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ON RIBBON 2-KNOTS II
THE SECOND HOMOTOPY GROUP OF THE COMPLEMENTARY DOMAIN

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1. Introduction

Concerning the problem(0) of "how to calculate the second homotopy group of the complementary domain of a 2-knot in R^4," there exist several results in [1], [2] and [3]. Especially, the result by C.H. Giffin in [3] seems to be conclusive, but the proof in his report is so brief that there are some parts which can not be understood straightforwards. In this paper, we will be concerned exclusively about only ribbon 2-knots(1) which have some nice properties both in the geometrical and in the algebraical sides in the 2-knot theory, see [5], [6], [7] and [8]. First in §3, we will discuss about the second homotopy group of the complementary domain of 2-nodes (D^2, H^4) with the properties defined in (2), (3) and (4) in §2, and we will prove the result $\pi_2(H^4-D^2)=(0)$ in Theorem (3.4). In §4, we will investigate a relation between the knot-group and the second homotopy group of the complementary doamin of the ribbon 2-knots, and as a consequence, we will prove the main theorem, Theorem (4.3), of this paper.

2. Preliminaries

We may suppose the following (1), (2) (3) and (4) with a slight modification for a ribbon 2-knot K^2:

(1) 2-balls $D^2_+=K^2\cap H^4_+$ and $D^2_-=K^2\cap H^4_-$ are symmetric each other with respect to the hyperplane R^3_0(2),

(2) D^2_+ has no minimal point,

(3) all saddle points p_i(3) of D^2_+ are at the level R^3_1, and in a small neigh-

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(0) See [4], p. 175, Problem 36.

(1) See [6], §4.

(2) $R^3_t \;=\{(x_1, x_2, x_3, x_4)\,|\,x_4=t\}$
$H^4_+ \;=\{(x_1, x_2, x_3, x_4)\,|\,x_4\geqq 0\}$
$H^4_- \;=\{(x_1, x_2, x_3, x_4)\,|\,x_4\leqq 0\}$
$H^4(J)=\{(x_1, x_2, x_3, x_4)\,|\,x_4\in J\}$.

(3) see [4] p. 133.

borhood of each saddle point, D_+^2 is a square B_i^2 at R_1^3 which is called *a saddle-band*(4), see Fig. (1),

(4) D_+^2 is in a general position with respect to the collection R_t^3 $(t \neq 1)$.

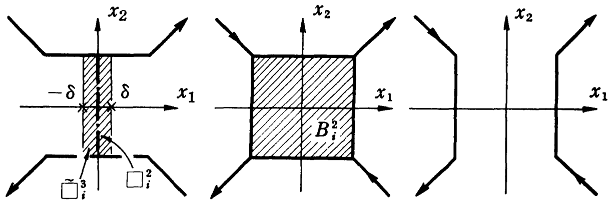


Fig. (1)

For each saddle point $p_i (i=1, 2, \cdots, n)$, we suppose that, for sufficiently small positive numbers ε and δ,

$$\begin{aligned} p_i &: (x_1^{(i)}, x_2^{(i)}, 0, 1) \\ B_i^2 &: |x_1 - x_1^{(i)}| \leqq 1, \quad |x_2 - x_2^{(i)}| \leqq 1, \quad x_3 = 0, \quad x_4 = 1 \\ \square_i^2 &: x_1 = 0, \quad |x_2 - x_1^{(i)}| \leqq 1, \quad x_3 = 0, \quad 1-\varepsilon \leqq x_4 \leqq 1 \\ \tilde{\square}_i^3 &: |x_1 - x_1^{(i)}| \leqq \delta, \quad |x_2 - x_2^{(i)}| \leqq 1, \quad x_3 = 0, \quad 1-\varepsilon \leqq x_4 \leqq 1, \end{aligned}$$

where $\square_i^2$ is a square and $\tilde{\square}_i^3$ is a cube, and $B_i^2 \cap B_j^2 = \phi$ and $\tilde{\square}_i^3 \cap \tilde{\square}_j^3 = \phi$ if $i \neq j$(5).

If we investigate the cross-sections of D_+^2 by $R_t^3 (1-\varepsilon \leqq t \leqq 1+\varepsilon)$ for a small positive number ε, we have the following (1), (2), (3) and (4):

(1) $D_+^2 \cap R_{1-\varepsilon}^3$ is a ribbon knot k in $R_{1-\varepsilon}^3$,

(2) $D_+^2 \cap R_{1+\varepsilon}^3$ is a trivial link $k_0 \cup k_1 \cup \cdots \cup k_n$ in $R_{1+\varepsilon}^3$,

(3) By the orthogonal projection θ of H_+^4 onto R_1^3,

$$\theta(D_+^2 \cap H^4(1, 1+\varepsilon]) \subset \partial(D_+^2 \cap R_1^3)^{(6)}$$
$$\theta(D_+^2 \cap H^4[1-\varepsilon, 1)) \subset \partial(D_+^2 \cap R_1^3)$$
$$\theta(\square_i^2) = \square_i^2 \cap B_i^2 \quad (i=1, \cdots, n).$$

(4) The band (square) B_i^2 spans $\theta(k_i)$ and $\theta(k_0)$ $(i=1, \cdots, n)$ coherently on its opposite, parallel edges.

3. Surgery

For a PL-map g'' of S^2 into $H_+^4 - D_+^2$, there is a PL-map g' of S^2 into

(4) This is a conventional word.

(5) These coordinate-presentations are not essential.

(6) ∂X means the boundary and $\mathring{X}$ the interior of a point set X. For convenience'sake, we denote $X - X \cap Y$ by $X - Y$.

$H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$ such that $g'(S^2)$ is homotopic to $g''(S^2)$ in $H^4_+-D^2_+$, since there exist only the maximal points of D^2_+ but no saddle point of D^2_+ in the exterior of $H^4[1-\varepsilon, 1+\varepsilon]$. If $g'(S^2)\cap(\widetilde{\square}_1\cup\cdots\cup\widetilde{\square}_n)=\phi$, we can jump to (3.3) without troubles in the following discussion. If $g'(S^2)\cap(\widetilde{\square}_1\cup\cdots\cup\widetilde{\square}_n)\neq\phi$, we consider a PL-map g of S^2 into $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$ satisfying (1)~(4) mentioned below:

(1) $g(S^2)$ is homotopic to $g'(S^2)$ in $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$,

(2) $g(S^2)\cap\square^2_i$ consists of at most a finite number of points on $R^3_{1-\varepsilon}$ denoted by $q^{(i)}_1, \cdots, q^{(i)}_{m_i}$, for which $g^{-1}(q^{(i)}_\lambda)$ is just one point, say $\tilde{q}^{(i)}_\lambda$, on S^2,

(3) there are 2-balls $\tilde{U}^{(i)}_\lambda$, $\tilde{V}^{(i)}_\lambda$ $(1\leqq i\leqq n, \lambda=1, \cdots, m_i)$ on S^2 such that

(3₁) $\mathring{\tilde{U}}^{(i)}_\lambda\supset\tilde{V}^{(i)}_\lambda\supset\mathring{\tilde{V}}^{(i)}_\lambda\ni\tilde{q}^{(i)}_\lambda$,

(3₂) $\tilde{U}^{(i)}_\lambda\cap\tilde{U}^{(j)}_\mu=\phi$ if either $i\neq j$ or $\lambda\neq\mu$,

(3₃) $g\,|\,\tilde{U}^{(i)}_\lambda$ is an imbedding,

(4) denote $g(\tilde{V}^{(i)}_\lambda)$ by $V^{(i)}_\lambda$ $(1\leqq i\leqq n, \lambda=1, \cdots, m_i)$, then

$$V^{(i)}_\lambda:\begin{cases} x_1^2+x_3^2\leqq 4 \\ x_2=\dfrac{1}{1+\lambda} \\ x_4=1-\varepsilon\,. \end{cases}$$

Since it is not difficult to see the existence of two deformation retractions ξ' and ξ'':

$$\xi':\ H^4[1-\varepsilon, 1+\varepsilon]-D^2_+\cup\widetilde{\square}_1\cup\cdots\cup\widetilde{\square}_n\to H^4[1, 1+\varepsilon]-D^2_+$$
$$\xi'':\ H^4[1, 1+\varepsilon]-D^2_+\to R^3_{1+\varepsilon}-D^2_+$$

we may suppose that the above PL-map g satisfies not only (1)~(4) but also (5), (6):

(5) $g(\overline{S^2\bigcup_{i,\lambda}\tilde{U}^{(i)}_\lambda})\subset R^3_{1+\varepsilon}-D^2_+$,

(6) 2-ball $U^{(i)}_\lambda=g(\tilde{U}^{(i)}_\lambda)$ satisfies that the annulus $\overline{U^{(i)}_\lambda-V^{(i)}_\lambda}$ is given by

$$\overline{U^{(i)}_\lambda-V^{(j)}_\lambda}:\begin{cases} x_1^2+x_3^2=4 \\ x_2=\dfrac{1}{1+\lambda} \\ 1-\varepsilon\leqq x_4\leqq 1+\varepsilon\,. \end{cases}$$

Let $c^{(i)}_\lambda$ be the simple closed curve $\partial U^{(i)}_\lambda$ $(i=1, \cdots, n, \lambda=1, \cdots, m_i)$. The orientation of $c^{(i)}_\lambda$ should be induced by that of $U^{(i)}_\lambda$ as a subcomplex of the oriented 2-sphere S^2, and the orientation of knot k_0 in $R^3_{1+\varepsilon}$ should be induced by that of D^2_+. We classify these simple closed curves $c^{(i)}_\lambda$ into two collections

$\Gamma_+^{(i)}$ and $\Gamma_-^{(i)}$:

$$\Gamma_+^{(i)} = \{c_\lambda^{(i)} \mid 1 \leqq \lambda \leqq m_i,\ \text{the linking number } c_\lambda^{(i)} \text{ and } k_0 = 1\}$$
$$\Gamma_-^{(i)} = \{c_\mu^{(i)} \mid 1 \leqq \mu \leqq m_i,\ \text{the linking number } c_\mu^{(i)} \text{ and } k_0 = -1\}$$

for each i ($i=1, \cdots, n$).

Lemma (3.1). *$\Gamma_+^{(i)}$ and $\Gamma_-^{(i)}$ contains the same number of circles for each i ($i=1, \cdots, n$).*

Proof. Consider a 2-node $D_+^{(i)}$ in $H^4[1-\varepsilon, \infty)$ given by cutting $D_+^2 \cap H^4[1-\varepsilon, \infty)$ along an arc $D_+^2 \cap \square_i^2$ and sewing by the 2-ball $\square_i^2$, where we suppose that $D_+^{(i)} \supset k_i$. Then, since the closed curve $c_\nu^{(j)}$ does not link with k_i in $R_{1+\varepsilon}^3 (j \neq i)$, the 2—ball $U_\nu^{(j)}$ bounded by $c_\nu^{(j)}$ is isolated from $D_+^{(i)}$ in $H^4[1-\varepsilon, \infty)-D_+^{(i)}$. Therefore $c_1^{(i)}+c_2^{(i)}+\cdots+c_{m_i}^{(i)}=0$ in $H_1(H^4[1-\varepsilon, \infty)-D_+^{(i)})$, as $c_1^{(i)} \cup \cdots \cup c_{m_i}^{(i)}$ bounds a 2-complex $g(S^2-\tilde{U}_1^{(i)} \cup \cdots \cup \tilde{U}_{m_i}^{(i)})$. Since $H_1(H^4[1-\varepsilon, \infty)-D_+^{(i)})=(t; -)$, and either $c_\lambda^{(i)}=t$ or $c_\mu^{(i)}=-t$ as $c_\lambda^{(i)} \in \Gamma_+^{(i)}$ or $c_\mu^{(i)} \in \Gamma_-^{(i)}$ respectively, the proof is now completed.

Lemma (3.2). *There exists an arc $\tilde{\gamma}$ on a perforated 2-ball $S^2-\bigcup_{i,\lambda} \mathring{\tilde{U}}_\lambda^{(i)}$ satisfying* (1) *and* (2) *as follows*:

(1) *The arc $\gamma=g(\tilde{\gamma})$ spans $c_\lambda^{(i)}$ and $c_\mu^{(i)}$, where $c_\lambda^{(i)} \in \Gamma_+^{(i)}$ and $c_\mu^{(i)} \in \Gamma_-^{(i)}$ $(1 \leqq i \leqq n)$,*

(2) *The arc $\gamma=g(\tilde{\gamma})$ is on E_i^2, where mutually disjoint 2-balls $E_0^2, E_1^2, \cdots, E_n^2$ satisfy that $\partial E_i^2=k_i$ ($i=0, 1, \cdots, n$) in $R_{1+\varepsilon}^3$.*

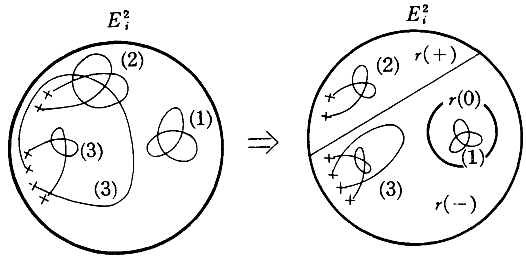
Proof. Let $\Sigma=g(S^2) \cap R_{1+\varepsilon}^3=g(S^2-\bigcup_{i,\lambda} \mathring{\tilde{U}}_\lambda^{(i)})$, then $\Sigma \subset R_{1+\varepsilon}^3-k_0 \cup k_1 \cup \cdots \cup k_n$, $\partial\Sigma=\bigcup_{i,\lambda} c_\lambda^{(i)}$. In this case, we may suppose with a slight modification if necessary that $\Sigma \cap E_i^2$ consists of the curves γ's of the following four types:

(1) $\gamma=g(\tilde{\gamma})$ for a closed curve $\tilde{\gamma}$ on S^2,
(2) $\gamma=g(\tilde{\gamma})$ for an arc $\tilde{\gamma}$ on S^2 spanning $\partial\tilde{U}_\lambda^{(i)}$ and $\partial\tilde{U}_\nu^{(i)}$, where either $c_\lambda^{(i)}, c_\nu^{(i)} \in \Gamma_+^{(i)}$ or $c_\lambda^{(i)}, c_\nu^{(i)} \in \Gamma_-^{(i)}$,
(3) $\gamma=g(\tilde{\gamma})$ for an arc $\tilde{\gamma}$ on S^2 spanning $\partial\tilde{U}_\lambda^{(i)}$ itself.
(4) $\gamma=g(\tilde{\gamma})$ for an arc $\tilde{\gamma}$ on S^2 spanning $\partial\tilde{U}_\lambda^{(i)}$ and $\partial\tilde{U}_\mu^{(i)}$, where $c_\lambda^{(i)} \in \Gamma_+^{(i)}$ and $c_\mu^{(i)} \in \Gamma_-^{(i)}$.

In the cases (1)$\sim$(4), γ may be a non-simple curve, but for an imbedding ψ_i of $E^2 \times [-\varepsilon, \varepsilon]$ into $R_{1+\varepsilon}^3$ such as $\psi_i(E^2 \times 0)=E_i^2$, we may suppose that

$$\psi_i^{-1} \cdot \psi_i(E^2 \times [-\varepsilon, \varepsilon] \cap \Sigma) = \psi_i^{-1}(E_i^2 \cap \Sigma) \times [-\varepsilon, \varepsilon]\,.$$

Leaving the points on $\psi_i(\partial(E^2 \times [-\varepsilon, \varepsilon]))$ fixed, we can homotopically carry the singularities of type (1), (2), and (3) into three regions $r(+)$, $r(0)$ and $r(-)$ on E_i^2, see Fig. (2) below:



(2_1) (2_2)

Fig. (2)

where we classify as follows:

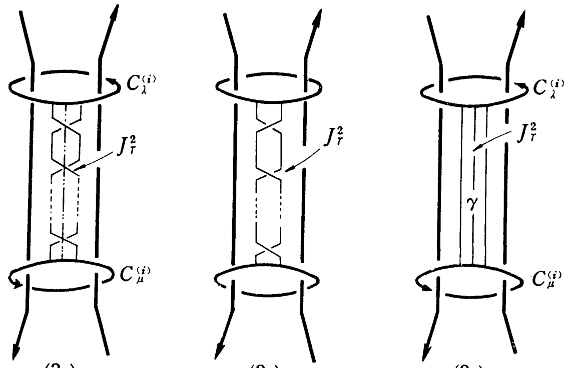
$\gamma \subset r(+)$, if it spans $c_\lambda^{(i)}$ and $c_\gamma^{(i)}$ of $\Gamma_+^{(i)}$,

$\gamma \subset r(-)$, if it spans $c_\lambda^{(i)}$ and $c_\mu^{(i)}$ of $\Gamma_-^{(i)}$,

$\gamma \subset r(0)$, if $\tilde{\gamma}$ is a closed curve.

If there exists no singularity of type (4), the trivial knot $\partial r(+)$ links only with $c_\lambda^{(i)}$ for $c_\lambda^{(i)} \in \Gamma_+^{(i)}$. Therefore, $c_{\lambda_1}^{(i)} + \cdots + c_{\lambda_x}^{(i)} = 0$ in $H_1(R_{1+\varepsilon}^3 - \partial r(+)) = (t; -)$, where $\{c_{\lambda_1}^{(i)}, \cdots, c_{\lambda_x}^{(i)}\} = \Gamma_+^{(i)}$, since $c_{\lambda_1}^{(i)} \cup \cdots \cup c_{\lambda_x}^{(i)}$ bounds a 2–complex $\Sigma \cup (\bigcup_{j,\lambda} \sigma_\lambda^{(j)})_{j=i, \lambda \neq \lambda_1, \cdots, \lambda_x}^{j \neq i}$ where the 2–ball $\sigma_\lambda^{(j)}$ is bounded by $c_\lambda^{(j)}$ in $R_{1+\varepsilon}^3 - \partial r(+)$. Then, we must say that $\Gamma_+^{(i)} = \phi$ and necessarily $\Gamma_-^{(i)} = \phi$ by (3.1). On the other hand, we have assumed that $g'(S^2) \cap (\widetilde{\square}_1 \cup \cdots \cup \widetilde{\square}_n) \neq \phi$, thus there must be at most one integer $i (1 \leqq i \leqq n)$ for which $\Gamma_\pm^{(i)} \neq \phi$. This is a contradiction, and there must be a singularity γ desired in (3.2).

By the result in (3.2), we can modify Σ homotopically in $R_{1+\varepsilon}^3 - k_0 \cup k_1 \cup \cdots \cup k_n$ leaving the circles $c_\lambda^{(i)}$ fixed for all i and λ so that there exists a band $J_\gamma^{(2)}$ containing γ and contained in Σ, see (3_1) in Fig. (3). Since Σ is an image



(3_1) (3_2) (3_3)

Fig. (3)

of a subset of S^2, there exist an even number of twists on J^2_γ, see (3_1). Nevertheless, it is not so difficult to move Σ homotopically in $R^3_{1-\varepsilon}-k_0\cup k_1\cup\cdots\cup k_n$ leaving the circles $c^{(i)}_\lambda$ fixed so that J^2_γ has no twist, see (3_2) and (3_3).

Consider a surgery on $g(S^2)$ in the following figure, Fig. (4).

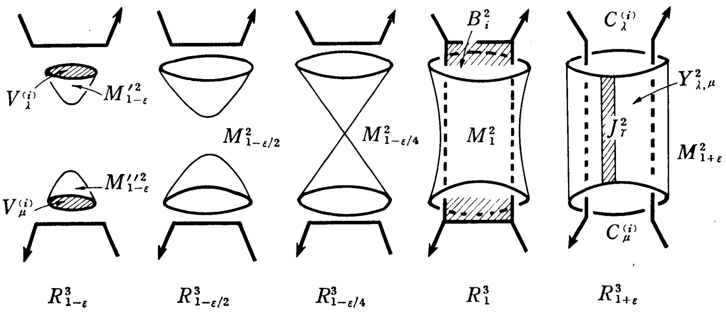


Fig. (4)

In Fig. (4), the boundary circles of a 2–surface M^2_t $(1-\varepsilon\leqq t\leqq 1+\varepsilon)$ are the circles $U^{(i)}_\lambda\cap R^3_t$ and $U^{(i)}_\mu\cap R^3_t$, therefore $\bigcup_{1-\varepsilon\leqq t\leqq 1+\varepsilon}\partial M^2_t\cup V^{(i)}_\lambda\cup V^{(i)}_\mu=U^{(i)}_\lambda\cup U^{(i)}_\mu$. Let $B'^3_{1-\varepsilon}$ and $B''^3_{1-\varepsilon}$ be the two 3–balls bounded by the 2–spheres $V^{(i)}_\lambda\cup M'^2_{1-\varepsilon}$ and $V^{(i)}_\mu\cup M''^2_{1-\varepsilon}$ in the level $R^3_{1-\varepsilon}$, then the 3-manifold $B'^3_{1-\varepsilon}\cup B''^3_{1-\varepsilon}\cup\bigcup_{1-\varepsilon\leqq t\leqq 1+\varepsilon} M^2_t$ is a 3-ball $X^3_{\lambda,\mu}$ for which we may suppose that $\partial X^3_{\lambda,\mu}=U^{(i)}_\lambda\cup U^{(i)}_\mu\cup M^2_{1+\varepsilon}$, $M^2_{1+\varepsilon}=J^2_\gamma\cup Y^2_{\lambda,\mu}$, where $Y^2_{\lambda,\mu}=\overline{M^2_{1+\varepsilon}-J^2_\gamma}$. Then, there is a PL-map f' of S^2 into $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$ satisfying the followings:

(1) $f'(S^2)$ is homotopic to $g(S^2)$ in $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$,

(2) $f'\,|\,S^2-\tilde{U}^{(i)}_\lambda\cup\tilde{U}^{(i)}_\mu=g\,|\,S^2-\tilde{U}^{(i)}_\lambda\cup\tilde{U}^{(i)}_\mu$,

(3) $f'(S^2)=g(S^2-\tilde{U}^{(i)}_\lambda\cup\tilde{U}^{(i)}_\mu\cup\tilde{J}^2_\gamma)\cup Y^2_{\lambda,\mu}$,

where $\tilde{J}^2_\gamma$ is a neighborhood of $\tilde{\gamma}$ such as $g(\tilde{J}^2_\gamma)=J^2_\gamma$.

Repeating these processes, we have finally a PL-map f of S^2 into $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$ such that $f(S^2)$ is homotopic to $g''(S^2)$ in $H^4[1-\varepsilon, 1+\varepsilon]-D^2_+$ and that $f(S^2)\subset R^3_{1+\varepsilon}-k_0\cup k_1\cup\cdots\cup k_n$.

Here, we want to get the consideration above into shape.

Lemma (3.3). *For any PL-map g'' of S^2 into $H^4_+-D^2_+$, there is a PL-map f of S^2 into $H^4_+-D^2_+$ satisfying* (1) *and* (2) *below*:

(1) *$f(S^2)$ is homotopic to $g''(S^2)$ in $H^4_+-D^2_+$,*

(2) $f(S^2)\subset R^3_{1+\varepsilon}-D^2_+\cap R^3_{1+\varepsilon}$.

Theorem (3.4). $\pi_2(H^4_+-D^2_+)=(0)$.

Proof. By the sphere-theorem[7] for 1–links (, for 3–manifolds,)

(7) See [9], [10].

$\pi_2(R^3_{1+\varepsilon}-D^2_+\cap R^3_{1+\varepsilon})$ is generated[(8)] by a collection of mutually disjoint non-singular 2-spheres $s^2_1, s^2_2, \cdots, s^2_n$ in $R^3_{1+\varepsilon}$, where a 2-sphere s^2_i is the boundary surface of a regular neighborhood of the 2-ball E^2_i in $R^3_{1+\varepsilon}$ $(i=1, \cdots, n)$. Since there is no saddle point of D^2_+ in $H^4[1+\varepsilon, \infty)$, we can easily contract the 2-sphere s^2_i to a point through $H^4[1+\varepsilon, \infty)-D^2_+$ $(i=1, \cdots, n)$. On the other hand, by (3.3), an arbitrary element s of $\pi_2(H^4_+-D^2_+)$ can be represented by the elements of $\pi_2(R^3_{1+\varepsilon}-D^2_+\cap R^3_{1+\varepsilon})$[(9)] which are contractible in $H^4_+-D^2_+$ as already mentioned. The proof is thus complete.

4. Covering spaces

Let u: $W\to R^4-K^2$ be a universal covering for the complementary domain of a ribbon 2-knot K^2 in R^4. Then,

$$u_+ = u\,|\,W_+ : \; W_+\to H^4_+-D^2_+$$
$$u_- = u\,|\,W_- : \; W_-\to H^4_--D^2_-$$

are both universal, since K^2 is symmetric with respect to the hyperplane R^3_0, and the inclusion-induced homomorphism of $\pi_1(H^4_+-D^2_+)$ into $\pi_1(R^4-K^2)$ is *onto* as the 2-node (D^2_+, H^4_+) has no minimal point. By (3.4) and the Hurewicz theorem, we have the followings:

$$(*)\;\begin{cases} H_1(W_+)=(0)\,, & H_2(W_+)=\pi_2(W_+)=\pi_2(H^4_+-D^2_+)=(0)\,,\\ H_1(W_-)=(0)\,, & H_2(W_-)=\pi_2(W_-)=\pi_2(H^4_--D^3_-)=(0)\,,\\ H_2(W)=\pi_2(W)=\pi_2(R^4-K^2)\,^{(10)}. \end{cases}$$

Consider the next Mayer-Vietoris sequence:

$$H_2(W_+)+H_2(W_-)\to \underset{\underset{\displaystyle H_2(W)}{\|}}{H_2(W_+\cup W_-)}\to H_1(W_+\cap W_-)\to H_1(W_+)+H_1(W_-)$$

By the relations in $(*)$, we have the following:

Lemma (4.1). $\pi_2(R^4-K^2)=H_1(W_+\cap W_-)\,.$

Now, we will consider the relation between $\pi_1(R^3_0-k)$ and $H_1(W_+\cap W_-)$, where $k=K^2\cap R^3_0$ is a 1-knot in R^3_0.

Lemma (4.2). *If $\pi_1(R^4-K^2)$ is torsion free, then $H_1(W_+\cap W_-)=\mathcal{K}/\mathcal{K}^{(1)}$, where the subgroup $\mathcal{K}$ of $\pi_1(R^3_0-k)$ is the kernel of the inclusion-induced homo-*

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(8) Consider $\pi_1(R^3_{1+\varepsilon}-D^2_+)$ as an operator.

(9) Consider $\pi_1(H^4_+-D^2_+)$ as an operator.

(10) "=" means *isomorphic to.*

morphism i_* *of* $\pi_1(R_0^3-k)$ *onto* $\pi_1(R^4-K^2)$(11).

Proof. Let $u_0=u|W_+\cap W_-\colon W_0\to R_0^3-k$, where $W_0=W_+\cap W_-$. Then u_0 is also a covering which is not always universal. Therefore $\pi_1(W_0)$ is isomorphic to a subgroup $\mathcal{H}$ of $\pi_1(R_0^3-k)$, so we have $\mathcal{H}=\mathcal{K}$ by the facts that $\pi_1(W)=1$ and the homomorphism i_* is onto(12). Abelianize $\mathcal{K}$ by the commutator subgroup $\mathcal{K}^{(1)}$ of $\mathcal{K}$, and we have (4.2).

By (4.1) and (4.2), we have

Theorem (4.3). *For a ribbon 2-knot* K^2, *if* $\pi_1(R^4-K^2)$ *is torsion free, then* $\pi_2(R^4-K^2)=\mathcal{K}/\mathcal{K}^{(1)}$, *where* $\mathcal{K}$ *is defined in* (4.2).

Question. If $\pi_1(R^4-K^2)$ is not torsion free, the subgroup $\mathcal{K}$ will be the subgroup of $\pi_1(R_0^3-k)$ generated by all the elements with finite orders of $i_*(\pi_1(R_0^3-k))$, therefore we have a question: "Is $\pi_1(R^4-K^2)$ torsion free for a ribbon 2-knot?"

REMARK. *If a ribbon 2-knot* K^2 *satisfies that* $\pi_1(R^4-K^2)=(t\colon -)$, *then* $\pi_2(R^4-K^2)=(0)$.

Proof. By the result in [8], for a ribbon 2-knot K^2 and the cross-sectional knot $k=K^2\cap R_0^3$, where the 2-nodes $(D_\pm^2, H_\pm^4)$ for the 2-balls $D_\pm^2=K^2\cap H_\pm^4$ satisfy the properties in §2, the Alexander polynomials satisfy that

$$\Delta_k(t)=\Delta_K(t)\cdot\Delta_K(\bar{t})\,.$$

Therefore, if $\pi_1(R^4-K^2)=(t;\ -)$, $\Delta_K(t)=1$, and necessarily $\Delta_k(t)=1$, then by the theorem (4, 9, 1) in [11], p. 46, $\mathfrak{G}^{(1)}=\mathfrak{G}^{(2)}$ for $\mathfrak{G}=\pi_1(R_0^3-k)$. On the other hand, since $\pi_1(R^4-K^2)=(t;\ -)=\mathfrak{G}/\mathfrak{G}^{(1)}$, the kernel $\mathcal{K}$ of i_* surely coincides with $\mathfrak{G}^{(1)}$. Thus, we have

$$\pi_2(R^4-K^2)=\mathfrak{G}^{(1)}/\mathfrak{G}^{(2)}=\mathfrak{G}^{(1)}/\mathfrak{G}^{(1)}=(0)\,.$$

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(11) $G^{(1)}=[G, G]$, $G^{(2)}=[G^{(1)}, G^{(1)}]$.
(12) This follows from the calculation of the knot-group of a 2-knot, see [4], p. 133~, §6.

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