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A NECESSARY AND SUFFICIENT CONDITION FOR A KERNEL TO BE A WEAK POTENTIAL KERNEL OF A RECURRENT MARKOV CHAIN

Yoichi ŌSHIMA

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1. Introduction

Let P be an irreducible recurrent transition probability on a denumerable space S with invariant measure α. Let c be an arbitrary (but fixed) state of S. Then from the work of Kondō [3] and Orey [8], there exist the class of *weak potential kernels* $A(x, y)$ defined by the property that, for every null charge f, Af is bounded and satisfies the equation

$$(I-P)Af=f. \tag{1.1}$$

Moreover Af is represented by

$$Af={}^cGf+1(f), \tag{1.2}$$

where f is a null charge, $1(\cdot)$ is an arbitrary linear functional on the space of null charges and ${}^cG$ is defined as follow;

$${}^cP(x, y) = \begin{cases} P(x, y) & x\neq c,\ y\neq c \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

$${}^cG(x, y) = \begin{cases} \sum_{n=0}^{\infty} {}^cP^n(x, y) & x\neq c,\ y\neq c \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Moreover A satisfies the following maximum principle [4], [5]:
(RSCM)[1] If m is a real number and f is a null charge then the relation that

$$m\geqq Af \qquad \text{on the set } \{f>0\} \tag{1.5}$$

implies that

$$m-f^{-}\geqq Af \qquad \text{everywhere,} \tag{1.6}$$

1) This is the abbreviation of "reinforced semi-complete maximum principle"; this maximum principle corresponds to the semi-complete M.P. as well as the reinforced M.P. (of Meyer) corresponds to the complete M.P.

where $f^{-}=(-f)\vee 0$.

In the present paper we are concerned with the following construction problem. Given a positive measure α and a (not necessarily positive) kernel A satisfying (RSCM), does there exist an irreducible recurrent transition probability which has α as its invariant measure, and A as its weak potential kernel? This is not true in general[2], but as Kondō [4] has proved, it is true if α is a finite measure. In section 2 we shall introduce another necessary condition for the weak potential kernel A (referred to as condition (*)). Then we shall prove (theorem 3.1) that, if the pair (A, α) satisfies maximum principle (RSCM) and condition (*), A is a weak potential kernel of a (unique) recurrent Markov chain with α as its invariant measure.

I should like to express my hearty gratitude to T. Watanabe for his kind advices.

2. Some potential theory for a kernel A satisfying (RSCM)

Let α be a strictly positive measure and A, a kernel on S. A function f on S is said to be a *null charge with respect to* α if $\sum\alpha(x)|f(x)|<\infty$ and $\sum\alpha(x)f(x)=0$. Let N be the space of null charges vanishing outside a finite subset of S. We assume that *the kernel A satisfies condition* (RSCM) *for $f\in N$*. Fix an arbitrary state c and define

$$(2.1)\qquad {}^{c}G(x, y) = A(x, y)-A(c, y)-(A(x, c)-A(c, c))\frac{\alpha(y)}{\alpha(c)}.$$

If A is a weak potential kernel then (2.1) is clearly satisfied by taking f in equation (1.2) as

$$(2.2)\qquad f(x) = \begin{cases} \dfrac{\alpha(y)}{\alpha(c)} & x=c \\ -1 & x=y \\ 0 & \text{otherwise,} \end{cases}$$

and calcurating $Af(x)-Af(c)$.

From definition (2.1) ${}^{c}G(c, x)={}^{c}G(x, c)=0$ for every $x\in S$.

Lemma 2.1 *For arbitrary elements x, y in S which are different from c*

$$I(x, y)\leqq {}^{c}G(x, y)\leqq {}^{c}G(y, y).$$

Proof. By taking f as (2.2) we have

$$Af(c) = A(c, c)\frac{\alpha(y)}{\alpha(c)}-A(c, y)$$

2) A counter example was given by Kondō and T. Watanabe.

$$Af(y) = A(y, c)\frac{\alpha(y)}{\alpha(c)} - A(y, y).$$

Hence, if we write $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, by (RSCM)

$$A(y, c)\frac{\alpha(y)}{\alpha(c)} - A(y, y) + f^+(x) \leqq Af(x) \leqq A(c, c)\frac{\alpha(y)}{\alpha(c)} - A(c, y) - f^-(x),$$

so that

$$A(y, c)\frac{\alpha(y)}{\alpha(c)} - A(y, y) \leqq A(x, c)\frac{\alpha(y)}{\alpha(c)} - A(x, y) \leqq A(c, c)\frac{\alpha(y)}{\alpha(c)} - A(c, y) - I(x, y), \tag{2.3}$$

which proves the lemma.

Corollary. *For every $x \in S$ there exists a constant C such that*

$$^cG(x, y) \leqq C \cdot \alpha(y) \qquad \textit{for every } y \in S. \tag{2.4}$$

Proof. Exchanging c and x in the second inequality in (2.3), it follows that

$$^cG(x, y) = A(x, y) - A(c, y) - (A(x, c) - A(c, c))\frac{\alpha(y)}{\alpha(c)}$$
$$\leqq \left(-\frac{A(x, c)}{\alpha(c)} - \frac{A(c, x)}{\alpha(x)} + \frac{A(c, c)}{\alpha(c)} + \frac{A(x, x)}{\alpha(x)}\right)\alpha(y).$$

Let ^cS be the set $S - \{c\}$, and $^c\boldsymbol{M}$ be the space of all functions on ^cS vanishing outside a finite subset of ^cS. Let $^c\boldsymbol{M}^+$ be the space of all non-negative functions in $^c\boldsymbol{M}$.

Theorem 2.1. *The kernel ^cG satisfies the reinforced maximum principle* [7]: (RM) *If a is a non-negative constant and if ^cf and ^cg are two elements of $^c\boldsymbol{M}^+$, then the relation that*

$$a + {}^cG^cf - {}^cf \geqq {}^cG^cg \qquad \textit{on the set } \{^cg > 0\} \textit{ implies that} \tag{2.5}$$

$$a + {}^cG^cf - {}^cf \geqq {}^cG^cg \qquad \textit{everywhere on } {}^cS. \tag{2.6}$$

Proof. Let f be the function on S such that $f \in \boldsymbol{N}$ and $f|_{^cS} = {}^cf$. Such f is obviously unique. The function $g \in \boldsymbol{N}$ is defined similarly. Then inequality (2.5) implies that

$$a + A(g - f)(c) \geqq A(g - f) \qquad \text{on the set } \{g - f > 0\}.$$

For, since ^cf and ^cg are non-negative, the set $\{g - f > 0\}$ is contained in the union of c and $\{^cg > 0\}$. Hence by (RSCM)

$$a+A(g-f)(c)-(g-f)^{-}\geqq A(g-f) \qquad \text{everywhere.}$$

Since the function $(g-f)^{-}$ is equal to ${}^cf$ on ${}^cS\cap\{g=0\}$, the above inequality, combined with (2.5), proves the theorem.

A non-negative function ${}^ch$ on ${}^cS$ is said to be *quasi-excessive*[3] if, for every ${}^cg\in{}^c\boldsymbol{M}$, the inequality

$${}^ch\geqq{}^cG^cg \qquad \text{on the set } \{{}^cg>0\}$$

implies that

$${}^ch-{}^cg^{-}\geqq{}^cG^cg \qquad \text{everywhere.}$$

Moreover Meyer introduced the notion of the *pseudo-rèduite* ${}^cH_E{}^ch$ for every quasi-excessive function ${}^ch$ and every subset E of ${}^cS$. This function ${}^cH_E{}^ch$ satisfies the following four conditions.

(2.7) ${}^cH_E{}^ch$ is quasi-excessive.
(2.8) ${}^cH_E{}^ch\leqq{}^ch$ on ${}^cS$ and ${}^cH_E{}^ch={}^ch$ on E.
(2.9) If ${}^ch_1$ and ${}^ch_2$ are two quasi-excessive functions such that ${}^ch_1\leqq{}^ch_2$ on E, then ${}^cH_E{}^ch_1\leqq{}^cH_E{}^ch_2$.
(2.10) If ${}^cf\in{}^c\boldsymbol{M}^{+}$ vanishes outside of E then ${}^cH_E{}^cG^cf={}^cG^cf$.

For example, the function ${}^cG^cf$, ${}^cf\in{}^c\boldsymbol{M}^{+}$, and every positive constant are quasi-excessive ([7] see also [5]).

Now we introduce a condition.

Condition (*): There exists a sequence of finite sets $\{E_n\}_{n=1,2,\cdots}$ increasing to S such that $c\in E_n$ for each n, and a sequence $\{h_n\}_{n=1,2,\cdots}$ of function on S satisfying the following conditions.

(*i*) $0\leqq h_n\leqq 1$, $h_n(c)=0$, $h_n=1$ on $F_n=S-E_n$, and $\lim h_n=0$.
(*ii*) For every $f\in\boldsymbol{N}$ and every real number m ($\geqq Af(c)$) the relation that

$$m+h_n\geqq Af \qquad \text{on the set } \{f>0\}.$$

implies that

$$m+h_n-f^{-}\geqq Af \qquad \text{everywhere on } {}^cS.$$

In section 3 we shall show that if A is a weak potential kernel of an irreducible recurrent Markov chain, it satisfies condition (*).

Theorem 2.2 *Condition* (*) *is equivalent to the condition that, there exists a sequence of finite sets* $\{{}^cE_n\}_{n=1,2,\cdots}$ *increasing to* ${}^cS$ *such that*

$$\lim {}^cH_{{}^cS-{}^cE_n}\cdot 1 = 0\,. \tag{2.11}$$

Proof. Suppose that condition (*) holds and let ${}^ch_n$ be the restriction of h_n to ${}^cS$ and ${}^cE_n={}^cS\cap E_n$. Obviously ${}^cS-{}^cE_n=F_n$, $0\leqq{}^ch_n\leqq 1$, and ${}^ch_n=1$ on F_n. It then follows that ${}^ch_n$ is a quasi-excessive function for every n. In fact,

3) This definition is slightly different from Meyer's one; this is the discrete version of Meyer's.

let ${}^cf$ be in ${}^c\boldsymbol{M}$ and f, the extention of ${}^cf$ to S such that $f\in\boldsymbol{N}$. If

$$ {}^ch_n\geqq {}^cG{}^cf \qquad \text{on the set } \{{}^cf>0\} $$

then

$$ h_n+Af(c)\geqq Af \qquad \text{on the set } \{f>0\}\,, $$

since $\{{}^cf>0\}$ is contained in $\{f>0\}\cup\{c\}$. Hence from condition $(*)$,

$$ h_n+Af(c)-f^-\geqq Af \qquad \text{everywhere on } {}^cS, $$

that is,

$$ {}^ch_n-{}^cf^-\geqq {}^cG{}^cf \qquad \text{everywhere on } {}^cS. $$

Since ${}^cH_{F_n}\cdot 1\leqq {}^ch_n$ by definition,

$$ \lim {}^cH_{F_n}\cdot 1=0\,. $$

Conversely, if (2.11) holds, set ${}^cE_n\cup\{c\}=E_n$, $F_n=S-E_n$ and

$$ h_n=\begin{cases} {}^cH_{F_n}\cdot 1 & \text{on } {}^cS \\ 0 & \text{at } c. \end{cases} $$

It is enough to show the property (*ii*) of condition $(*)$. Suppose that, for some $f\in\boldsymbol{N}$ and some real number m $(\geqq Af(c))$

$$ m+h_n\geqq Af \qquad \text{on } \{f>0\}\,. $$

Then one has

$$ m-Af(c)+{}^cH_{F_n}\cdot 1\geqq {}^cG{}^cf \qquad \text{on } \{{}^cf>0\}\,, $$

where ${}^cf$ is the restriction of f to ${}^cS$. The fact that $m-Af(c)+{}^cH_{F_n}\cdot 1$ is a quasi-excessive function implies that

$$ m-Af(c)+{}^cH_{F_n}\cdot 1-{}^cf^-\geqq {}^cG{}^cf \qquad \text{everywhere on } {}^cS, $$

which is nothing but condition $(*)$.

Note. *If α is a finite measure, then condition $(*)$ is satisfied.*

Let I_F be the indicator function of a set F, then from lemma 2.1 ${}^cGI_F\geqq 1$ on F. Hence from (2.8) and (2.9) ${}^cH_F\cdot 1\leqq {}^cGI_F$. Hence if F_n decrease to empty set, inequality

$$ {}^cH_{F_n}\cdot 1(x)\leqq {}^cGI_{F_n}(x)\leqq \textstyle\sum_{y\in F_n} C\cdot\alpha(y)\,, $$

implies that

$$ \lim {}^cH_{F_n}\cdot 1(x)=0\,. $$

Where the second inequality follows from the corollary of lemma 2.1.

3. Main result

Let A be a weak potential kernel of an irreducible recurrent transition probability P with invariant measure α. We shall now prove that A satisfies condition $(*)$ of section 2.

Define ${}^cP$ and ${}^cG$ as (1.3) and (1.4) respectively. Let ${}^cH_F$ be the réduite defined by ${}^cP$. Since ${}^cH_F\cdot 1$ is the pseudo-réduite associated with the above ${}^cG$ (see [5] P. 37, theorem 1.3), it is enough to show that for a sequence of finite sets $\{{}^cE_n\}_{n=1,2,\cdots}$ increasing to ${}^cS$, $\lim {}^cH_{F_n}\cdot 1=0$ ($F_n={}^cS-{}^cE_n$) by theorem 2.2. One can easily seen that the function ${}^ch(x)=\lim {}^cH_{F_n}\cdot 1(x)$ is an invariant function for ${}^cP$ (i.e. ${}^cP{}^ch={}^ch$) and bounded by 1. On the other hand,

$$1 = {}^cG(1-{}^cP\cdot 1)(x)+\lim {}^cP^n\cdot 1(x)$$

and

$$\lim {}^cP^n\cdot 1(x) = \lim \boldsymbol{P}_x[\sigma_{\{c\}}>n] = 0\,,$$

implies that 1 is a potential of non-gengative function (where $\sigma_{\{c\}}$ is the hitting time of the Markov chain with transition probability P). Hence ${}^ch$ is also a potential. The fact that ${}^ch$ is an invariant function and also a potential shows that ${}^ch=0$.

The main result of the present paper is this.

Theorem 3.1. *Given a positive measure α and a kernel A satisfying maximum principle* (RSCM) *and condition* $(*)$, *there exists a unique irreducible recurrent transition probability P which has α as its invariant measure, and A as its weak potential kernel.*

Uniqueness was proved by Kondō [4]. We shalll divide the proof of existence into several lemmas. In the following we shall use the notation of section 2 with no further reference.

Lemma 3.1. *There exists a sub-Markov transition probability ${}^cP(x, y)$ on ${}^cS$ such that*

$${}^cG(x, y) = \sum_{n=0}^{\infty} {}^cP^n(x, y) \qquad \textit{for every } x, y \textit{ in } {}^cS.$$

Proof. See Meyer [7] P. 238 lemma 10.

Lemma 3.2. *For every $y\in {}^cS$, $\sum_{x\neq c} \alpha(x){}^cP(x, y)\leqq\alpha(y)$.*

Proof. To the contrary, suppose that there exists some state $y\in{}^cS$ such that

$$\sum_{x\neq c} \alpha(x){}^cP(x, y)-\alpha(y)>0\,.$$

Then there exists a finite subset F of ${}^cS$ containing y and satisfying

$$\sum_{x\in F} \alpha(x){}^cP(x, y)-\alpha(y) = a>0\,.$$

Define a function $f\in \boldsymbol{N}$ by

$$f(x)=\begin{cases} {}^cP(x, y)-I(x, y) & x\in F \\ -\dfrac{a}{\alpha(c)} & x=c \\ 0 & \text{otherwise.}\end{cases}$$

Since $Af+f^-$ attains its maximum on the set $\{f>0\}$ and since $f(c)<0$, there exists a state $x_0\in F$ such that,

$$Af(x_0)\geqq Af+f^- \qquad \text{everywhere on } S.$$

In particular,

$$Af(x_0)\geqq Af(c)+\frac{a}{\alpha(c)}.$$

Hence,

$$0>-\frac{a}{\alpha(c)}\geqq Af(c)-Af(x_0)={}^cG(-{}^cf)(x_0),$$

where ${}^cf$ is the restriction of f to ${}^cS$. On the other hand,

$$\begin{aligned}{}^cG(-{}^cf)(x_0) &= {}^cG(x_0, y)-\textstyle\sum_{z\in F}{}^cG(x_0, z){}^cP(z, y)\\ &\geqq {}^cG(x_0, y)-({}^cG(x_0, y)-I(x_0, y))=I(x_0, y)\geqq 0.\end{aligned}$$

This lead us to a contradiction.

Lemma 3.3. ${}^cG(1-{}^cP\cdot 1)=1$ *on* ${}^cS$.

Proof. For any positive integer n, we have

$$1=\textstyle\sum_{k=0}^{n}{}^cP^k(1-{}^cP\cdot 1)(x)+{}^cP^{n+1}\cdot 1(x).$$

Passing to the limit we obtain

$$1={}^cG(1-{}^cP\cdot 1)(x)+r(x),$$

where $r(x)=\lim {}^cP^{n+1}\cdot 1(x)$. It remains to show that $r(x)=0$. From condition $(*)$ for arbitrary $\varepsilon>0$ there exists a number M such that for any integer $m\geqq M$,

$${}^cH_{F_m}\cdot 1(x)<\varepsilon.$$

Hence

$$\textstyle\sum_{y\neq c}{}^cP^{n+1}(x, y)={}^cP^{n+1}I_{F_m}(x)+{}^cP^{n+1}I_{E_m}(x)\leqq {}^cH_{F_m}\cdot 1(x)+{}^cP^{n+1}I_{E_m}(x),$$

where I_F is the indicator function of F. Tending n to infinity we obtain $r(x)\leqq\varepsilon$.

Lemma 3.4. $\sum_{x\neq c}\alpha(x)(1-{}^cP\cdot 1)(x)\leqq\alpha(c)$.

Proof. Let F be an arbitrary finite subset of ${}^cS$, and define

$$f(x)=\begin{cases}1-{}^cP\cdot 1(x) & x\in F\\ -\sum_{y\in F}\dfrac{\alpha(y)}{\alpha(c)}(1-{}^cP\cdot 1(y)) & x=c\\ 0 & \text{otherwise.}\end{cases}$$

As noted in the proof of lemma 3.2, there exists a state $x_0\in F$ such that

$$Af(x_0)\geqq Af+f^- \qquad \textit{on } S.$$

In particular,

$${}^cG^cf(x_0)=Af(x_0)-Af(c)\geqq f^-(c)=\sum_{y\in F}\alpha(y)(1-{}^cP\cdot 1))/\alpha(c)\,,$$

and by lemma 3.3, the left side of the above inequality is bounded by 1.

Now we can define the desired transition probability P.

$$P(x,y)=\begin{cases}{}^cP(x,y) & x\neq c,\ y\neq c\\ 1-{}^cP\cdot 1(x) & x\neq c,\ y=c\\ (\alpha(y)-\alpha^cP(y))/\alpha(c) & x=c,\ y\neq c\\ 1-\sum_{z\neq c}P(c,z) & x=c,\ y=c\,.\end{cases} \tag{3.1}$$

From lemmas 3.2 and 3.4, P is a transition probability on S.

Lemma 3.5. *$\alpha P=\alpha$ and $(I-P)Af=f$ for any $f\in N$.*

Proof. If $x\neq c$, then

$$\alpha P(x)=\sum_{y\neq c}\alpha(y)^cP(y,x)+\alpha(x)-\alpha^cP(x)=\alpha(x)$$

and

$$(I-P)Af(x)=(I-P)(Af(c)+{}^cGf)(x)=f(x)\,.$$

By the same argument for $x=c$, lemma follows.

Lemma 3.6. *The transition probability P is recurrent and irreducible.*

Proof. Let $\sigma_{\{x\}}$ be the hitting time for x of the Markov chain with transition probability P. Then for every $x\neq c$,

$$\boldsymbol{P}_x[\sigma_{\{c\}}<\infty]=\sum_{y\neq c}{}^cG(x,y)P(y,c)={}^cG(1-{}^cP\cdot 1)(x)=1\,,$$

by lemma 3.3. Hence,

$$\boldsymbol{P}_c[\sigma^+_{\{c\}}<\infty]=\sum_{x\in S}P(c,x)\boldsymbol{P}_x[\sigma_{\{c\}}<\infty]=1\,,$$

where $\sigma^+_{\{c\}}$ is the positive hitting time for state c. Thus c is a recurrent state for P and hence also for $\hat{P}$, where $\hat{P}$ is defined by,

$$\hat{P}(x,y)=\frac{\alpha(y)}{\alpha(x)}P(y,x)\,. \tag{3.2}$$

Moreover,

$$\hat{P}^n(c, x) = \frac{\alpha(x)}{\alpha(c)} P^n(x, c), \text{ and } \boldsymbol{P}_x[\sigma_{\{c\}} < \infty] = 1 ,$$

shows that

$$\hat{\boldsymbol{P}}_c[\sigma_{\{x\}} < \infty] > 0 \text{ for all } x \in {}^cS.$$

Hence x is a recurrent state for $\hat{P}$ and hence for P. Since x is recurrent and $\boldsymbol{P}_x[\sigma_{\{c\}} < \infty] = 1$, it follows that $\boldsymbol{P}_c[\sigma_{\{x\}} < \infty] = 1$ for all $x \in S$. Irreducibility follows from the fact that, $\boldsymbol{P}_x[\sigma_{\{c\}} < \infty] = 1$ and $\boldsymbol{P}_c[\sigma_{\{y\}} < \infty] = 1$ for every x, y in S.

OSAKA CITY UNIVERSITY

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