bundle $E(J_\tau)$ may be interpreted in the following way [6]. Let X_u be the hermitian symmetric space of compact type associated with X. The representation τ of K defines a "homogeneous" vector bundle $E_u(\tau)$ over X_u. Now X is imbedded in X_u as an open submanifold and Γ acts on $E_u(\tau)|X$ as a group of bundle automorphisms, where $E_u(\tau)|X$ denotes the portion of $E_u(\tau)$ over X. Then the quotient of $E_u(\tau)|X$ by the action of Γ is a holomorphic vector bundle over M which is isomorphic to $E(J_\tau)$.

Let $\boldsymbol{E}(J_\tau)$ denote the sheaf of germs of holomorphic sections of $E(J_\tau)$. Each cohomology class in $H^q(M, \boldsymbol{E}(J_\tau))$ is represented by an $E(J_\tau)$-valued harmonic q-form which we shall call an automorphic harmonic q-form of type J_τ. In particular, for $q=0$ an automorphic harmonic 0-form is nothing but a holomorphic automorphic form of type J_τ in the usual sense.

In Part I of this paper we shall show that an automorphic harmonic q-form η of type J_τ is identified with a set $(f_S)_{S\in W_1^\Lambda(q)}$ of holomorphic automorphic forms f_S of type J_{τ_S} provided that the highest weight Λ of the representation τ of K satisfies a certain condition; here W_Λ^1 denotes a subset of the Weyl group of the Lie algebra $\mathfrak{g}^C$ uniquely determined by Λ and τ_S is a representation of K determined by Λ and S in a certain way. We shall show also that, for $q=q_\Lambda$, where q_Λ is a number uniquely determined by Λ, the set $W_\Lambda^1(q)$ consists of a single element; thus every automorphic harmonic q-form η of type J_τ is identified with a holomorphic automorphic form of type J_{τ_S} for $q=q_\Lambda$.

In Part II of this paper we shall prove a formula which expresses the dimension of the space of "automorphic forms" in terms of the unitary representation of G in $L^2(\Gamma\backslash G)$. Combined with the results obtained in Part I we obtain a formula on the dimension of the space of automorphic harmonic forms which might be interesting in view of a conjecture stated by Langlands in [3].

PART I

1. We retain the notation introduced in the introduction. We have a decomposition of the Lie algebra

$$\mathfrak{g}^C = \mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{k}^C,$$

where $\mathfrak{k}^C$ is the complexification of the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ corresponding K and $\mathfrak{n}^\pm$ are abelian subalgebras of $\mathfrak{g}^C$ such that

$$[\mathfrak{k}^C, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm, \ [\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{k}^C.$$

Moreover, $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are spanned by root vectors corresponding to the roots of $\mathfrak{g}^C$ (with respect to the Cartan subalgebra $\mathfrak{h}$) (see [6, Part II]). We denote by Ψ the set of all roots α such that the root vector X_α belongs to $\mathfrak{n}^+$. Then

$$\mathfrak{n}^+ = \sum_{\alpha\in\Psi} \boldsymbol{C} X_\alpha \quad \text{and} \quad \mathfrak{n}^- = \sum_{\alpha\in\Psi} \boldsymbol{C} X_{-\alpha}$$

with $X_{-\alpha}=\bar{X}_{\alpha}$, where - denotes the conjugation of $\mathfrak{g}^C$ with respest to the real form $\mathfrak{g}$. We choose X_α in such a way that

$$\varphi(X_\alpha, X_{-\alpha}) = 1 ,$$

where φ denotes the Killing form of $\mathfrak{g}^C$.

We know that there exists an ordering of the roots such that the roots in Ψ are all positive. We fix once and for all such an ordering of roots. We denote by Θ the set of all positive roots not belonging to Ψ. Then the root vector of $\beta\in\Theta$ belongs to $\mathfrak{k}^C$. We call a root α belonging to Ψ (resp. Θ) as a non-compact (resp. compact) positive root. We shall denote by Σ the set of all roots and by Σ^+ (resp. Σ^-) the set of all positive (resp. negative) roots.

Let W be the Weyl group of $\mathfrak{g}^C$. W is a group of linear transformations of the dual space $\mathfrak{h}_0^*$ of the real vector space $\mathfrak{h}_0=\sqrt{-1}\,\mathfrak{h}$ in $\mathfrak{g}^C$ and W is generated by the reflections $S_\alpha(\alpha\in\Sigma^+)$ with respect to the hyperplanes $P_\alpha=\{\lambda\,|\,(\alpha, \lambda)=0\}$.

We shall denote by W_1 the subgroup of W generated by S_β with $\beta\in\Theta$. W_1 is isomorphic to the Weyl group of $\mathfrak{k}^C$. For $T\in W$, let

$$\Phi_T = T(\Sigma^-)\cap\Sigma^+$$

and let

$$n(T) = \text{the number of roots in } \Phi_T .$$

Let W^1 be the subset of the Weyl group W consisting of all $T\in W$ such that $\Phi_T\subset\Psi$. It is easy to see that T belongs to W^1 if and only if $T^{-1}(\Theta)\subset\Sigma^+$.

Now let τ be an irreducible representation of K in a complex vector space S. Then τ defines an irreducible representation of the complex reductive Lie algebra $\mathfrak{k}^C$ which we shall denote by the same letter τ. Let Λ be the highest weight of τ. Then we have

$$(\Lambda, \beta)\geq 0 \qquad \text{for all} \quad \beta\in\Theta .$$

We shall assume that τ satisfies the following condition:

$$(*) \qquad (\Lambda, \alpha)\geq 0 \qquad \text{for all} \quad \alpha\in\Psi .$$

Then $(\Lambda, \gamma)\geq 0$ for all $\gamma\in\Sigma^+$ and hence there exists an irreducible representation of $\mathfrak{g}^C$ whose highest weight is Λ. We shall denote this representation of $\mathfrak{g}^C$ by ρ and by Λ' the lowest weight of ρ.

Now put

$$W_\Lambda^1 = \{T\in W^1\,|\,T\Lambda' = R_1\Lambda\} ,$$

where R_1 is the unique element of W_1 such that

$$R_1(\Theta) = -\Theta .$$

Further we let

$$W_\Lambda^1(q)=\{T\in W_\Lambda^1 \mid n(T)=q\}\,.$$

For $T\in W^1$ we shall denote by τ_T the irreducible representation of $\mathfrak{k}^C$ whose highest weight is $-\xi'_T$, where

$$\xi'_T=T\Lambda'+\langle\Phi_T\rangle=T(\Lambda'-\delta)+\delta;$$

here, for any subset Φ of Σ, $\langle\Phi\rangle$ denotes the sum of the roots belonging to Φ and $\delta=\langle\Sigma^+\rangle/2$.

Now we can state

Theorem 1. *In the notation introduced above, to each automorphic harmonic q-form η of type J_τ, where τ satisfies the assumption* (*), *we can associate uniquely a set $(f_S)_{S\in W_\Lambda^1(q)}$ of holomorphic automorphic forms of type J_{τ_S} and each of such a set corresponds to an automorphic harmonic q-form η of type J_τ. In other words we have the following isomorphism of the cohomology groups*:

$$H^q(M,\boldsymbol{E}(J_\tau))\cong\sum_{S\in W_\Lambda^1(q)}H^0(M,\boldsymbol{E}(J_{\tau_S}))\,.$$

REMARK 1. If $q=N=\dim_C M$, this theorem reduces to the duality theorem of Serre as we shall see later.

REMRAK 2. Actually we shall prove Theorem 1 without the assumption that Γ acts freely on M. See Theorem 1′ in § 2.

Under the same assumption (*) on Λ, let q_Λ be the number of positive roots $\alpha\in\Psi$ such that $(\Lambda,\alpha)>0$.

Then we have

Theorem 2. *If $q<q_\Lambda$, then $W_\Lambda^1(q)$ is empty. Moreover, $W_\Lambda^1(q_\Lambda)$ consists of a single element T_0 such that*

$$\Phi_{T_0}=\{\alpha\in\Psi\mid(R_1\Lambda,\alpha)>0\}\,.$$

Thus we have

$$H^q(M,\boldsymbol{E}(J_\tau))=0\qquad for\quad q<q_\Lambda;$$
$$H^{q_\Lambda}(M,\boldsymbol{E}(J_\tau))\cong H^0(M,\boldsymbol{E}(J_{\tau_{T_0}}))\,,$$

where the highest weight of the irreducible representation τ_{T_0} of $\mathfrak{k}^C$ is

$$-\xi'_{T_0}=-T_0\Lambda'-\langle\Phi_{T_0}\rangle\,.$$

REMRAK 3. The vanishing of the cohomology groups $H^q(M,E(J_\tau))$ for $q<q_\Lambda$ has been already proved in [5]. We also remark that $R_1\Lambda$ is the lowest weight of the representation τ of $\mathfrak{k}^C$.

The proof of these theorems will be given in the following sections.

2. We shall recall here some results proved in [4] and [5]. Let ρ be an irreducible representation of $\mathfrak{g}^C$ in a complex vector space F with highest weight Λ.

A*) Restricting the representation ρ of $\mathfrak{g}^C$ to $\mathfrak{n}^-$, we may consider F as an $\mathfrak{n}^-$-module. Let $C(\mathfrak{n}^-, F)=\sum_q C^q(\mathfrak{n}^-, F)$ be the cochain complex of the abelian Lie algebra $\mathfrak{n}^-$ with coefficients in F, where $C^q(\mathfrak{n}^-, F)$ is the vector space of all q-linear alternating maps of $\mathfrak{n}^-$ into F. By the Killing form φ of $\mathfrak{g}^C$ we can identify $\mathfrak{n}^+$ with the dual space of $\mathfrak{n}^-$ and hence $C(\mathfrak{n}^-, F)$ with $F\otimes\wedge\mathfrak{n}^+$ and $C^q(\mathfrak{n}^-, F)$ with $F\otimes\overset{q}{\wedge}\mathfrak{n}^+$. Let $(\ ,\ )_F$ be the inner product in F such that $(\rho(x)u, v)_F=(u, \rho(\bar{x})v)_F$ for all $x\in\mathfrak{n}^+\oplus\mathfrak{n}^-$ and $(\rho(y)u, v)_F=-(u, \rho(\bar{y})v)_F$ for all $y\in\mathfrak{k}^C$, where - denotes the conjugation of $\mathfrak{g}^C$ with respest to $\mathfrak{g}$. The Killing form φ of $\mathfrak{g}^C$ defines an inner product in $\mathfrak{n}^+$ such that $\{X_\alpha \mid \alpha\in\Psi\}$ is an orthonormal basis. Using these inner products in F and $\mathfrak{n}^+$ we can define an inner product in $C(\mathfrak{n}^-, F)$ which we denote by $(\ ,\ )$. Let d^- be the coboundary operator. Then there exists an operator δ^- of degree -1 such that $(d^- c, c')=(c, \delta^- c')$ for all $c, c'\in C(\mathfrak{n}^-, F)$. Let $\Delta^-=d^-\delta^-+\delta^-d^-$. An element $c\in C^q(\mathfrak{n}^-, F)$ is called a harmonic q-cocyle if $\Delta^-c=0$. Every cohomology class of $H^q(\mathfrak{n}^-, F)$ is represented by a unique harmonic cocyle. Let $\mathcal{H}^q$ be the space of all harmonic q-cocyles.

Now $C^q(\mathfrak{n}^-, F)=F\otimes\overset{q}{\wedge}\mathfrak{n}^+$ is a $\mathfrak{k}^C$-module, where $y\in\mathfrak{k}^C$ operates on F and $\mathfrak{n}^+$ respectively by $\rho(y)$ and $\mathrm{ad}(y)$. Let

$$F\otimes\overset{q}{\wedge}\mathfrak{n}^+ = \Sigma_{\xi'} m_{\xi'} U_{\xi'} \tag{1}$$

be a decomposition of $F\otimes\overset{q}{\wedge}\mathfrak{n}^+$ into direct sum of irreducible $\mathfrak{k}^C$-modules $U_{\xi'}$, where ξ' denotes the lowest weight of the irreducible representation $\tau_{\xi'}$ of $\mathfrak{k}^C$ in $U_{\xi'}$ and $m_{\xi'}$ denotes the multiplicity of $\tau_{\xi'}$.

For $T\in W^1$, let

$$\xi'_T = T\Lambda'+\langle\Phi_T\rangle .$$

Then the mapping $T\to\xi'_T$ is an injection of W^1 into the set $\{\xi'\}$ of lowest weights appearing in (1) and we have:

$$\mathcal{H}^q = \sum_{T\in W^1(q)} U_{\xi'_T} , \qquad m_{\xi'_T} = 1 ,$$

where $W^1(q)=\{T\in W^1 \mid n(T)=q\}$. This result is due to Kostant [2].

B. The $\mathfrak{g}^C$-module F decomposes into sum $F=S_1+S_2+\cdots+S_m$ of mutually orthogonal $\mathfrak{k}^C$-submodules such that

* See [4] §§8, 10 or [6].

1) $\rho(X)S_t \subset S_{t-1}$ for $X \in \mathfrak{n}^+$ and $\rho(Y)S_t \subset S_{t+1}$ for $Y \in \mathfrak{n}^-$ for $t=1, 2, \cdots, m$, where $S_0 = S_{m+1} = (0)$;

2) S_1 and S_m are simple $\mathfrak{k}^C$-modules and the highest weight of the representation of $\mathfrak{k}^C$ in S_1 is Λ. Moreover

$$S_1 = \{u \in F \mid \rho(X)u = 0 \quad \text{for all} \quad X \in \mathfrak{n}^+\} ,$$
$$S_m = \{u \in F \mid \rho(Y)u = 0 \quad \text{for all} \quad Y \in \mathfrak{n}^-\} .$$

(See [4, Lemma 5.2] or [6, Lemma 6.1].)

Let

$$\mathcal{H}^{0,q} = (S_1 \otimes \overset{q}{\wedge} \mathfrak{n}^+) \cap \mathcal{H}^q .$$

Then

$$\mathcal{H}^{0,q} = \sum_{T \in W^1_\Lambda(q)} U_{\xi'_T} \tag{2}$$

where, as in §1, $W^1_\Lambda(q) = \{T \in W^1 \mid T\Lambda' = R_1\Lambda,\ n(T) = q\}$.

Proof. Since $\mathcal{H}^{0,q}$ is a $\mathfrak{k}^C$-submodule of $\mathcal{H}^q$ and $\mathcal{H}^q$ is a direct sum of simple $\mathfrak{k}^C$-modules $U_{\xi'_T}$ which are not isomorphic to each other, we have

$$\mathcal{H}^{0,q} = \sum_{T \in A} U_{\xi'_T}$$

where A is a subset of $W^1(q)$. Then $U_{\xi'_T}$ is contained in $S_1 \otimes \overset{q}{\wedge} \mathfrak{n}^+$ with multiplicity 1 for $T \in A$.

Now let

$$F = \sum_{\mu'} n_{\mu'} F_{\mu'}$$

be the decomposition of F into direct sum of simple $\mathfrak{k}^C$-submodules $F_{\mu'}$ with lowest weight μ' and with multiplicity $n_{\mu'}$. Then $S_1 = F_{R_1\Lambda}$ and $n_{R_1\Lambda} = 1$. For any $T \in W^1$, $T\Lambda$ appears as one of μ' with $n_{T\Lambda'} = 1$. Indeed, as $T^{-1}(\Theta) \subset \Sigma^+$, $\langle T\Lambda', \beta \rangle = \langle \Lambda', T^{-1}\beta \rangle \leq 0$ for all $\beta \in \Theta$ and therefore $T\Lambda'$ is the lowest weight of an irreducible respresentation of $\mathfrak{k}^C$ which is contained in F and the eigenspace for the weight $T\Lambda'$ is of dimension 1. Now let $T \in A$. Then $\xi'_T = T\Lambda' + \langle \Phi_T \rangle$ and hence the 1-dimensional eigenspace for the weight ξ'_T is contained in $F_{T\Lambda'} \otimes \overset{q}{\wedge} \mathfrak{n}^+$. On the other hand, it is contained in $S_1 \otimes \overset{q}{\wedge} \mathfrak{n}^+$. Therefore we should have $S_1 = F_{T\Lambda'}$ and hence $R_1\Lambda = T\Lambda'$. Thus $T \in W^1_\Lambda(q)$. Let, conversely, $T \in W^1_\Lambda(q)$. Then $F_{T\Lambda'} = S_1$, because $T\Lambda' = R_1\Lambda$ and $F_{T\Lambda'} \otimes \overset{q}{\wedge} \mathfrak{n}^+ \supset U_{\xi'_T}$ and therefore $T \in A$. Thus we have proved that $A = W^1_\Lambda(q)$ and hence our assertion.

C. Now let τ be a representation of K in a complex vector space S and let Γ be a discrete subgroup of G such that $\Gamma \backslash G$ is compact. We don't need to assume that Γ acts freely on $X=G/K$. Let $A^{0,q}(\Gamma, X, J_\tau)$ denote the vector space of all S-valued differential forms of type $(0, q)$ on X such that

$$(\eta\circ L_\gamma)_x = J_\tau(\gamma, x)\eta_x$$

for all $\gamma\in\Gamma$ and $x\in X$, where L_γ denotes the transformation of X by γ. Then $\Sigma_q A^{0,q}(X, \Gamma, J_\tau)$ is a complex with coboundary operator d'' and we denote by $H^{0,q}(X, \Gamma, J_\tau)$ the q-th cohomology group of this complex. Each cohomology class of $H^{0,q}(X, \Gamma, J_\tau)$ is represented by a harmonic form which we shall call an *automorphic harmonic q-form of type J_τ*. In the case $q=0$, an automorphic harmonic form is a holomorphic function f on X such that $f(\gamma x)=J_\tau(\gamma, x)f(x)$ for all $\gamma\in\Gamma$ and $x\in X$, i.e. a holomorphic automorphic form of type J_τ.

If Γ acts freely on X, then the cohomology group $H^{0,q}(X, \Gamma, J_\tau)$ is isomorphic to $H^q(M, \boldsymbol{E}(J_\tau))$.

D. From now on we assume that τ is an irreducible representation of K such that the highest weight Λ of τ satisfies the condition (*) in §1, i.e. $(\Lambda, \alpha)\geq 0$ for all roots $\alpha\in\Psi$. There exists then an irreducible representation ρ of $\mathfrak{g}^C$ in a complex vector space F whose highest weight is Λ. Then we have a decomposition $F=S_1+S_2+\cdots+S_m$ of F into direct sum of $\mathfrak{k}^C$-submodules such that the representation space S of τ is isomorphic to S_1 as $\mathfrak{k}^C$-module.

We assume that the representation ρ of $\mathfrak{g}^C$ is induced from a representation ρ of the group G. Let $A(X, \Gamma, \rho)$ be the vector space of all F-valued r-forms ω on X such that $\omega\circ L_\gamma=\rho(\gamma)\omega$ for all $\gamma\in\Gamma$. Then $\Sigma_r A^r(X, \Gamma, \rho)$ is a complex with coboundary operator d (d being the operator of exterior differentiation) and each element of the cohomology group $H^r(X, \Gamma, \rho)$ is represented by a unique harmonic form; moreover $H^r(X, \Gamma, \rho)=\sum_{r=p+q} H^{p,q}(X, \Gamma, \rho)$, where $H^{p,q}(X, \Gamma, \rho)$ denotes the cohomology classes represented by harmonic forms of type (p, q) (see [4], [5], [6]). We have proved in [5] the following results:

a) $H^{0,q}(X, \Gamma, J_\tau)\cong H^{0,q}(X, \Gamma, \rho)$;

b) The space of harmonic forms of type $(0, q)$ in $A^q(X, \Gamma, \rho)$ is identified with the space of all $F\otimes\overset{q}{\wedge}\mathfrak{n}^+$ valued smooth functions f on $\Gamma\backslash G$ satisfying the following condition:

(i) $Yf=-(\rho\otimes\mathrm{ad}_+^q)(Y)f$ for all $Y\in\mathfrak{k}$.

(ii) For every point $x\in\Gamma\backslash G$, the value $f(x)(\in F\otimes\overset{q}{\wedge}\mathfrak{n}^+)$ is a harmonic cocycle of $C^q(\mathfrak{n}^-, F)=F\otimes\overset{q}{\wedge}\mathfrak{n}^+$ and moreover $f(x)\in S_1\otimes\overset{q}{\wedge}\mathfrak{n}^+$.

(iii) $X_\alpha f=0$ for all $\alpha\in\Psi$;

here we consider an element $X\in\mathfrak{g}^C$ as a complex vector field on $\Gamma\backslash G$ which is a projection of the left invariant complex vector field X on G. For the details see [5, Theorem 7.1 and Lemma 6.1].

Thus we may identify an automorphic harmonic q-form η with an $F\otimes\overset{q}{\wedge}\mathfrak{n}^+$ valued function f on $\Gamma\backslash G$ satisfying the above three conditions. We denote by $\mathcal{L}$ the vector space consisting of all these functions. It follows from (ii) that every $f\in\mathcal{L}$ is actually $\mathcal{H}^{0,q}$ valued. $\mathcal{H}^{0,q}$ is a direct sum of simple $\mathfrak{k}^C$-modules $U_{\xi'_T}(T\in W^1_\Lambda(q))$. To simplify the notation we put $U_T=U_{\xi'_T}$ and let τ'_T denote the representation of $\mathfrak{k}^C$ in U_T. Then $\tau'_T(Y)=(\rho\otimes\mathrm{ad}\,{}^q_+)(Y)$ on U_T for all $Y\in\mathfrak{k}^C$.

Let $f_T(x)$ denote the U_T-component of $f(x)$ for $x\in\Gamma\backslash G$. Then $f=\Sigma f_T$ and $\mathcal{L}$ decomposes into direct sum

$$\mathcal{L}=\Sigma_T\mathcal{L}_T\,,$$

where $\mathcal{L}_T$ consists of all U_T-valued functions f_T on $\Gamma\backslash G$ satisfying

$$1_T)\quad Yf_T=-\tau'_T(Y)f_T\,,\qquad \text{for all}\quad Y\in\mathfrak{k};$$
$$2_T)\quad X_\alpha f=0\qquad \text{for all}\quad \alpha\in\Psi\,.$$

The inner product in $F\otimes\wedge\mathfrak{n}^+$ defines an inner product in U_T such that $(\tau'_T(Y)u,v)+(u,\tau'_T(\bar{Y})u)=0$ $(Y\in\mathfrak{k}^C)$. Let $\sharp$ be the conjugate linear isomorphism of U_T onto the dual space U^*_T defined by

$$(\sharp u)(v)=(v,u)\qquad \text{for all}\quad v\in U_T\,.$$

Let τ_T denote the representation of $\mathfrak{k}^C$ in U^*_T contragredient to τ'_T. Then we have

$$\tau_T(Y)\sharp u=\sharp(\tau'_T(\bar{Y})u)$$

for $Y\in\mathfrak{k}^C$. For every U_T-valued function f on $\Gamma\backslash G$ we define U^*_T-valued function $\sharp f$ on $\Gamma\backslash G$ by putting

$$(\sharp f)(x)=\sharp f(x)\,.$$

Then, for $X\in\mathfrak{g}^C$, we have

$$\sharp(Xf)=\bar{X}(\sharp f)\,.$$

Thus $\sharp$ defines a conjugate linear isomorphism of $\mathcal{L}_T$ onto the complex vector space $\mathcal{L}^*_T$ consisting of all U^*_T valued functions h on $\Gamma\backslash G$ satisfying

$$1^*_T)\quad Yh=-\tau_T(Y)h\qquad \text{for all}\quad Y\in\mathfrak{k};$$
$$2^*_T)\quad X_{-\alpha}h=0\qquad \text{for all}\quad \beta\in\Psi\,.$$

On the other hand by [4], a function $h\in\mathcal{L}^*_T$ is identified with a holomorphic automorphic form of type J_{τ_T}; in fact for a holomorphic automorphic form $a(x)$ on X let $\tilde{h}(s)=J_{\tau_T}(s,x_0)^{-1}a(\pi(s))$, for $s\in G$, where $\pi\colon G\to X$ denotes the canonical projection. Then the function $\tilde{h}$ on G is left invariant by Γ and hence $\tilde{h}$ defines a function h on $\Gamma\backslash G$ and this function h satisfies the above two conditions.

Thus we have proved the following Theorem 1'.

Theorem 1'. *Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Let τ be an irreducible representation of K with highest weight Λ such that $(\Lambda, \alpha)\geq 0$ for all $\alpha\in\Psi$. Then the space of automorphic harmonic q-forms of type J_τ is isomorphic to the direct sum of the spaces of holomorphic automorphic forms of type J_{τ_T}, where T ranges over the subset $W_\Lambda^1(q)$ of the Weyl group W of $\mathfrak{g}^C$ and where τ_T denotes the irreducible representation of $\mathfrak{k}^C$ with the highest weight $-T\Lambda'-\langle\Phi_T\rangle$, Λ' being the lowest weight of the irreducible representation ρ of $\mathfrak{g}^C$ with highest weight Λ.*

REMARK 1. Let $q=N=\dim_C X$. There exists a unique element $R\in W$ such that $R(\Sigma^+)=\Sigma^-$. Let $R^1=R_1^{-1}R$. Then $R^1\in W^1$ and $\Phi_{R^1}=\Psi$. In fact, $(R^1)^{-1}(\Theta)=R^{-1}R_1(\Theta)=R^{-1}(-\Theta)\subset\Sigma^+$ and hence $R^1\in W^1$. Moreover since Ψ is the set of weights of the $\mathfrak{k}^C$-module $\mathfrak{n}^+$ and since $R_1\in W_1$, we have $R_1(\Psi)=\Psi$ and hence $R((R^1)^{-1}\Psi)=\Psi\subset\Sigma^+$ and hence $(R^1)^{-1}\Psi\subset\Sigma^-$. Thus $\Psi\subset R^1(\Sigma^-)$ and hence $\Psi=\Phi_{R^1}$. Thus $R^1\in W^1(N)$. Next we show that $R^1\in W_\Lambda^1(N)$, i.e. $R^1\Lambda'=R_1\Lambda$. In fact, we have $R\Lambda'=\Lambda$, i.e. $R_1R^1\Lambda'=\Lambda$. But $R_1^2=1$ and hence $R^1\Lambda'=R_1\Lambda$. Moreover, it is clear that $W_\Lambda^1(N)$ consists of a single element and hence $W_\Lambda^1(N)=\{R^1\}$. Now we show that

$$\tau_{R_1}=\sigma\otimes\tau^*,$$

where σ is the 1-dimensional representation of $\mathfrak{k}$ in $\overset{N}{\wedge}\mathfrak{n}^-$ defined by $\sigma(Y)=\mathrm{tr}(\mathrm{ad}_-(Y))$ $(Y\in\mathfrak{k}^C)$ and τ^* is the contragredient of τ. In fact, the highest weight of τ_{R_1} is $-R^1\Lambda'-\langle\Phi_{R_1}\rangle=-R_1\Lambda-\langle\Psi\rangle$. On the other hand, the weight of σ is $-\langle\Psi\rangle$ and the highest weight of τ^* is $-R_1\Lambda$ and hence $\tau_{R_1}=\sigma\otimes\tau^*$. By Theorem 1, we have $H^{0,N}(X, \Gamma, J_\tau)\cong H^{0,0}(X, \Gamma, J_{\sigma\otimes\tau^*})$. If Γ acts freely on X, we have $E(J_{\sigma\otimes\tau^*})=K\otimes E(J_\tau)^*$, where K denotes the canonical bundle of $M=\Gamma\backslash X$ and $E(J_\tau)^*$ is the dual vector bundle of $E(J_\tau)$ and hence the isomorphism $H^N(M, \boldsymbol{E}(J_\tau))\cong H^0(M, \boldsymbol{K}\otimes\boldsymbol{E}(J_\tau)^*)$ and this is a special case of Serre duality theorem.

REMARK 2. We have $(-\xi'_T, -\xi'_T+2\delta)=(\Lambda, \Lambda+2\delta)$ for any T, where $\delta=\Sigma_{\alpha>0}(\alpha/2)$. In fact $\xi'_T=T(\Lambda'-\delta)+\delta$ and hence $(-\xi'_T, -\xi'_T+2\delta)=(T(\Lambda'-\delta), T(\Lambda'-\delta))-(\delta, \delta)=(\Lambda'-\delta, \Lambda'-\delta)-(\delta, \delta)=(\Lambda', \Lambda')-2(\Lambda', \delta)=(R\Lambda, R\Lambda)-2(R\Lambda, R(R^{-1}\delta))=(\Lambda, \Lambda)-2(\Lambda, -\delta)=(\Lambda, \Lambda+2\delta)$.

3. In this section we shall prove Theorem 2. Let R and R_1 be the elements in W and W_1 respectively such that $R(\Sigma^+)=-\Sigma^+$ and $R_1(\Theta)=-\Theta$. Then $R^2=R_1^2=1$. Let

$$V_\Lambda=\{T\in W\mid T\Lambda=\Lambda\}.$$

Then we see easily that $R_1V_\Lambda R=\{T\in W\mid T\Lambda'=R_1\Lambda\}$ and hence

$$(1) \qquad W^1_\Lambda = W^1 \cap R_1 V_\Lambda R\,.$$

On the other hand we have

$$(2) \qquad W^1 = R_1 W^1 R\,.$$

In fact, let $T\in W^1$ and $\alpha\in\Theta$. Then $(R_1TR)^{-1}\alpha=RT^{-1}R_1\alpha\in RT^{-1}(-\Theta)$ $\subset -R\Sigma^+=\Sigma^+$ and hence $(R_1\ TR)^{-1}\Theta\subset\Sigma^+$ which shows that R_1TR is in W^1. Thus $R_1W^1R\subset W^1$. But we have $R_1^2=R^2=1$ and hence we get $W^1\subset R_1W^1R$ and hence (2). From (1) and (2) we obtain:

$$(3) \qquad W^1_\Lambda = R_1(W^1 \cap V_\Lambda)R\,.$$

Let

$$(W^1\cap V_\Lambda)(q) = \{T\in W^1\cap V_\Lambda \,|\, n(T)=q\}\,.$$

For a subset Φ of Σ^+, Φ^c will denote the complement of Φ in Σ^+.

Lemma 1. *Let $T\in W^1$. Then $\Phi_{R_1TR}=R_1(\Psi\cap\Phi^c_T)$ and hence $n(R_1TR)$ $=N-n(T)$, where N denotes the number of roots in Ψ.*

Proof. We first remark that $R_1(\Psi)=\Psi$ and that, as R_1TR and T are in W^1, Φ_{R_1TR} and Φ_T are subsets of Ψ. Φ_{R_1TR} consists of all $\alpha\in\Psi$ such that $(R_1TR)^{-1}\alpha<0$. Now $(R_1TR)^{-1}\alpha=RT^{-1}R_1\alpha$ $(\alpha\in\Psi)$ is negative if and only if $T^{-1}R_1\alpha$ is positive. But $R_1\alpha\in\Psi$ and hence $T^{-1}R_1\alpha$ is positive if and only if $R_1\alpha\notin\Phi_T$ and this proves Lemma 1.

From (3) and Lemma 1 we get

$$(4) \qquad W^1_\Lambda(q) = R_1(W^1\cap V_\Lambda)(N-q)R\,.$$

Let now

$$(5) \qquad \Psi^0_\Lambda = \{\alpha\in\Psi \,|\, (\Lambda,\alpha)=0\}\,.$$

Then the number of roots in Ψ^0_Λ is $N-q_\Lambda$.

Lemma 2. *If $T\in W^1\cap V_\Lambda$, then $\Phi_T\subset\Psi^0_\Lambda$.*

Proof. Let $\alpha\in\Phi_T$. Then $\alpha\in\Psi$ and $T^{-1}\alpha<0$ and hence $(\Lambda, T^{-1}\alpha)=$ $(T\Lambda,\alpha)\leq 0$. Since $T\in V$, we have $T\Lambda=\Lambda$ and hence $(\Lambda,\alpha)\leq 0$. By our assumption on Λ, $(\Lambda,\alpha)\geq 0$ and therefore $(\Lambda,\alpha)=0$ and this shows that $\alpha\in\Psi^0_\Lambda$.

From Lemma 1 and 2 we see that, if $T\in W^1_\Lambda$, then $n(T)\geq q_\Lambda$ which proves the first part of Theorem 2.

Now we are going to show that there exists a unique elements $S\in W^1\cap V_\Lambda$ such that $\Phi_S=\Phi^0_\Lambda$. Then $T_0=R_1SR$ will be the unique element in $W^1_\Lambda(q_\Lambda)$ and $\Phi_{T_0}=\{R_1\alpha\,|\,\alpha\in\Psi, (\Lambda,\alpha)>0\}=\{\alpha\,|\,\alpha\in\Psi, (R_1\Lambda,\alpha)>0\}$.

First we remark that the uniqueness of T such that $\Phi_T=\Psi_\Lambda^0$ follows from a results of Kostant [2, Prop. 5.10]*): The mapping $T\to\Phi_T$ $(T\in W)$ defines an injection of W into the set of subsets of Σ^+.

Now let

$$\mathfrak{g}_u=\sqrt{-1}\mathfrak{m}+\mathfrak{k},$$

where $\mathfrak{g}=\mathfrak{m}+\mathfrak{k}$ is the Cartan decomposition of $\mathfrak{g}$ so that $\mathfrak{m}^C=\mathfrak{n}^++\mathfrak{n}^-$. Then $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}^C$. Let H_0 be the element in $\mathfrak{h}$ such that $\varphi(H, H_0)=\sqrt{-1}\Lambda(H)$ for all $H\in\mathfrak{h}$. Then $[H_0, X_\alpha]=\sqrt{-1}(\Lambda, \alpha)X_\alpha$ for all root α.

Let $\mathfrak{l}$ be the centralizer of H_0 in $\mathfrak{g}_u$. Then $\mathfrak{l}$ is the Lie algebra of a compact Lie group and $\mathfrak{l}^C$ is identified with the centralizer of H_0 in $\mathfrak{g}^C$. We see easily that

$$\mathfrak{l}^C=\mathfrak{h}^C+\sum_{\alpha\in\Xi}(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}),$$

where $\Xi=\{\alpha\in\Sigma^+\,|\,(\Lambda, \alpha)=0\}$ and $\mathfrak{g}_\alpha$ denotes the 1-dimensional eigenspace for the root α.

Let G_u be the adjoint group of $\mathfrak{g}_u$ and L the subgroup of G_u consisting of all $\sigma\in G_u$ such that $\sigma(H_0)=H_0$. Then the Lie algebra of L is $\mathfrak{l}$ and, L being the centralizer of a 1-parameter subgroup, L is connected. Now let $T\in V_\Lambda$.

We consider W as a group of linear transformations of $\mathfrak{h}$ by identifying roots α with elements H_α in $\mathfrak{h}$ such that $\sqrt{-1}\,\alpha(H)=\varphi(H, H_\alpha)$ for all $H\in\mathfrak{h}$. Then we know that for $T\in W$ there exists an element $t\in G_u$ such that $t(X)\in T(X)$ for all $X\in\mathfrak{h}$. Then $T\Lambda=\Lambda$ implies $t(H_0)=H_0$ and hence t belongs to L. Thus T belongs to the Weyl group of $\mathfrak{l}^C$. It follows then that V_Λ is the subgroup of W generated by S_α with $\alpha\in\Xi$, where S_α denotes the reflection with respect to the hyperplane $\alpha=0$.

Now let

$$\Omega=\{\alpha\in\Sigma^+\,|\,(\Lambda, \alpha)>0\}.$$

Then $\Xi\cup(-\Xi)\cup\Omega$ is a closed system, i.e. if α and β belong to this set of roots and $\alpha+\beta$ is also a root, then $\alpha+\beta$ belongs also to this set. Then

$$\begin{aligned}\mathfrak{u}&=\mathfrak{h}^C+\sum_{\alpha\in\Xi}(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha})+\sum_{\alpha\in\Omega}\mathfrak{g}_\alpha\\&=\mathfrak{l}^C+\sum_{\alpha\in\Omega}\mathfrak{g}_\alpha\end{aligned}$$

is a parabolic subalgebra and $\mathfrak{l}^C$ and $\sum_{\alpha\in\Omega}\mathfrak{g}_\alpha$ are the reductive part and the nilpotent part of $\mathfrak{u}$ respectively. Then by a theorem of Kostant [2, Prop. 5.13], every $T\in W$ is written uniquely

* The existence of $T\in W$ such that $\Phi_T=\Psi_\Lambda{}^0$ follows also the same proposition, but it is not easy to see that $T\in W_\Lambda$.

$$T = T_\Lambda T^\Lambda,\ T_\Lambda \in V_\Lambda,\ T^\Lambda \in V^\Lambda,$$

where $V^\Lambda = \{T \in W \mid \Phi_T \subset \Omega\}$.

Lemma 3. *Let $U \in W^1$ and let*

$$U = ST,\ S \in V_\Lambda,\ T \in V^\Lambda.$$

Then $\Phi_S = \Phi_U \cap \Psi_\Lambda^0$.

Proof. Let $\alpha \in \Omega$. Then $(\Lambda, \alpha) > 0$ and $(\Lambda, \alpha) = (S^{-1}\Lambda, S^{-1}\alpha) = (\Lambda, S^{-1}\alpha)$, because $S^{-1}\Lambda = \Lambda$. Then $S^{-1}\alpha$ is positive and belongs to Ω. Hence $\Phi_S \cap \Omega = \phi$.

Next let $\alpha \in \Xi \cap \Theta$. Since $U \in W^1$, we have $U^{-1}\alpha > 0$. Suppose $S^{-1}\alpha$ is negative. Then $-S^{-1}\alpha = T(-U^{-1}\alpha) > 0$ and $-U^{-1}\alpha > 0$ and hence $T(-U^{-1}\alpha)$ belongs to Φ_T. Since $T \in V^\Lambda$, Φ_T is contained in Ω and hence $-S^{-1}\alpha \in \Omega$. One the other hand, as $\alpha \in \Xi$ and $S \in V_\Lambda$, we have $(\Lambda, -S^{-1}\alpha) = (S\Lambda, -\alpha) = -(\Lambda, \alpha) = 0$ and this contradicts the fact $-S^{-1}\alpha \in \Omega$. Therefore $S^{-1}\alpha$ must be positive for all $\alpha \in \Xi \cap \Theta$ and this shows that $\Phi_S \cap \Xi \cap \Theta = \phi$.

Finally let $\alpha \in \Xi \cap \Psi$. Suppose $U^{-1}\alpha < 0$ and $S^{-1}\alpha > 0$. Then $S^{-1}\alpha = TU^{-1}\alpha$ and hence $S^{-1}\alpha \in \Xi \cap \Phi_T \subset \Xi \cap \Omega = \phi$ and this is a contradiction. Therefore if $U^{-1}\alpha < 0$, then $S^{-1}\alpha$ must be negative. Analogously we can show that if $U^{-1}\alpha > 0$, then $S^{-1}\alpha > 0$. These show that

$$\Phi_S \cap \Xi \cap \Psi = \Phi_U \cap \Xi \cap \Psi = \Phi_U \cap \Psi_\Lambda^0 .$$

On the other hand we have

$$\Sigma^+ = \Xi \cup \Omega = (\Xi \cap \Psi) \cup (\Xi \cap \Theta) \cup \Omega \text{ (disjoint)}$$

Therefore we get from what we have proved so far

$$\Phi_S = \Phi_U \cap \Psi_\Lambda^0 .$$

Lemma 4. *An element U of W belongs to $W^1 \cap V_\Lambda$ if and only if $\Phi_U \subset \Psi_\Lambda^0$.*

Proof. By Lemma 2, if $U \in W^1 \cap V_\Lambda$, Φ_U is contained Ψ_Λ^0. Conversely, let $\Phi_U \subset \Psi_\Lambda^0$ and let $U = S \cdot T$ as in Lemma 3. Then $\Phi_S = \Phi_U$ by Lemma 3. But the mapping $T \to \Phi_T$ is bijective and hence $U = S \in V_\Lambda$. But $\Phi_U \subset \Psi_\Lambda^0 \subset \Psi$ and hence $U \in W^1$. Thus $U \in W^1 \cap V_\Lambda$.

Now let $R^1 = R_1 R$. Then $R^1 \in W^1$ and $\Phi_{R^1} = \Psi$ (see Remark 1 in § 2). Let $R^1 = ST$ as in Lemma 3. Then S belongs to $W^1 \cap V_\Lambda$ and $\Phi_S = \Phi_{R^1} \cap \Psi_\Lambda^0 = \Psi_\Lambda^0$. Thus S is the unique element in $(W^1 \cap W)\,(N - q_\Lambda)$ and Theorem 2 is proved.

From Theorems 1′ and 2 we get the following theorem.

Theorem 2′. *The assumptions and the notation being as in Theorems 1′ and 2, the space of automorphic harmonic q_Λ-forms of type J_τ is isomorphic to the* *space of holomorphic automorphic forms of type* $J_{\tau_{T_0}}$, *where* T_0 *is the unique element in* $W_\Lambda^1(q_\Lambda)$. *We have*

$$\Phi_{T_0} = \{\alpha \in \Psi \mid (R_1\Lambda, \alpha) > 0\} .$$

The highest weight of τ_{T_0} *is*

$$-T_0 R\Lambda - \langle \Phi_{T_0} \rangle .$$

PART II

We retain the notation introduced in Part I*). Let $\mathfrak{g}=\mathfrak{m}+\mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$ and let $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_r\}$ be the bases of $\mathfrak{m}$ and $\mathfrak{k}$ respectively such that $\varphi(X_i, X_j)=\delta_{ij}$, $\varphi(Y_a, Y_b)=-\delta_{ab}$ $(i, j=1, \cdots, m;$ $a, b=1, \cdots, r)$. Then the Casimir operator C is the differential operator on G given by

$$C = \sum_{i=1}^{m} X_i^2 - \sum_{a=1}^{r} Y_a^2 .$$

We denote by $C^\infty(G, V)$ the complex vector space of all C^∞-functions on G with values in a finite dimensional complex vector space V.

1. Let τ be a representation of K in a complex vector space V and let Γ be a discrete subgroup of G. By an automorphic form of type $(\Gamma, \tau, \lambda_\tau)$ we mean a function $f \in C^\infty(G, V)$ satisfying the following three conditions (cf. [1]):

1) $f(gk)=\tau(k^{-1})f(g)$, $k \in K$, $g \in G$;
2) $f(\gamma g)=f(g)$, $\gamma \in \Gamma$, $g \in G$;
3) $Cf=\lambda_\tau f$, where λ_τ is a complex constant depending only on τ.

We denote by $A(\Gamma, \tau, \lambda_\tau)$ the vector space of all automorphic forms of type $(\Gamma, \tau, \lambda_\tau)$.

Proposition 1. *Assume* $\Gamma \backslash G$ *is compact. Then the dimension of the vector space* $A(\Gamma, \tau, \lambda_\tau)$ *is finite.*

Proof. ([1]). It follows from the condition 1) for automorphic forms that $Yf=-\tau(Y)f$ for $Y \in \mathfrak{k}$ and $f \in A(\Gamma, \tau, \lambda_\tau)$. Put

$$C' = \sum_{a=1}^{r} Y_a^2 .$$

Then $C'f=\tau(C')f$, where $\tau(C')=\sum_{a=1}^{r} \tau(Y_a)^2$. Let

$$L = C+2C' = \sum_{i=1}^{m} X_i^2 + \sum_{a=1}^{r} Y_a^2 .$$

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* In Part II we don't need to assume that G/K has a G-invariant complex structure.

Then L is a left invariant elliptic differential operator on G. For $f\in A(\Gamma, \tau, \lambda_\tau)$ we have

$$Lf = Mf,$$

where M denotes the endomorphism of V defined by

$$M = \lambda_\tau I + 2\tau(C').$$

Let $F(x)$ be a polynomial such that $F(M)=0$ and let $P=F(L)$. Then P is also a left invariant elliptic operator on G and we have $Pf=0$ for $f\in A(\Gamma, \tau, \lambda_\tau)$.

Now by the second condition on automorphic forms, we may consider f as a V-valued function on $\Gamma\backslash G$ and by the left invariance of P, we may consider P as an elliptic operator on $\Gamma\backslash G$. The manifold $\Gamma\backslash G$ being compact, the vector space of all V-valued functions on $\Gamma\backslash G$ satisfying the equation $Pf=0$ is finite dimensional and in particular $A(\Gamma, \tau, \lambda_\tau)$ is finite dimensional.

EXAMPLE. Let us assume that G/K is a bounded symmetric domain in $\boldsymbol{C}^N$ as in Part I and let J_τ be the canonical automorphic factor of type τ. Assume τ is irreducible and let Λ be the highest weight of τ. Then the space of all holomorphic automorphic forms of type J_τ on G/K is identified with the space of all automorphic forms of type $(\Gamma, \tau, \lambda_\tau)$ with

$$\lambda_\tau = (\Lambda, \Lambda+2\delta),$$

where

$$\delta = \sum_{\alpha>0} \alpha/2 = \langle\Sigma^+\rangle/2.$$

2. Let T be a unitary representation of G in a Hibert space H and let C_T be the Casimir operator of the representation T. C_T is a self-adjoint operator of H with a dense domain and if T is irreducible, there exists a complex number λ_T such that $C_T\varphi=\lambda_T\varphi$ for all φ in the domain of C_T.

Let T be irreducible and let T_K be the restriction of T onto K. Then T_K is a unitary representation of K and it is known that T_K decomposes into a countable sum of irreducible representations of K and each irreducible representation τ of K enters in T_K with finite multiplicity which we shall denote by $(T_K: \tau)$.

Let U be the unitary representation of G in the Hibert space $L_2(\Gamma\backslash G)$: $(U(g)f)(x)=f(xg)$, $x\in\Gamma\backslash G$, $g\in G$. We know that U decomposes into sum of a countable number of irreducible unitary representations in which each irreducible representation T enters with a finite multiplicity which we shall denote by $(U: T)$.

Note that we have $C_Uf=Cf$ for $f\in L_2(\Gamma\backslash G)\cap C^\infty(\Gamma\backslash G)$.

Theorem 3. *Assume that $\Gamma\backslash G$ is compact and τ is irreducible. Then*

$$\dim A(\Gamma, \tau, \lambda_\tau) = \sum_{T \in D_{\lambda_\tau}} (U: T)(T_K: \tau^*),$$

where τ^* *denotes the irreducible representation of* K *contragredient to* τ *and* D_{λ_τ} *denotes the set of irreducible representations* T *of* G *such that* $\lambda_T = \lambda_\tau$, λ_T *being the constant such that* $C_T\varphi = \lambda_T\varphi$ *for all* φ *in the domain of the Casimir operator* C_T.

From Theorem 2′ and Remark 1 in Part I, §2 and Example in Part II, §1, we obtain the following corollary.

Corollary. *The notation and the assumptions being as in Theorems* 2′ *and* 3, *the dimension of the space of automorphic harmonic* q_Λ*-forms of type* J_τ *is equal to*

$$\sum_{T \in D_{<\Lambda, \Lambda+2\delta>}} (U: T)(T_K: \tau'_{T_0}),$$

where τ'_{T_0} *denotes the irreducible representation of* K *with lowest weight*

$$\xi'_{T_0} = T_0\Lambda' + \langle\Phi_{T_0}\rangle,$$

where

$$\Phi_{T_0} = \{\alpha \in \Psi \mid (R_1\Lambda, \alpha) > 0\}.$$

3. Proof of Theorem 3. Let

$$L^2(\Gamma\backslash G) = \sum_{a=1}^{\infty} \oplus H_a$$

be the decomposition of $L^2(\Gamma\backslash G)$ into direct sum of irreducible invariant closed subspaces and let

$$U_a = U \mid H_a.$$

Let a be an index such that $C_{U_a}\varphi = \lambda_\tau\varphi$ for φ in the domain of C_{U_a} and let

$$m_a = ((U_a)_K: \tau^*).$$

Further let

$$H_a = \sum_{b=1}^{\infty} \oplus H_{a,b}$$

be the decomposition of H_a into direct sum of irreducible K-invariant subspaces. We may assume that for $b = 1, 2, \cdots, m$, the irreducible representation of K in $H_{a,b}$, is equivalent to τ^*.

Take a basis $\{v_1, \cdots, v_n\}$ of the representation space V of τ and let

$$\tau(k)v_\lambda = \sum_\mu \tau^\mu_\lambda(k)v_\mu, \qquad k \in K.$$

Then there exists a basis $\{f^1_{a,b}, \cdots, f^n_{a,b}\}$ of $H_{a,b}$ such that

$$(1) \qquad f^{\lambda}_{a,b}(xk) = \sum_{\mu} \tau^{\lambda}_{\mu}(k^{-1}) f^{\mu}_{a,b}(x)$$

for $k \in K$ and $x \in \Gamma \backslash G$. Define a V-valued function $f_{a,b}$ on $\Gamma \backslash G$ by putting

$$f_{a,b}(x) = \sum_{\lambda} f^{\lambda}_{a,b}(x) v_{\lambda} .$$

Then

$$f_{a,b}(xk) = \tau(k^{-1}) f_{a,b}(x) .$$

Put

$$\tilde{f}_{a,b} = f_{a,b} \circ \pi ,$$

where π denotes the projection of G onto $\Gamma \backslash G$. Then $\tilde{f}_{a,b}$ satisfies the conditions 1) and 2) of automorphic forms of type $(\Gamma, \tau, \lambda_\tau)$. If $\{g^1_{a,b}, \cdots, g^n_{a,b}\}$ is another basis of $H_{a,b}$ satisfying the condition (1), then there exists a complex number α such that $\alpha\, g^{\lambda}_{a,b} = f^{\lambda}_{a,b}$ for $\lambda = 1, \cdots, n$ by Schur's Lemma. Therefore the function $\tilde{f}_{a,b}$ is well defined up to constant multiple.

Let us prove that $\tilde{f}_{a,b}$ is differentiable and satisfies the equation $C\tilde{f}_{a,b} = \lambda_\tau \tilde{f}_{a,b}$. To show this it is sufficient to show that every function $\varphi \in H_{a,b}$ is differentiable. In fact, φ is then in the domain of the operator $C_U = C$ and $C_U \varphi = C_{U_a} \varphi = \lambda_\tau \varphi$ (Remark that C is a differential operator on G left invariant by G and hence we may consider C as a differential operator on $\Gamma \backslash G$. To show the differentiability of $\varphi \in H_{a,b}$, we remark first that for any $h \in C^\infty(\Gamma \backslash G)$ and $\psi \in H_a$ we have

$$(2) \qquad (Ch, \psi) = (h, \lambda_\tau \psi) .$$

In fact, let ψ be an element in the domain of the operator C_{U_a}. Then $(Ch, \psi) = (C_U h, \psi) = (h, C_{U_a} \psi) = (h, \lambda_\tau \psi)$. The elements ψ being dense in H_a, the equality (2) holds for any $\psi \in H_a$. Now let $\varphi \in H_{a,b}$. Then for any $Y \in \mathfrak{k}$,

$$\lim_{t \to 0} \frac{1}{t} (U_a(\exp tY)\varphi - \varphi) = U'_a(Y)\varphi$$

exists. In fact, $U'_a(Y)\varphi = -\tau^*(Y)\varphi$. In particular

$$U'(C')\varphi = \tau^*(C')\varphi ,$$

where, as in the proof of Proposition 1 in §1, we put $C' = \sum_a Y_a^2$. Let

$$L = C + 2C' .$$

Then for any $\varphi \in H_{a,b}$ and $h \in C^\infty(\Gamma \backslash G)$ we get from (2):

$$\begin{aligned}(Lh, \varphi) &= (Ch, \varphi) + 2(C'h, \varphi) = (h, \lambda_\tau \varphi) + (h, 2\tau^*(C')\varphi) \\ &= (h, B\varphi) ,\end{aligned}$$

where $B=\lambda_\tau I+2\tau^*(C')$ is an endomorphism of the finite dimensional vector space $H_{a,b}$. By induction we get $(L^k h, \varphi)=(h, B^k\varphi)$ for $k=1, 2, \cdots,$. Let $F(x)$ be a polynomial such that $F(B)=0$ and let $P=F(L)$. Then $(Ph, \varphi)=0$. This shows that the distribution D_φ defined by $D_\varphi(h)=(h, \varphi)$ satisfies the elliptic equation $PD_\varphi=0$. Then φ is differentiable and in fact $P\varphi=0$.

Thus we have shown that for each index a such that $U_a \in D_{\lambda_\tau}$, we get m_a functions $\tilde{f}_{a,b}(b=1, \cdots, m_a)$ belonging to $A(\Gamma, \tau, \lambda_\tau)$. If c is another index such that $U_c \in D_{\lambda_\tau}$, then we get also functions $\tilde{f}_{c,d}$ $(d=1, 2, \cdots, m_c; m_c=((U_c)_K\colon \tau^*))$ belonging to $A(\Gamma, \tau, \lambda_\tau)$ and it is easy to see that $\tilde{f}_{a,b}$ and $\tilde{f}_{c,d}$ are linearly independent. By Proposition 1, the dimension $A(\Gamma, \tau, \lambda_\tau)$ is finite. It follows then that the number of the irreducible unitary representations T of G belonging to D_{λ_τ} such that $(U\colon T)\neq 0$ and $(T\colon \tau^*)\neq 0$ is finite. We may therefore assume that $U_1, U_2, \cdots, U_t$ are these unitary representations. For each a, $1\leq a\leq t$. We get functions $\tilde{f}_{a,b}$ $(1\leq b\leq ((U_a)_K\colon \tau^*))$ in $A(\Gamma, \tau, \lambda_\tau)$ and these functions are linearly independent. The number of these functions equals $\sum_{a=1}^{t}((U_a)_K\colon \tau^*)$ which is equal to $\sum_{T\in D_T}(U\colon T)(T_K\colon \tau^*)$. Thus we get

$$\dim A(\Gamma, \tau, \lambda_\tau)\geq \sum_{T\in D_T}(U\colon T)(T_K\colon \tau)\,.$$

Now let $\tilde{f}\in A(\Gamma, \tau, \lambda_\tau)$ and let

$$\tilde{f}(g)=\sum_\lambda \tilde{f}^\lambda(g)v_\lambda\,, \qquad g\in G\,.$$

Then $\tilde{f}^\lambda$ is a differentiable function on G such that $\tilde{f}^\lambda(\gamma g)=\tilde{f}^\lambda(g)$ for all $\gamma\in\Gamma$. Then there exists a differentiable function f^λ on $\Gamma\backslash G$ such that $f^\lambda=\tilde{f}^\lambda\circ\pi$. Then we have

$$f^\lambda(xk)=\sum_\mu \tau^\lambda_\mu(k^{-1})f^\mu(x)$$

for all $k\in K$ and $x\in\Gamma\backslash G$. Moreover

$$Cf^\lambda=\lambda_\tau f^\lambda\,.$$

Let P_a be the projection of $L^2(\Gamma\backslash G)$ onto H_a. If h is differentiable, we have $P_aXh=\lim_{t\to 0}\frac{1}{t}P_a(U(\exp tX)h-h)=\lim_{t\to 0}\frac{1}{t}(U_a(\exp tX)P_ah-P_ah)=U_a'(X)P_ah.$ It follows that P_ah is in the domain of $C_{U_a}=\sum_i U_a'(X_i)^2-\sum_b U_b'(Y_b)^2$ and $P_aCh=C_{U_a}P_ah$. Thus we get:

$$C_{U_a}P_af^\lambda=\lambda_\tau P_af^\lambda\,.$$

It follows that, if $U_a\notin D_{\lambda_\tau}$, then $P_af^\lambda=0$. Therefore we have:

$$f^\lambda=P_1f^\lambda+P_2f^\lambda+\cdots+P_tf^\lambda. \tag{3}$$

Let $a\in[1, t]$ and assume $P_a f^\lambda \neq 0$ for some λ. Let F be the linear subspace of H_a spanned by $\{P_a f^1, P_a f^2, \cdots, P_a f^n\}$. Then $F\neq(0)$ and F is K-invariant. In fact

$$U_a(k)P_a f^\mu = P_a U(k) f^\mu = P_a \sum_\nu \tau^\mu_\nu(k^{-1}) f^\nu$$
$$= \sum_\nu \tau^\mu_\nu(k^{\mu-1}) P_a f^\nu .$$

Let $\{\xi^1, \cdots, \xi^n\}$ be the basis of V^* dual to the basis $\{v_1, \cdots, v_n\}$ of V. The linear map of V^* onto F defined by $\xi^\nu \rightarrow P_a f^\nu$ is a K-homorphism and in fact a K-isomorphism, because V^* is irreducible. It follows that $P_a f^1, \cdots, P_a f_u$ are linearly independent and F is contained in $\sum_{b=1}^{m_a} \oplus H_{a,b}$ because of the orthogonality relation. Then we can write:

$$P_a f^\lambda = \sum_{b=1}^{m_a} \sum_{\mu=1}^{n} \alpha(b)^\lambda_\mu f^\mu_{a,b}$$

Then

$$(P_a f^\lambda)(xk) = \sum_{b,\mu} \alpha(b)^\lambda_\mu f^\mu_{a,b}(xk)$$
$$= \sum_{b,\mu,\tau} \alpha(b)^\lambda_\mu \tau^\mu_\nu(k^{-1}) f^\nu_{a,b}(x) .$$

On the other hand

$$(P_a f^\lambda)(xk) = \sum_\mu \tau^\lambda_\mu(k^{-1})(P_a f^\mu)(x)$$
$$= \sum_\mu \tau^\lambda_\mu(k^{-1}) \sum_{b,\nu} \alpha(b)^\mu f^\nu_{a,b}(x) .$$

Therefore we get:

$$\sum_\mu \alpha(b)^\lambda_\mu \tau^\mu_\nu(k^{-1}) = \sum_\mu \tau^\lambda_\mu(k^{-1})\alpha(b)^\mu_\nu \qquad (\lambda,\ \nu=1, \cdots, n) .$$

By Schur's Lemma we have $\alpha(b)^\lambda_\mu = \alpha_{a,b}\delta^\lambda_\mu$ where $\alpha_{a,b}$ is a constant depending on a and b. Thus

$$P_a f^\lambda = \sum_{b=1}^{m_a} \alpha_{a,b} f^\lambda_{a,b} ,$$

whence $f^\lambda = \sum_a P_a f^\lambda = \sum_{a,b} \alpha_{a,b} f^\lambda_{a,b}$. Therefore $\sum_\lambda f^\lambda v_\lambda = \sum_{a,b} \alpha_{a,b} f_{a,b}$. It follows then $\tilde{f}$ is a linear combination of $\tilde{f}_{a,b}$'s and therefore $\{\tilde{f}_{a,b}\}$ form a basis of the vector space $A(\Gamma, ,\tau\ \lambda_t)$ and the theroem is proved.

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