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ON ISOTROPY ALGEBRAS OF A LIE ALGEBRA OF VECTOR FIELDS WHICH SATISFIES A CERTAIN CONVERGENCE CONDITION

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Let M be a C^∞-manifold and let $L(M)$ be the Lie algebra of all C^∞-vector fields on M. For a Lie subalgebra L of $L(M)$, Matsuda [3] gave a sufficient condition for its integrability. Precisely speaking, for a Lie subalgebra L which satisfies the convergence condition (C) stated below, Matsuda proved that through every point of M there passes a maximal integral manifold of L.

For two elements u and v of $L(M)$, defining $(\operatorname{ad} v)^k u$ inductively as $[v, (\operatorname{ad} v)^{k-1}u]$, we put

$$g_t(u, v) = \sum_{j=0}^{\infty} (-1)^j \frac{t^j}{j!} (\operatorname{ad} v)^j u .$$

Let $c(u, v; x)$ be the radius of convergence of $g_t(u, v)$ at x of M.

(C) *For any pair of u and v in L and any compact set K in M, there exists a positive number $c(u, v; K)$ such that*

(i) *we have $c(u, v; x) \geqslant c(u, v; K)$ at every $x \in K$, and*

(ii) *$g_t(u, v)$ is continuously differentiable with respect to (t, x) term by term at every (t, x) which satisfies $|t| < c(u, v; K)$ and $x \in K^i$, the interior of K.*

In this paper we say a Lie subalgebra L satisfies the condition (C_k), if '*continously differentiable*' in (ii) of Matsuda's condition (C) can be replaced by '*$(k+1)$-th continuously differentiable*'. Here k is a non-negative integer.

For any Lie subalgebra L of $L(M)$ and a point x of M, we shall define $\mathfrak{g}_k^l(x)$, the *(l, k)-isotropy algebra of L at x*, as follows: Let L_x^k denote the subalgebra of L consisting of vector fields whose coefficients vanish at x with all their derivatives through order k (in one and hence all coordinate systems). Then L_x^k is an ideal in L_x^l for $k \geqq l \geqq 0$. We shall denote the factor algebra L_x^l/L_x^k by $\mathfrak{g}_k^l(x)$ for $k \geqq l \geqq 0$. In particular $\mathfrak{g}_1^0(x)$ is the linear isotropy algebra of L at x.

Theorem 1. *Let M be a connected, paracompact manifold and let L be a Lie subalgebra satisfying the condition (C_k) with $k>0$. Then the (t, s)-isotropy algebras*

$\mathfrak{g}_s^t(x)$ and $\mathfrak{g}_s^t(y)$ of L are isomorphic for $k \geqq s > t \geqq 0$, if x and y lie on the same orbit under L. In particular the linear isotropy algebra of L is isomorphic each other on every orbit under L.

Here an orbit under L is the set of all points of M that can be joined each other by finite number of integral curves of L.

Every finite dimensional Lie subalgebra L satisfies the condition (C_k) for all k (see [3]). Also if we are in the real analytic category, then every Lie subalgebra L of $L(M)$ satisfies the condition (C_k) for all k(see [3]).

A subalgebra L is called transitive if $L(x)$ equals the tangent space $T_x(M)$ at every x of M where $L(x)$ is the subspace of $T_x(M)$ defined by $L(x)=\{u(x); u \in L\}$.

Theorem 2. *Under the same assumptions as Theorem 1, if moreover L is transitive, then there exists a G-structure of order l with $1 \leqq l \leqq k$ such that the Lie algebra of G is isomorphic to $\mathfrak{g}_l^0(x)$ for every $x \in M$.*

A G-structure of order l is by definition a reduction of the bundle $F^l(M)$ of l frames of M to the group G (see [4]).

The conclusion of Theorem 2 was obtained by Singer and Sternberg ([6], p. 39) under the assumption that L (Lie algebra sheaf in their case) is invariant by a local one parameter transformation group generated by any element of L. Theorem 2 will be proved through the proof of Theorem 1.

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Proof of Theorem 1. For simplicity, we shall prove the case $t=0$ and $s=1$. The proof for the general case proceeds similarly, if we replace M by $F^t(M)$ and $F(M)$ *by* $F^{s-t}(F^t(M))$ respectively in the following argument. Let $N(x)$ be the orbit under L through $x \in M$. By the result of Matsuda, $N(x)$ is a maximal connected integral manifold of L through x. We shall denote by $F(M)$ the bundle of linear frames of M with π as projection. Since $N(x)$ is a submanifold of M, $\pi^{-1}(N(x))$ is a submanifold of $F(M)$. Each vector field u on M induces a vector field $F(u)$ on $F(M)$ as follows: Let $\phi_t(u)$ be a local one parameter transformation group of M generated by u and let $F(\phi_t(u))$ be a local one parameter transformation group of $F(M)$ defined by

$$F(\phi_t(u))\cdot w = \phi_t(u)_{*z}\cdot w \qquad w \in F(M) \quad \text{and} \quad z = \pi(w)\,.$$

We shall denote its infinitesimal generator by $F(u)$. The mapping F: $u \to F(u)$ of $L(M)$ into $L(F(M))$ is a Lie algebra homomorphism and has following properties:

(i) $\pi_* F(u) = u$

(ii) $R_{a*}F(u) = F(u)$ for any $a \in GL(m, R)$, where R_a is the right translation of $F(M)$ by an element a and m is dim M. We put $F(L)=\{F(u)\colon u \in L\}$, which is a Lie algebra. For $u \in L$ a vector field $F(u)$ on $F(M)$ is, as is clear from the definition, tangent to the submanifold $\pi^{-1}(N(x))$ at all points and so defines a vector field on $\pi^{-1}(N(x))$. The set of vector fields on $\pi^{-1}(N(x))$ so obtained is a Lie algebra which we shall denote by $\overline{F(L)}$. If L satisfies the condition (C_1) on M, $F(L)$ satisfies the condition (C_0) on $F(M)$ and so $\overline{F(L)}$ satisfies the condition (C_0) on $\pi^{-1}(N(x))$. Let p be a point of $F(M)$ with $\pi(p)=x$. We denote the orbit under $\overline{F(L)}$ through p by $W(p)$. Then again by the result of Matsuda, $W(p)$ is a maximal integral manifold of $\overline{F(L)}$. We have clearly $\pi(W(p))=N(x)$. Put $n=\dim N(x)$ and $r=\dim W(p)-n$.

(1) The intersection $W(p)\cap\pi^{-1}(x)$ is a r-dimensional submanifold of three manifolds $\pi^{-1}(N(x))$, $\pi^{-1}(x)$ and $W(p)$. It has at most countably many connected components.

We shall prove this statement (1). For $w\in\pi^{-1}(N(x))$ the differential π_{*w} at w of π maps $\overline{F(L)}(w)$ onto the tangent space $T_{\pi(w)}(N(x))$ at $\pi(w)$, where $\overline{F(L)}\,(w)=\{\bar{u}(w)\colon \bar{u}\in\overline{F(L)}\}$, since $\pi_*F(u)=u$ and $L(z)=T_z(N(x))$. Hence we obtain $T_w(\pi^{-1}(N(x)))=T_w(W(p))+T_w(\pi^{-1}(x))$ for any $w\in W(p)\cap\pi^{-1}(x)$, i.e., two submanifolds $W(p)$ and $\pi^{-1}(x)$ of $\pi^{-1}(N(x))$ are transverse at their intersection. From this fact the first statement follows (see [5], p. 30). To prove the second part of (1), we first show that $\overline{F(L)}$ defines an involutive distribution in the sense of Chevalley [1] on $\pi^{-1}(N(x))$. For this, it suffices to show that the dimension of $\overline{F(L)}(w)$ is constant on $\pi^{-1}(N(x))$. The orbit $W(p)$ is an integral manifold of $\overline{F(L)}$ and so $\dim\overline{F(L)}(w)$ is constant on $W(p)$. Since $\pi(W(p))=N(x)$, and since each vector field of $F(L)$ is invariant by the action of $GL(m, \mathrm{R})$, we have $\dim\overline{F(L)}(w)=$*constant* on $\pi^{-1}(N(x))$. Hence each point $w\in\pi^{-1}(x)$ has a neighbourhood U in $\pi^{-1}(N(x))$ with coordinates $(x_1, \cdots, x_{n+m^2})$ such that the slices given by $x_{n+r+1}=$const., $\cdots$, $x_{n+m^2}=$const. are integral manifolds of $\overline{F(L)}$ ([1], p. 89, Theorem 1). Since two submanifolds $W(p)$ and $\pi^{-1}(x)$ of $\pi^{-1}(N(x))$ are transverse at their intersection, we can change this coordinates such that, in addition to the above condition, the set $V=\{q\in U\colon x_1(q)=\cdots=x_n(q)=0\}$ together with the restriction of $(x_{n+1}, \cdots, x_{n+m^2})$ to V form a local chart on $\pi^{-1}(x)$ containing w (compare the proof in [5], p. 30, Lemma 6). Hence $W(p)\cap V$ is the union of certain sets of planes represented by $x_1=0, \cdots, x_n=0, x_{n+r+1}=$const., $\cdots$, $x_{n+m^2}=$const., each of which is an open set of $W(p)\cap\pi^{-1}(x)$. The intersection $W(p)\cap U$ is the union of certain sets of slices given by $x_{n+r+1}=$const., $\cdots$, $x_{n+m^2}=$const. The paracompactness of M implies that the connected submanifold $N(x)$ of M and hence $\pi^{-1}(N(x))$ satisfies the second axiom of countability and so does the connected submanifold $W(p)$ of $\pi^{-1}(N(x))$. Then it follows that the intersection $W(p)\cap U$ is the union of at most countably many of slices and so the neighbourhood V of w in $\pi^{-1}(x)$ can meet at most countably many connected components of $W(p)\cap\pi^{-1}(x)$. Thus we have shown that each $w\in\pi^{-1}(x)$ has a neighbourhood V in $\pi^{-1}(x)$ which meets at most countably many connected components of $W(p)\cap\pi^{-1}(x)$. Since $\pi^{-1}(x)$ can be covered with countably many of these open sets, we have the second part of (1).

Each element of $F(L_x^0)=\{F(u)\colon u\in L_x^0\}$ is tangent to submanifold $\pi^{-1}(x)$ at all points and so defines a vector field on $\pi^{-1}(x)$. The vector fields on $\pi^{-1}(x)$ so obtained is a Lie algebra which we shall denote by $\overline{F(L_x^0)}$. Then the orbit $W_0(w)$ under $\overline{F(L_x^0)}$ through w is the connected component of $W(p)\cap\pi^{-1}(x)$ containing w and so a maximal integral manifold of $\overline{F(L_x^0)}$ through w. Since $\overline{F(L_x^0)}$ is invariant by the action of $GL(m, \mathrm{R})$, the translations R_a ($a\in GL(m, \mathrm{R})$) permute among themselves the maximal integral manifolds of $\overline{F(L_x^0)}$. Thus for $w=p\cdot a$ with $a\in GL(m, \mathrm{R})$ we have $W_0(w)=W_0(p)\cdot a$ and so $W(p)\cap\pi^{-1}(x)$ can be written as the disjoint union

$$W(p)\cap\pi^{-1}(x)=\bigcup_{a\in A}W_0(p)\cdot a\,. \tag{2}$$

By (1), the index set A is at most countable. Next consider the diffeomorphism $\gamma\colon p\cdot a\to a$ of $\pi^{-1}(x)$ onto $GL(m, \mathrm{R})$. We put $\gamma(W(p)\cap\pi^{-1}(x))=G(x, p)$. By the definitions of γ and $G(x, p)$, we have $G(x, p)=\{a\in GL(m, \mathrm{R})\colon p\cdot a\in W(p)\}$. Since $W(p)$ is the set of all points that can be joined to p by finite number of integral curves of $\overline{F(L)}$ and since each vector field of $\overline{F(L)}$ is invariant by the action of $GL(m, \mathrm{R})$, it follows that $G(x, p)$ is a group. If we introduce the differential structure on $G(x, p)$ by γ, $G(x, p)$ is also a submanifold of $GL(m, \mathrm{R})$.

(3) $G(x, p)$ is a Lie subgroup of $GL(m, \mathrm{R})$ and the Lie algebra of $G(x, p)$ is isomorphic to $\mathfrak{g}_1^0(x)$.

To prove this statement, we put $\mathfrak{g}(x, p)=\{\gamma_*\bar{u}\colon \bar{u}\in\overline{F(L_x^0)}\}$. If we regard $\mathfrak{gl}(m, \mathrm{R})$ as the Lie algebra of right invariant vector fields on $GL(m, \mathrm{R})$, $\mathfrak{g}(x, p)$ is a subalgebra of $\mathfrak{gl}(m, \mathrm{R})$. Let $G_0(x, p)$ be the connected Lie subgroup of $GL(m, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}(x, p)$. Since $G_0(x, p)$ is the maximal integral manifold of $\mathfrak{g}(x, p)$ containing the identity element of $GL(m, \mathrm{R})$, we have $G_0(x, p)=\gamma(W_0(p))$. Corresponding to (2), the submanifold $G(x, p)$ of $GL(m, \mathrm{R})$ is the disjoint union of integral manifolds $G_0(x, p)\cdot a$ ($a\in A$) of $\mathfrak{g}(x, p)$

$$G(x, p)=\bigcup_{a\in A}G_0(x, p)\cdot a\,.$$

Since A is countable, $G(x, p)$ satisfies the second axiom of countability and so the mapping $(a, b)\to ab^{-1}$ of $G(x, p)\times G(x, p)\to G(x, p)$ is differentiable (see [1], p. 95, Proposition 1. To prove this fact the connectedness of $G(x, p)$ is not needed. See also [2], p. 10, Proposition 1.3). Hence $G(x, p)$ is a Lie subgroup of $GL(m, R)$. To prove the second part of (3), let $\bar{F}$ denote the mapping of $L_x^0\to\overline{F(L_x^0)}$ induced by $F\colon L(M)\to L(F(M))$. The mapping $\bar{F}$ is a Lie algebra homomorphism of L_x^0 onto $\overline{F(L_x^0)}$ and the kernel of $\bar{F}$ is precisely

L_x^1. Thus we have

$$\mathfrak{g}_1^0(x) \simeq \overline{F(L_x^0)} \simeq \mathfrak{g}(x, p)\,.$$

Now we shall conclude the proof of Theorem 1. For any $y \in N(x)$, take $q \in W(p)$ with $\pi(q)=y$. Then $W(p)$ is also a maximal integral manifold of $\overline{F(L)}$ through q. If we put $G(y, q)=\{a \in GL(m, R): q \cdot a \in W(p)\}$, then by the same argument as (3), $G(y, q)$ is a Lie subgroup of $GL(m, R)$ having at most countably many connected components and whose Lie algebra is isomorphic to $\mathfrak{g}_1^0(y)$. Since a differential system $\overline{F(L)}$ on $\pi^{-1}(N(x))$ is invariant by the action of $GL(m, R)$, the translations R_a $(a \in GL(m, R))$ permute among themselves the maximal integral manifolds of $\overline{F(L)}$. It follows that two Lie subgroups $G(x, p)$ and $G(y, q)$ of $GL(m, R)$ coincide as set. Since $G(x, p)$ and $G(y, q)$ satisfy the second axiom of countability, they coincide as Lie subgroups (see [2], p. 40) and so do their Lie algebras. Hence we have $\mathfrak{g}_1^0(x) \simeq \mathfrak{g}_1^0(y)$. This completes the proof.

REMARKS 1. If L is transitive, the submanifold $W(p)$ of $F(M)$ and the group $G(x, p)$ in the preceding proof form the required G-structure of order 1 in Theorem 2. The higher order case can be proved, if we put $t=0$ and $s=l$ in the above proof of Theorem 1.

2. If we restrict vector fields of L to an orbit $N(x)$, we get a transitive Lie algebra $\bar{L}$ of vector fields on $N(x)$. In general the (t, s)-isotropy algebras $\mathfrak{g}_s^t(y)$ of L are different from the (t, s)-isotropy algebras $\bar{\mathfrak{g}}_s^t(y)$ of $\bar{L}$ for $y \in N(x)$. For example let L be the Lie algebra generated by a vector field $-y\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}$ on $\boldsymbol{R}^2$. Then the origin o is an orbit under L and the linear isotropy algebra $\mathfrak{g}_1^0(o)$ of L at o is not zero, while $\bar{\mathfrak{g}}_1^0(o)$ is zero. Thus in order to prove Theorem 1 it is not sufficient to prove only the transitive case.

3. Let us give an example L with is not invariant under the automorphism $\phi_t(u)_*$ of $L(M)$ generated by any non-zero element u of L. Let M be $\boldsymbol{R}^1$ with a coordinate (x) and let L be the Lie algebra consisting of vector fields of the form $x^2 f(x)\dfrac{\partial}{\partial x}$, where $f(x)$ is a polynomial function on $\boldsymbol{R}^1$. Then this L has the desired property. In fact if we put $\phi_t(u)_* x^2 f(x)\dfrac{\partial}{\partial x}=g(t, x)\dfrac{\partial}{\partial x}$, then $g(t, x)$ is a meromorphic function of x and t.

4. There exists a Lie algebra L to which our theorems cannot be applied. Let M be $\boldsymbol{R}^1$ with a coordinate (x). Take a C^∞-function $f(x)$ on $\boldsymbol{R}^1$ with a compact support which is not identically zero. Let L be the Lie algebra generated by $\dfrac{\partial}{\partial x}$ and $f(x)\dfrac{\partial}{\partial x}$. Then L is transitive and so L has only one orbit M. We have $\mathfrak{g}_k^0(x)=\{0\}$ for $x \notin \operatorname{supp} f$ and all k, while $\mathfrak{g}_k^0(x) \neq \{0\}$ for x with $\dfrac{d}{dx}f(x) \neq 0$ and all k.

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