Sasaki, T.
Osaka J. Math.
7 (1970), 527-534

ON SOME EXTREMAL QUASICONFORMAL MAPPINGS OF DISC

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(Received June 9, 1970)

1. Introduction

A quasiconformal mapping $w(z)$ of the unit disc $\Delta=\{z \mid |z|<1\}$ onto itself is known to have continuous boundary values, hence we may consider the class $Q(w; \Delta, \Delta)$ of all quasiconformal mappings of Δ onto itself that coincide with $w(z)$ on the boundary $\partial\Delta=\{z \mid |z|=1\}$. In $Q(w; \Delta, \Delta)$ there is at least one quasiconformal mapping whose maximal dilatation is a minimum. Such a quasiconformal mapping is called extremal in the class $Q(w; \Delta, \Delta)$. If there exists a regular single-valued analytic function φ defined on Δ and if the complex dilatation μ of a quasiconformal mapping is written in the form

$$\mu = k\frac{\bar{\varphi}}{|\varphi|} \quad (0<k<1), \tag{1}$$

except at zeros of φ, then it is called a Teichmüller mapping corresponding to φ. It was studied by K. Strebel [4] whether a quasiconformal mapping $f(z)$ with the complex dilatation of the form (1) is extremal in the class $Q(f; \Delta, \Delta)$ or not.

In section 2 and section 3 we prove two distortion theorems which serve to show some extremality. In section 4 some extremal quasiconformal mappings which are not Teichmüller mappings in general are considered.

2. Distortion of argument (1)

Let $w(z)$ be a K-quasiconformal mapping which maps $|z|<1$ onto $|w|<1$ with $w(0)=0$ and $w(1)=1$ and let $\arg w(z)=\arg w(re^{i\theta})$ a continuous branch with $\arg w(1)=0$. Then we have

Theorem 1. *For all K-quasiconformal mappings which map $|z|<1$ onto $|w|<1$ with $w(0)=0$ and $w(1)=1$, we have*

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w(r)}{\log r}\right| \leqq \frac{1}{2}\left(K-\frac{1}{K}\right). \tag{2}$$

This bound is best possible.

We begin with some preliminary considerations. For $|w(r)|$ it is well known that $\frac{1}{4^K} r^K < |w(r)| < 4r^{K^{-1}}$. Therefore for each r we have

$$|w(r)| = c(r) r^{f(r)} \text{ with } \frac{1}{4^K} < c(r) < 4,\ \frac{1}{K} \leqq f(r) \leqq K. \tag{3}$$

Also the next inequality is known to hold (see for example [1])

$$\frac{\max_{0 \leqq \theta < 2\pi} |w(re^{i\theta})|}{\min_{0 \leqq \theta < 2\pi} |w(re^{i\theta})|} \leqq \lambda(K) = \frac{1}{16} e^{\pi K} - \frac{1}{2} + O(e^{-\pi K}),$$

so that we see at once

$$\begin{aligned} \max_{0 \leqq \theta < 2\pi} |w(re^{i\theta})| &\leqq \lambda(K) c(r) r^{f(r)}, \\ \min_{0 \leqq \theta < 2\pi} |w(re^{i\theta})| &\geqq \lambda(K)^{-1} c(r) r^{f(r)}. \end{aligned} \tag{4}$$

In order to estimate $\max_{0 \leqq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)|$, we need

Lemma. *For K-quasiconformal mapping $w(z)$ which maps $|z| < 1$ onto $|w| < 1$ with $w(0) = 0$ we have*

$$\max_{0 \leqq \theta < 2\pi} |\arg w(re^{i\theta}) - \arg w(r)| < N_1 < \infty, \tag{5}$$

where N_1 is a constant depending only on K.

Proof. The lemma is a consequence of the following

Theorem ([2], theorem 2). *Suppose that w is a K-quasiconformal mapping of the extended plane and that $w(\infty) = \infty$. Then for each triple of distinct finite points z_1, z_0, z_2*

$$\sin \frac{1}{2} \beta \geqq \varphi_K\left(\sin \frac{1}{2} \alpha\right),$$

where $\varphi_K(r) = \mu^{-1}(K\mu(r))$ and

$$\alpha = \arc \sin\left(\frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|}\right),\ \beta = \arc \sin\left(\frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_0)| + |f(z_2) - f(z_0)|}\right). \tag{*}$$

Here $\mu(r)$ is the modulus of the unit disc slit along the real axis from 0 to r, and μ^{-1} is the inverse of μ.

Proof of lemma. The mapping $w(z)$ in our lemma is extended by reflexion to a K-quasiconformal mapping of the extended plane with $w(\infty)=\infty$, and we apply the above theorem to the inverse w^{-1}.

Suppose that w satisfies the inequalities;

$$n\pi \leqq \max_{0\leqq\theta<2\pi} |\arg w(re)^{i\theta} - \arg w(r)| < (n+1)\pi ,$$

where n is a positive integer. Then there exists at least $2n-1$ points such that $0=\theta_1<\theta_2<\cdots<\theta_{2n-1}=2\pi$ and that $|\arg w(re^{i\theta}) - \arg w(re^{i\theta_2})| = |\arg w(re^{i\theta_2}) - \arg w(re^{i\theta_3})| = \cdots = |\arg w(er^{i\theta_{2n-2}}) - \arg w(re^{i\theta_{2n-1}})| = \pi$. If we put $w(re^{i\theta_\nu})$, $w(0)$, $w(re^{i\theta_{\nu+1}})$ in place of z_1, z_0, z_2 in $(*)$ respectively and $z_\nu=re^{i\theta_\nu}$, $z_0=0$, $z_{\nu+1}=re^{i\theta_{\nu+1}}$ in place of $f(z_1)$, $f(z_0)$, $f(z_2)$ in $(*)$ respectively, $(\nu=1, 2, \cdots, 2n-2)$, then

$$\alpha = \arc\sin\left(\frac{|w(re^{i\theta_\nu})-w(re^{i\theta_{\nu+1}})|}{|w(re^{i\theta_\nu})-w(0)|+|w(re^{i\theta_{\nu+1}})-w(0)|}\right) = \arc\sin 1 = \frac{\pi}{2},$$

$$\beta = \arc\sin\left(\frac{|re^{i\theta_\nu}-re^{i\theta_{\nu+1}}|}{|re^{i\theta_\nu}-0|+|re^{i\theta_{\nu+1}}-0|}\right) = \arc\sin\frac{1}{2}\,|1-e^{i(\theta_{\nu+1}-\theta_\nu)}|$$
$$= \frac{\theta_{\nu+1}-\theta_\nu}{2},$$

and

$$\sin\frac{\theta_{\nu+1}-\theta_\nu}{4} \geqq \varphi_K\left(\sin\frac{\pi}{4}\right) = \varphi_K\left(\frac{1}{\sqrt{2}}\right), \quad (\nu = 1, 2, \cdots, 2n-2).$$

Therefore we have

$$\theta_{\nu+1}-\theta_\nu \geqq 4\arc\sin\varphi_K\left(\frac{1}{\sqrt{2}}\right), \quad (\nu = 1, 2, \cdots, 2n-2).$$

On adding these $2n-2$ inequalities we obtain

$$2\pi \geqq 4(2n-2)\arc\sin\varphi_K\left(\frac{1}{\sqrt{2}}\right)$$

or

$$n+1 \leqq \frac{\pi}{4\arc\sin\varphi_K\left(\frac{1}{\sqrt{2}}\right)} + 2.$$

Putting $N_1=\left(\frac{\pi}{4\arc\sin\varphi_K\left(\frac{1}{\sqrt{2}}\right)}+2\right)\pi$ we have

$$\max_{0\leqq\theta<2\pi} |\arg w(re^{i\theta}) - \arg w(r)| < N_1,$$

thus the lemma is proved.

Proof of theorem 1. By $Z=\log z$ and $W=\log w$ we map $|z|<1$ and $|w|<1$

conformally onto half strips of height 2π such that $Z(1)=\log 1=0$ and $W(1)=\log 1=0$, in the Z and W planes. Then a quadrilateral $q=\{z \mid r<|z|<1, 0<\arg z<2\pi\}$ in the z plane will be mapped onto a quadrilateral $Q=\{Z \mid \log r<X<0, 0<Y<2\pi\}$ in the $Z=X+iY$ plane. Let Q' be the quadrilateral in the W plane to which Q corresponds, and let $M(Q)$ and $M(Q')$ denote the moduli of Q and Q'. Then

$$\frac{1}{K}M(Q) \leqq M(Q') \leqq KM(Q),\ M(Q)=\frac{1}{2\pi}\log\frac{1}{r}. \tag{6}$$

On the other hand, applying Rengel's inequality ([3], p. 24) to Q', we have

$$\frac{(S_b(Q'))^2}{m(Q')} \leqq M(Q') \leqq \frac{m(Q')}{(S_a(Q'))^2}, \tag{7}$$

where $S_a(Q')$ and $S_b(Q')$ denote the distances between a-sides and b-sides in Q' respectively and $m(Q')$ the area of Q'.

The lemma and (4) imply that left one of b-sides in Q' must lie in $\{W=U+iV \mid U \leqq \log \lambda(K)c(r)r^{f(r)}, |V|>|\arg w(r)|-N_1\}$ when $|\arg w(r)|>N_1$. Put $N_2=(N_1+2\pi)\operatorname{sgn}\arg w(r)$, when $|\arg w(r)|>N_1+2\pi$. Then it follows by use of the Pythagorus equality that

$$(S_b(Q'))^2 \geqq |\arg w(r)-N_2|^2+|\log \lambda(K)c(r)r^{f(r)}|^2.$$

For $m(Q')$ we have by (4)

$$m(Q') \leqq 2\pi|\log \lambda(K)^{-1}c(r)r^{f(r)}|.$$

On using these inequalities, (7) becomes

$$\frac{|\arg w(r)-N_2|^2+\log \lambda(K)c(r)r^{f(r)}|^2}{2\pi|\log \lambda(K)^{-1}c(r)r^{f(r)}|} \leqq M(Q').$$

If we combine above inequality with (6), it holds that

$$\begin{aligned}|\arg w(r)-N_2|^2 \leqq f(r)(K-f(r))|\log r|^2+(2|\log \lambda(K)c(r)|f(r)\\ +K|\log \lambda(K)^{-1}c(r)|)|\log r|+|\log \lambda(K)c(r)|^2.\end{aligned}$$

Dividing both sides by $|\log r|^2$ and letting r tend to 0, we have

$$\varlimsup_{r\to 0}\left|\frac{\arg w(r)}{\log r}\right| \leqq \varlimsup_{r\to 0}\sqrt{f(r)(K-f(r))} \leqq \varlimsup_{r\to 0}\frac{K}{2}=\frac{K}{2}. \tag{8}$$

Next, we suppose that

$$\varlimsup_{r\to 0}\left|\frac{\arg w(r)}{\log r}\right|=\frac{1}{2}\left(K-\frac{1}{K}+\delta\right),\quad \delta \geqq 0.$$

We introduce an auxiliary K_1-quasiconformal mapping $w_1(re^{i\theta})=r^{\alpha}e^{i(\theta-\beta\log r)}$, where $\beta^2=-\alpha^2+\left(K_1+\frac{1}{K_1}\right)\alpha-1$, $\frac{1}{K_1}\leqq\alpha\leqq K_1$ and sgn $\beta=\pm 1$ according as

$$\overline{\lim_{r\to 0}}\frac{\arg w(r)}{|\log r|}=\frac{1}{2}\left(K-\frac{1}{K}+\delta\right) \text{ or } \underline{\lim_{r\to 0}}\frac{\arg w(r)}{|\log r|}=-\frac{1}{2}\left(K-\frac{1}{K}+\delta\right).$$

The composed KK_1-quasiconformal mapping $w\circ w_1$ maps $|z|<1$ onto $|w|<1$ so that $w\circ w_1(0)=0$ and $w\circ w_1(1)=1$. On account of (8) it satisfies

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w\circ w_1(r)}{\log r}\right|\leqq\frac{KK_1}{2}.$$

By (5) its argument is written in the form $\arg w\circ w_1(r)=\arg w(r^{\alpha})+\arg w_1(r)+A(r)$, where $A(r)$ satisfies $|A(r)|<N_1$. There is a sequence $\{r_i\}$ such that

$$\frac{KK_1}{2}\geqq\lim_{i\to\infty}\left|\frac{\arg w(r_i^{\alpha})+\arg w_1(r_i)+A(r_i)}{\log r_i}\right|=\overline{\lim_{r\to 0}}\left|\frac{\arg w(r^{\alpha})}{\log r}\right|+|\beta|$$
$$=\frac{\alpha}{2}\left(K-\frac{1}{K}+\delta\right)+|\beta|$$

or

$$\delta\leqq\frac{KK_1}{\alpha}-\frac{2|\beta|}{\alpha}-K+\frac{1}{K}.$$

If we put $\alpha=\frac{K^2}{K^2+1}K_1$, $\left(\alpha\geqq\frac{1}{K_1}\right)$ then

$$\frac{KK_1}{\alpha}=K+\frac{1}{K},\ \frac{2|\beta|}{\alpha}=\frac{2(K^2+1)}{K^2}\sqrt{\left(\frac{K}{K^2+1}\right)^2+\left(\frac{K^2}{K^2+1}-1\right)\frac{1}{K_1^2}}.$$

Letting $K_1\to\infty$ we have

$$\delta\leqq K+\frac{1}{K}-\frac{2(K^2+1)}{K^2}\frac{K}{K^2+1}-K+\frac{1}{K}=0$$

Hence

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w(r)}{\log r}\right|\leqq\frac{1}{2}\left(K-\frac{1}{K}\right).$$

The bound $\frac{1}{2}\left(K-\frac{1}{K}\right)$ is attained by $w(re^{i\theta})=r^{\alpha}e^{i(\theta-\beta\log r)}$, where $\alpha=\frac{1}{2}\left(K+\frac{1}{K}\right)$ and $\beta=\frac{1}{2}\left(K-\frac{1}{K}\right)$.

q. e. d.

3. Distortion of argument (2)

Theorem 2. *Suppose that $w(z)$ is a K-quasiconformal mapping of $|z|<1$ onto $|w|<1$ with $w(0)=0$ and $w(1)=1$ and that*

$$|w(r)| = c(r)r^a,\ \log c(r) = o(\log r)\,. \tag{9}$$

Then

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w(r)}{\log r}\right| \leqq \sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)}\,. \tag{10}$$

For each a this bound is best possible.

Proof. We remark first that a lies between $\frac{1}{K}$ and K. We use the same reasoning as in section 2 to obtain

$$|w(r)| = c(r)r^a,\ \log c(r) = o(\log r) \tag{3'}$$

$$\begin{aligned}&\max_{0\leqq\theta<2\pi}|w(re^{i\theta})| \leqq \lambda(K)c(r)r^a,\ \log c(r) = o(\log r)\\ &\min_{0\leqq\theta<2\pi}|w(re^{i\theta})| \geqq \lambda(K)^{-1}c(r)r^a,\ \log c(r) = o(\log r)\,.\end{aligned} \tag{4'}$$

Putting $f(r)=a$ in (8) we have

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w(r)}{\log r}\right| \leqq \sqrt{a(K-a)}\,. \tag{8'}$$

We suppose that

$$\overline{\lim_{r\to 0}}\left|\frac{\arg w(r)}{\log r}\right| = \sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)}+\delta,\ \delta\geqq 0\,.$$

Using an auxiliary K_1-quasiconformal mapping $w(re^{i\theta})=r^{\alpha}e^{i(\theta-\beta\log r)}$, we have in the same manner as in section 2

$$\alpha\left(\sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)}+\delta\right)+\sqrt{\alpha\left(K_1+\frac{1}{K_1}\right)-(\alpha^2+1)} \leqq \sqrt{a\alpha(KK_1-a\alpha)}\,,$$

because the composed KK_1-quasiconformal mapping satisfies (9) with $a\alpha$ in place of a. Therefore we have

$$\delta \leqq \sqrt{a\left(K\frac{K_1}{\alpha}-a\right)}-\sqrt{\frac{K_1}{\alpha}+\frac{1}{\alpha K_1}-\left(1+\frac{1}{\alpha^2}\right)}-\sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)}\,. \tag{11}$$

For $a\neq\frac{1}{K}$ we put $\alpha=\frac{aK-1}{a(K-K^{-1})}K_1$ and let $K_1\to\infty$, so that we have

$$\delta \leqq \sqrt{\frac{a^2K(K-a)}{aK-1}}-\sqrt{\frac{1-aK^{-1}}{aK-1}}-\sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)} = 0\,.$$

Hence we have the desired inequality (10). For $a=\frac{1}{K}$ (11) becomes

$$\delta \leqq \sqrt{\frac{K_1}{\alpha}-\frac{1}{K^2}}-\sqrt{\frac{K_1}{\alpha}+\frac{1}{\alpha K_1}-\left(1+\frac{1}{\alpha^2}\right)}. \tag{11'}$$

On putting $\alpha=\sqrt{K_1}$ and letting $K_1\to\infty$, we have also $\delta=0$. For each a the bound is attained by $w(re^{i\theta})=r^a e^{i(\theta-b\log r)}$, $b^2=a\left(K+\frac{1}{K}\right)-(a^2+1)$. q. e. d.

4. Some extremal mappings

Let $w(z)$ be an extremal quasiconformal mapping in $Q(w;\Delta,\Delta)$. If we map Δ conformally onto a Jordan region D by φ and map Δ onto an another Jordan region D' by ψ, then we obtain a boundary correspondence given by $\psi\circ w\circ\varphi^{-1}$ between D and D'. In $Q(\psi\circ w\circ\varphi^{-1}; D, D')$, $\psi\circ w\circ\varphi^{-1}$ is again extremal and if W is extremal in $Q(\psi\circ w\circ\varphi^{-1}; D, D')$ then $\varphi^{-1}\circ W\circ\psi$ is extremal in $Q(w;\Delta,\Delta)$. By this reason extremal quasiconformal mappings which we are going to deal with are those which have the boundary correspondence between D and D'.

Theorem 3. *Let $W_0(Z)=f(X)+i(Y+g(X))$ be a K-quasiconformal mapping of $D=\{Z=X+iY\,|\,X<0,\ 0<Y<2\pi\}$ such that $f(0)=g(0)=0$ and that $f(X)\to-\infty$ as $X\to-\infty$. If*

$$\overline{\lim_{X\to-\infty}}\left|\frac{g(X)}{X}\right|=\frac{1}{2}\left(K-\frac{1}{K}\right).$$

then $W_0(Z)$ is extremal in $Q(W_0; D, D')$, where $D'=W_0(D)$.

Proof. Let $W(Z)$ be a K'-quasiconformal mapping in $Q(W_0; D, D')$. We map D into Δ by $z=e^Z$ and D' into another Δ. Then $e^{W(\log z)}$ with $\log 1=0$ is a K'-quasiconformal mapping of Δ onto Δ with $e^{W(\log 0)}=0$ and $e^{W(\log 1)}=1$, because it is topological on Δ and K'-quasiconformal in $\Delta-\{z=x+iy\,|\,0\leqq x<1,\ y=0\}$, ([3], I. Satz 8, 3). Theorem 1 asserts that

$$\frac{1}{2}\left(K'-\frac{1}{K'}\right)\geqq\overline{\lim_{r\to0}}\left|\frac{\arg e^{W(\log r)}}{\log r}\right|=\overline{\lim_{r\to0}}\left|\frac{\arg e^{W_0(\log r)}}{\log r}\right|=\overline{\lim_{X\to-\infty}}\left|\frac{g(X)}{X}\right|$$
$$=\frac{1}{2}\left(K-\frac{1}{K}\right).$$

Therefore we have $K'\geqq K$ and it is shown that $W_0(Z)$ is extremal in $Q(W_0; D, D')$. q. e. d.

The complex dilatation of $W_0(Z)$ is of the form as follows;

$$\mu_{W_0}(Z)=\frac{f'(X)+ig'(X)-1}{f'(X)+ig'(X)+1}.$$

If $\mu_{W_0}(Z)=k\frac{\overline{\varphi(Z)}}{|\varphi(Z)|}$, $k=\frac{K-1}{K+1}$, with analytic φ in D, then arg φ=constant on each $X=c$, $-\infty<c<0$, and $\varphi=e^{aZ+b}$. But it is not difficult to see that $a=0$. We conclude that $W_0(Z)$ is not Teichmuller mapping except the case when $W_0(Z)$ is an affine mapping.

By the same resoning as before and by theorem 2 we obtain the following

Theorem 4. *Let $W_a(Z)=f(X)+i(Y+g(X))$ be K-quasiconformal mapping of $D=\{Z=X+iY\,|\,X<0,\ 0<Y<2\pi\}$ such that $f(0)=g(0)=0$, $f(X)\to\infty$ as $X\to-\infty$ and that $\lim_{X\to-\infty}\left|\frac{f(X)}{X}\right|=a$, $\frac{1}{K}\leqq a\leqq K$.* If

$$\overline{\lim_{X\to-\infty}}\left|\frac{g(X)}{X}\right|=\sqrt{a\left(K+\frac{1}{K}\right)-(a^2+1)},$$

then $W_a(Z)$ is extremal in $Q(W_a;\ D,\ D')$, where $D'=W_a(D)$.

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