EXTENDIBLE VECTOR BUNDLES OVER LENS SPACES MOD 3

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1. Introduction. In [5] Schwarzenberger investigated the problem of determining whether a real vector bundle over the real projective space $\mathbb{R}P^n$ can be extended to a real vector bundle over $\mathbb{R}P^m$ ($n < m$). In [3], he also investigated the case of the complex tangent bundle of the complex projective space.

The purpose of this note is to prove the non-extendibility of a bundle over lens spaces mod 3 by making use of Schwarzenberger's technique ([5]).

Let $S^{2n+1}$ be the unit $(2n+1)$-sphere. That is

$$S^{2n+1} = \{(z_0, \cdots, z_n); \sum_{i=0}^{n} |z_i|^2 = 1, z_i \in \mathbb{C} \text{ for all } i\}$$

Let $\gamma$ be the rotation of $S^{2n+1}$ defined by

$$\gamma(z_0, \cdots, z_n) = (e^{\pi i/p} z_0, \cdots, e^{\pi i/p} z_n).$$

Then $\gamma$ generates the differentiable transformation group $\Gamma$ of $S^{2n+1}$ of order $p$, and lens space mod $p$ is defined to be the orbit space $L^n(p) = S^{2n+1}/\Gamma$. It is a compact differentiable $(2n+1)$-manifold without boundary and $L^n(2) = \mathbb{R}P^{2n+1}$.

The Grothendieck rings $KO(L^n(p))$, $K(L^n(p))$ were determined by T. Kambe [4]. We recall them in 2. Let $\{z_0, \cdots, z_n\} \in L^n(p)$ denote the equivalence class of $(z_0, \cdots, z_n) \in S^{2n+1}$. $L^n(p)$ is naturally embedded in $L^{n+1}(p)$ by identifying $\{z_0, \cdots, z_n\}$ with $\{z_0, \cdots, z_n, 0\}$. Hence $L^n(p)$ is embedded in $L^m(p)$ for $n < m$.

Throughout this note we suppose $p = 3$. Now we state our theorems which shall be proved in 3 and 4.

Let $\xi$ be any $t$-dimensional real bundle over $L^n(3)$. Let $p(\xi)$ be the mod 3 Pontryagin class of $\xi$

$$p(\xi) = \sum_{j} p_j(\xi) \text{ where } p_j(\xi) = (-1)^j C_{\xi}(\xi \otimes C) \mod 3.$$
where $d_j \in \mathbb{Z}$ and $x$ is a generator of $H^2(L^n(3); \mathbb{Z})$. Then there exists an integer $s$ such that

$$p(\zeta) = 1 + d_1x^2 + \cdots + d_sx^{2s} \text{ for } 0 \leq 2s \leq t. \quad (1)$$

Then we have the following

**Theorem 1.** Let $\zeta$ be a $t$-dimensional real vector bundle over $L^n(3)$. If $2t < n+1$, then we have

$$p(\zeta) = \left(1 + x^2\right)^s \mod 3 \text{ for some integer } s \quad 0 \leq 2s \leq t.$$

**Corollary 2.** Under the assumptions of Theorem 1,

$$p(\zeta) = p(\eta_L^m \oplus \cdots \oplus \eta_L^n) \text{ for some } 0 \leq 2s \leq t,$$

where we denote by $\oplus$ a Whitney sum of $\eta_L^n$. (See [2] for the definition of $\eta_L^n$.)

For a pair $(X, Y)$ of compact spaces, a bundle $\zeta_Y$ over $Y$ is said to be extendible to $X$ provided there exists a bundle $\zeta_X$ over $X$ such that

$$\zeta_X|_Y \cong \zeta_Y,$$

where we denote by $|_Y$ the restriction to $Y$.

Let $a$ be a real number. We denote by $[a]$ the integral part of $a$. Let $b$ be an integer. We denote by $\nu_s(b)$ an integer $q$ such that

$$b = r \cdot 3^q, \quad \text{where } (r, 3) = 1.$$

For integers $t$ and $m$, define

$$\beta_s(t, m) = \text{Min}\left(\left(i - \left[\frac{i}{2}\right] - 1\right) - \nu_s\left(i - \left[\frac{i}{2}\right]\right) + \nu_s\left(\left(i - \left[i/2\right]\right)\right)\right)$$

where $t < i < m$, $i \equiv 0 \mod 2$ and $i \equiv 1 \mod 6$.

**Theorem 3.** Assume that $n$, $m$ and $t$ are the positive integers such that

- $(2) \quad 2t < m + 1$
- $(3) \quad n \equiv 0 \mod 4$
- $(4) \quad m \equiv 0 \mod 4$
- $(5) \quad \left[\frac{m}{2}\right] \geq \left[\frac{n}{2}\right] + \beta_s(t, m).$

Let $\zeta$ be a $t$-dimensional real vector bundle over $L^n(3)$ which is extendible to $L^m(3)$ ($n < m$). Then $\zeta$ is stably equivalent to
\[ \eta_{L_n} \oplus \cdots \oplus \eta_{L_n} \text{ for some integer } s (0 \leq 2s \leq t). \]

As an application of Th. 3, we obtain the following

**Theorem 4.** Let \( \zeta \) be a \( t \)-dimensional real vector bundle over \( L^n(3) \) \((n \equiv 0 \mod 4)\). Assume that \( \zeta \) is stably equivalent to

\[ \eta_{L_n} \oplus \cdots \oplus \eta_{L_n} \text{ for some } s \geq \left[ \frac{t}{2} \right]. \]

Then \( \zeta \) is not extendible to \( L^{(t,n)}(3) \), where

\[ \phi(t, n) = \text{Min} \left\{ m \geq 2t; m \equiv 0 \mod 4, \left[ \frac{m}{2} \right] - \beta(t, m) \geq \left[ \frac{n}{2} \right] \right\}. \]

Next we show

**Theorem 5.** The tangent bundle \( \tau(L^n(3)) \) of \( L^n(3) \) is not extendible to \( L^{(t,n)}(3) \) for \( n \equiv 0 \mod 4 \) and \( \tau(L^n(p)) \) is not extendible to \( L^{(t,n)}(3) \) \((n \equiv 0 \mod 4)\).

2. **The structure of \( \widetilde{KO}(L^n(p)) \).** The structure of \( \widetilde{KO}(L^n(p)) \) is stated as follows [4]. Let \( CP^n \) be the complex projective space of complex \( n \)-dimension. Let \( \eta \) be the canonical complex line bundle over \( CP^n \), \( r(\eta) \) the real restriction of \( \eta \). Consider the natural projection

\[ \pi: L^n(p) \to CP^n. \]

Define \( \eta_{L^n} = \pi^*(r(\eta)) \in KO(L^n(p)) \) where \( \pi^*: KO(CP^n) \to KO(L^n(p)) \) is the induced homomorphism of \( \pi \). Let \( \sigma \) denote the stable class of \( \eta_{L^n} \), i.e., \( \sigma = \eta_{L^n} - 2 \in KO(L^n(p)) \). We recall \( \tau_{L^n} = (n+1) \eta_{L^n} \) where \( \tau_{L^n} \) is the tangent bundle of \( L^n(p) \). The theorem of T. Kambe (Th. 2, [4]) is as follows:

**Theorem (Kambe).** Let \( p \) be an odd prime, \( q = (p-1)/2 \) and \( n = s(p-1)+r \) \((0 \leq r < p-1)\). Then

\[ \widetilde{KO}(L^n(p)) \cong \{ (Z_{p^r+1})^{[r/2]} + (Z_p)^{q-[r/2]} \cdots \text{ (if } n \equiv 0 \mod 4) \}
\[ Z_2 + (Z_{p^r+1})^{[r/2]} + (Z_p)^{q-[r/2]} \cdots \text{ (if } n \equiv 0 \mod 4) \]

and the direct summand \( (Z_{p^r+1})^{[r/2]} \) and \( (Z_p)^{q-[r/2]} \) are additively generated by \( \sigma_n, \ldots, \sigma_{n^{[r/2]}} \) and \( \sigma_n^{[r/2]+1}, \ldots, \sigma_n^q \) respectively. Moreover its ring structure is given by

\[ \sigma_n^{q+1} = \sum_{i=1}^{q} \frac{-(q+i-1)(q-i-1)}{(2i-1)(2i-2)} \sigma_n^i \sigma_n^{(n/2)+1} = 0. \]

In the theorem, \((Z_a)^b\) indicates the direct sum of \( b \)-copies of cyclic group of order \( a \). Let \( p = 3 \) in the above theorem. If \( n \equiv 0 \mod 4 \) then
\[ \widetilde{KO}(L^n(3)) \simeq \mathbb{Z}_3, \quad s = \left[ \frac{n}{2} \right]. \]

and \( \mathbb{Z}_3 \) is generated by \( \sigma_n \). Its ring structure is given by

\[ \sigma_n^2 = (-3) \sigma_n, \quad \sigma^{s+1} = 0. \]

3. The proofs of Theorem 1 and Corollary 2. From Th. 11.3 in [2], we obtain the following equality. (For the proof, see Proposition 5, in the last part of this section.) Let \( \delta_k^s \colon H^q(L^n(3); \mathbb{Z}_3) \rightarrow H^{q+k}(L^n(3); \mathbb{Z}_3) \) the \( k \)-th reduced power operation mod 3. Then we have

\[
(6) \quad \delta_k^s(p_s(\zeta)) = \left( \sum_{n+m=k} p_n(\zeta) p_m(\zeta) \right) p_s(\zeta) + \sum_{i>1} p_i(\zeta) (\cdots)
\]

for \( 0 \leq k \leq s \).

Let \( s \) be an integer such as (1) in 1. Since \( d_j \equiv 0 \) for all \( j > s \) and \( d_s \neq 0 \), then (6) gives

\[
(7) \quad \delta_k^s(p_s(\zeta)) = \left( \sum_{n+m=k} p_n(\zeta) p_m(\zeta) \right) p_s(\zeta).
\]

For an element \( x^{2s} \) of \( H^q(L^n(3); \mathbb{Z}_3) \), we have

\[
\delta_k^s(x^{2s}) = \left( \binom{2s}{k} \right) x^{2s+2k} \quad \text{and} \quad d_s \left( \binom{2s}{k} \right) x^{2s+2k} = \left( \sum_{n+m=k} d_n d_m \right) d_s x^{2s+2k}.
\]

From \( 2s + 2k \leq 4s \leq 2t < n+1, \ x^{2s+2k} \neq 0 \). Hence \( \binom{2s}{k} = \sum_{n+m=k} d_n d_m \).

By induction, we obtain \( d_j \equiv \binom{s}{j} \mod 3 \). Therefore

\[
p(\zeta) = 1 + \binom{s}{1} x^2 + \cdots + \binom{s}{s} x^{2s} \mod 3
\]

\[= (1 + x^2)^s \mod 3.\]

The proof of Theorem 1 is completed if we prove Proposition 5. Now, it is well known that the bundle \( \eta_{L^n} \) over \( L^n(3) \) has the total Pontryagin class mod 3 \( p(\eta_{L^n}) = 1 + x^2 \). Thus the proof of Corollary 2 is completed.

Now, in order to prove the formula (6) in the proof of Theorem 1, we consider a following symmetric polynomial. Let \( \sum x_1^2 x_2^2 \cdots x_{k+1}^2 \cdots x_s^2 \) be a homogeneous symmetric polynomial in variables \( x_1, x_2, \ldots, x_t \) of degree \( N = (p-1)k+s \) where \( p, k \) and \( N \) are positive integers.

To prove (6), we show the following propositions.
Proposition 1.

\[ \sum x_1^2 \cdots x_{k+1}^2 = \sum_{i=0}^{k} A(i) \sigma_{k-i} \sigma_{s+i} \]

where \( A(i) = (-1)^i \sum_{j=1}^{k} \left( s - k + 2j \right) A(i-j), \) \( 1 \leq i \leq k \) and \( A(0) = 1. \)

Proof. Put \( f(k, s) = \sum x_1^2 \cdots x_{k+1}^2. \) By an easy calculation,

\[ f(k, s) = (\sum x_1 \cdots x_k)(\sum x_1 \cdots x_s) - \sum_{i=1}^{k} \left( s - k + 2j_1 \right) f(k-j, s+j). \]

By making use of (8) repeatedly, we have

\[ f(k, s) = \sigma_k \sigma_s + \sum_{i=1}^{k} F_i \]

where \( F_i = (-1)^i \sum_{j=1}^{k-i} \cdots \sum_{j_{k-i} \leq s-j_1 \leq k-(l-i)} \left( s - k + 2j_1 \cdots j_{k-i} \right) \sigma_{k-j_1} \sigma_{s+j_1} \) and

\[ 1 \leq j_i \leq k - \sum_{i=1}^{k-i} j_i \leq k - (l-i). \] If \( l=k \), then \( k = \sum_{i=1}^{k} j_i. \)

Let \( A_i(i) \) be the coefficient of \( \sigma_{k-i} \sigma_{s+i} \) in \( F_i \), then

\[ A_i(i) = (-1)^i \sum_{j_1, \ldots, j_{k-i}} \left( s - k + 2j_1 \cdots j_{k-i} \right) \left( s - k + 2i \right). \]

Put \( A(i) = A_i(i) + \cdots + A_{k-i}(i). \) Then

\[ A(i) = (-1)^i \sum_{j_1, \ldots, j_{k-i}} \left( s - k + 2i \right) A_{i-j} - \left( s - k + 2i \right). \]

Since \( \sum_{i=1}^{k} A_i(j) \) is a coefficient \( A(j) \) of \( \sigma_{k-j} \sigma_{s+j} \) in \( f(k, s) \), we have

\[ A(i) = (-1)^i \sum_{j=1}^{k} \left( s - k + 2j \right) A(i-j). \]

This completes the proof of Proposition 1.

Proposition 2.

\[ \sum x_1^2 \cdots x_s^2 = \sum_{i=0}^{s} A(i) \sigma_{k-i} \sigma_{k+i} \]

where

\[ (a) \quad \sum_{j=0}^{s} A(i-j) \binom{2i}{j} = 0 \quad \text{and} \quad A(0) = 1 \]

\[ (b) \quad A(i) \equiv (-1)^{i+1} \mod 3 \quad \text{for} \quad i \neq 0. \]
Proof. The part of (a) is completed by Proposition 1. The proof of (b) is obtained by induction. For \( i=1, 2 \), \( A(1)=(-1)\left(\begin{array}{c} 2 \\ 1 \end{array}\right) \equiv 1 \) and \( A(2)=2 \).

Assuming the equation (b) for integers \( i \leq 2q \), we have

\[
A(i+1) = 2 \sum_{j=1}^{i} \binom{2i+2}{j} (-1)^{i-j+2} + 2 \binom{2i+2}{i+1}.
\]

By making use of \( \sum_{j=1}^{i} \binom{2i+2}{j} (-1)^{i-j+2} + 2 \binom{2i+2}{i+1} = 0 \), we have

\[
A(i+1) \equiv 1 \mod 3.
\]

Assuming the equation (b) for integers \( i \leq 2q+1 \), we can obtain \( A(i+1) \equiv 2 \mod 3 \). Thus Proposition 2 is obtained by induction.

**Proposition 3.**

\[
\sum x_1^2 x_2^2 \cdots x_k^2 x_{k+1} \cdots x_s = (\sum x_1^2 \cdots x_k^2) \sigma_s - \sum_{i=1}^s \sigma_i(\cdots).
\]

**Proof.** Put \( f(a, b) = \sum x_1^2 \cdots x_k^2 x_{k+1} \cdots x_s \) with \( c=N-2a-b \). By calculation, we have the following equality;

\[
f(k, 0) = (\sum x_1^2 \cdots x_k^2) \sigma_s - \sum_{i=1}^s f(k-a_i, \alpha_i).
\]

Define \( a_0=k, b_0=0 \) and \( c_0=N-2a_0-b_0 \). Then (9) is reformed as follows:

\[
f(a_0, b_0) = f(0, a_0) \sigma_{c_0} - \sum_{a_i=1}^{a_0} f(a_1, b_1) \text{ where } a_i = a_0 - \alpha_i, \quad b_i = \alpha_i + \beta_i(\beta_i=0)
\]

Now for each term \( f(a_i, b_i) \) in (10), we obtain

\[
f(a_i, b_i) = f(0, a_i) \sigma_{c_i} - \sum_{a_i=1}^{a_0} \sum_{b_i=0}^{b_i} A(\alpha_i, \beta_i) f(a_i, \beta_i)
\]

for some integers \( A(\alpha_i, \beta_i) \) and \( A(0, b_i)=0 \). We can inductively define two sequences \( \{a_i\}, \{b_i\} \) satisfying the followings

\[
f(a_{i-1}, b_{i-1}) = f(0, a_{i-1}) \sigma_{c_{i-1}} - \sum_{a_i=0}^{a_{i-1}} \sum_{\beta_i=0}^{b_{i-1}} A(\alpha_i, \beta_i) f(a_i, \beta_i)
\]

with some integers \( A(\alpha_i, \beta_i) \) and \( A(0, b_i)=0 \). Put \( c_{i-1}=N-2a_{i-1}-b_{i-1} \). Then we have \( c_1 < c_2 < \cdots < c_i < \cdots \). From (13), \( a_i \leq a_i \) for all \( i \). Hence consider the following cases:

(14) there exists an integer \( n \) such as \( a_i < a_i \) for all \( i \geq n \),

(15) there exists an integer \( m \) such as \( a_m = \cdots = a_i = \cdots \) for all \( i \geq m \).
If (14) is satisfied, then \( a_q = 0 \) for some integer \( q \). From (12) and Proposition 1, we have

\[
(16) \quad f(a_{q-1}, b_{q-1}) = f(0, a_{q-1}) \sigma_{e_{q-1}} \sum_{\beta_0=1}^{k_{q-1}} A(0, \beta_q)f(0, b_q)
\]

\[
= f(0, a_{q-1}) \sigma_{e_{q-1}} - \sum_{\beta_0=1}^{k_{q-1}} \sum_{i=0}^{k_{q-1}} A(0, \beta_q) A(i) \sigma_{b_{q-i} \sigma_{e_{q+i}}}
\]

If (15) is satisfied, then \( b_i > b_{i+1} \) for all \( i \geq m \). Therefore \( b_r = 0 \) for some integer \( r \). From (12) we have

\[
f(a_{r-1}, b_{r-1}) = f(0, a_{r-1}) \sigma_{a_{r-2}} - \sum_{\alpha_q=0}^{a_{r-1}} A(\alpha_r, 0)f(\alpha_r, 0).
\]

Since \( a_r < a_q \), the above discussion is also applied to \( f(a_r, 0) \) in this case. Hence, by making use of (9) repeatedly, we have finally

\[
f(k, 0) = f(0, k) \sigma_z - \sum_{i=1}^{i \geq k} (\cdots) \sigma_i.
\]

From Proposition 2 and Proposition 3 we have the following

**Proposition 4.**

\[
\sum x_1^2 \cdots x_k^2 x_{k+1} \cdots x_s = (\sigma^2 + \sum_{j=1}^{k} (-1)^{i+1} \sigma_{k-i} \sigma_{k+i}) \sigma_z + \sum_{i=1}^{s} (\cdots) \sigma_i.
\]

Now, we can prove the formula (6) in the proof of Theorem 1.

**Proposition 5.** \( \partial^s_\delta(p_s(\xi)) = (\sum_{n \geq m \geq s} \partial^s_{\delta}(\xi) \partial^m(\xi)) \partial_s(\xi) + \sum_{i=1}^{\ldots} \partial_i(\xi)(\cdots) \).

Proof. Let \( C_i \in H^s(B_{U(\xi)} : Z_3) \) be the \( i \)-th Chern class mod 3. By Th. 11.3 ([2]) and Proposition 4, we have

\[
(17) \quad \partial^s_\delta(C_i) = (C_i^2 + \sum_{j=1}^{k} (-1)^{k+j} C_{k-j} C_{k+j}) C_i + \sum_{i=1}^{\ldots} C_i.
\]

Let \( p_s(\xi) \) be the \( s \)-th Pontrjagin class mod 3 of a real bundle \( \xi \). Then \( p_s(\xi) = (-1)^s C_{2s}(\xi \otimes C) \mod 3 \) where \( C_{2s}(\xi \otimes C) \) is a 2s-th Chern class of \( \xi \otimes C \). From (17) we obtain

\[
\partial^s_\delta(p_s(\xi)) = (p_s^2(\xi) + 2 \sum_{j=1}^{k} p_{i+j}(\xi) p_{i-j}(\xi)) p_s(\xi) + \sum_{i=1}^{\ldots} p_i(\xi)
\]

and \( \partial^s_\delta(p_s^2(\xi)) = 2(\sum_{j=1}^{k} p_{i-j}(\xi) p_{i+j}(\xi)) p_s(\xi) + \sum_{i=1}^{\ldots} p_i(\xi) \). This completes the proof of Proposition 5.

4. Proofs of Theorem 3, 4 and 5. To prove Theorem 3, we discuss the following lemmas. The proofs of Lemma 1, 2 and 3 are omitted.
Lemma 1. Let \( A_0, A_1, \ldots, A_n \) be integers with \( \nu_3(A_j) > 0 \) for all \( j \neq n \) and \( \nu_3(A_n) \geq 0 \). If \( \nu_3(A_n) < \nu_3(A_j) \) for all \( j \neq n \), then

\[
\nu_3\left( \sum_{j=0}^{n} A_j \right) = \nu_3(A_n).
\]

Lemma 2. If \( r, s, a \) and \( u \) are positive integers with \( s < a < 3^u \) and \( (r, 3) = 1 \) then the following hold.

\[
\begin{align*}
(18) & \quad \nu_3\left( \frac{rs^u + s}{a} \right) = \nu_3\left( \frac{3^u + s}{a} \right) \\
(19) & \quad \nu_3\left( \frac{3^u}{a} \right) = u - \nu_3(a).
\end{align*}
\]

Lemma 3. If \( u \) and \( n \) are positive integers, then

\[
\begin{align*}
(20) & \quad \nu_3((3^u)!) = \frac{3^u - 1}{2} \\
(21) & \quad \nu_3((2n+1)!) \leq n \\
(22) & \quad \nu_3((2n)!) < n.
\end{align*}
\]

Put \( A_j = (-1)^j(-3)^{i-j} \binom{q}{i-j} \left( \binom{i-j}{j} \right) (j = 0, 1, \ldots, \left[ \frac{i}{2} \right]) \) for some positive integers \( q, i > 2 \) with \( q > i - j \).

Lemma 4. Let \( A_j \) be above integers. Then

\[
\nu_3\left( \sum_{j=0}^{i/3} A_j \right) = \nu_3(A_{i/3}) \quad \text{for } i \equiv 1 \text{ mod } 6 \text{ and } i \equiv 0 \text{ mod } 2.
\]

Proof. If \( i = 2n \), then for each \( l = 1, 2, \ldots, n-1 \)

\[
\nu_3(A_{n-l}) = (n+l-1) - \nu_3((2l)!) - \nu_3((n-l)!) + \nu_3(q) + \cdots + \nu_3(q-n-l+1).
\]

From Lemma 3 (22) \( \nu_3(A_{n-l}) > \nu_3(A_n) > l - \nu_3((2l)!) \). Then we have

\[
\nu_3(A_j) > \nu_3(A_n) \quad \text{and} \quad \nu_3(A_j) > 0 \quad \text{for all } j \neq n.
\]

Therefore by Lemma 1 we obtain \( \nu_3\left( \sum_{j=0}^{i/3} A_j \right) = \nu_3(A_{i/3}) \) for \( i \equiv 0 \text{ mod } 2 \).

From Lemma 3 (21), we obtain

\[
\nu_3(A_{n-l}) - \nu_3(A_n) > \nu_3(n!) - \nu_3((n-l)!) > 0
\]

under the condition \( i \equiv 1 \text{ mod } 6, \left[ \frac{i}{2} \right] = n = 3m \).

Now we prove the theorems.
Proof of Theorem 3. Let $\zeta'$ be the extension over $L^m(3)$ of $\zeta$. By the structure of $\widetilde{KO}$-ring of the lens space ([4]), $\zeta'$ is stably equivalent to $q\eta_{L^m}$, for some $q \in \mathbb{Z}_{\leq 2l}$. Since $\zeta' - t = q\eta_{L^m} \in \widetilde{KO}(L^m(3))$, we have

\[ \zeta' - t = q(i*\eta_{L^m} - 2) \in \widetilde{KO}(L^m(3)) \]

where $i* : \widetilde{KO}(L^m(3)) \to \widetilde{KO}(L^n(3))$ is the induced homomorphism of natural embedding $i : L^n(3) \to L^m(3)$. If $2q \leq t$, then $\zeta$ is stably equivalent to $\eta_{L^n} \oplus \ldots \oplus \eta_{L^n}$ for some integer $q$ ($0 \leq 2q \leq t$). If $2q > t$, $\gamma'(q\sigma_m) = 0$ for all $i \geq \dim(q\sigma_m)$ ([1] Prop. 2.3). Since $t \geq \dim(q\sigma_m)$, we have

\[ \gamma'(q\sigma_m) = 0 \quad \text{for all } i > t. \]

According to the Theorem of Kambe ([4] Lemma 4.8),

\[ \gamma_*(q\sigma_m) = (1 + \sigma_m(t - t^2))^q \]

\[ = \sum_{\alpha=0}^q \left( \sum_{j=0}^{[\alpha/2]} A_j \right) \sigma_m t^\alpha \]

where $A_j = (-1)^j(-3)^{\alpha - j} \binom{q}{\alpha - j} \binom{\alpha - j}{j}$. Then we have $\gamma'(q\sigma_m) = \sum_{j=0}^{[\alpha/2]} A_j \sigma_m$. From (23),

\[ \sum_{j=0}^{[\alpha/2]} A_j \sigma_m = 0 \in \widetilde{KO}(L^m(3)) = \mathbb{Z}_{\leq 2l} \quad \text{for all } i > t. \]

Therefore

\[ \nu_3(\sum_{j=0}^{[\alpha/2]} A_j) \geq \left\lceil \frac{m}{2} \right\rceil \quad \text{for all } i > t. \]

Now, according to Lemma 4, we have

\[ \nu_3(\sum_{j=0}^{[\alpha/2]} A_j) = \nu_3(A_{t/3}) \quad \text{for } i > t \ (i \equiv 0 \mod 2 \text{ and } i \equiv 1 \mod 6) \]

And so we have

\[ (i - \left\lfloor \frac{i}{2} \right\rfloor - 1) + \nu_3 \left( \binom{q}{i/2} \right) + \nu_3 \left( \binom{i - \left\lceil \frac{i}{2} \right\rceil}{i/2} \right) \geq \left\lceil \frac{m}{2} \right\rceil \quad \text{for } i > t, \ i \equiv 0 \mod 2 \]

and $i \equiv 1 \mod 6$.

Now the total Pontrjagin class mod 3 of $q\eta_{L^m}$ is given by the equation $p(q\eta_{L^m}) = (1 + x^2)^q$. Since $m > 2t - 1$, Theorem 1 implies that there exists an integer $s$ such that

\[ p(\xi') = (1 + x^2)^s, \quad 0 \leq 2s \leq t. \]

Hence we have
$(1+\varepsilon)^r \equiv (1+\varepsilon)^s \mod 3$, i.e.,
$$1+\left(\frac{q-s}{1}\right)x^2+\cdots+\left(\frac{q-s}{[m/2]}\right)x^{[m/2]} \equiv 1 \mod 3.$$ 

This implies that there exists an integer $u$ such that

$$q-s = 3^ru, \ (r, 3) = 1 \ \text{and} \ 3^u > [m/2].$$

Then we obtain the following

$$\nu_s\left(\begin{pmatrix} q \\ i-\lfloor i/2 \rfloor \end{pmatrix}\right) = \nu_s\left(\begin{pmatrix} 3^u+s \\ i-\lfloor i/2 \rfloor \end{pmatrix}\right)$$

$$\leq \nu_s\left(\begin{pmatrix} 3^u \\ i-\lfloor i/2 \rfloor \end{pmatrix}\right) \ \text{for} \ t < i < m \ (\text{by Lemma 2})$$

$$= u - \nu_s(i-\lfloor i/2 \rfloor).$$

Hence from (25) $u+(i-\lfloor i/2 \rfloor)-\nu_s(i-\lfloor i/2 \rfloor)+\nu_s\left(\begin{pmatrix} i-\lfloor i/2 \rfloor \end{pmatrix}\right) \geq \left\lfloor \frac{m}{2} \right\rfloor$ for $t < i < m$ and $i \equiv 0 \mod 2$, $i \equiv 1 \mod 6$. By the assumption (5) of Theorem 3, we have

$$u \geq [m/2] - \text{Min} \left[ \left( i-\lfloor i/2 \rfloor - 1 \right) - \nu_s(i-\lfloor i/2 \rfloor) + \nu_s\left(\begin{pmatrix} i-\lfloor i/2 \rfloor \end{pmatrix}\right) \right]$$

$$= [m/2] - \beta_s(t, m) \geq \left[ \frac{n}{2} \right].$$

According to (23), (26) and (27), there exists an integer $s$ such that

$$0 \leq 2s \leq t,$$

$$\xi - t = (r3^u+s)\sigma_n$$

$$= s\sigma_n.$$

This completes the proof of Theorem 3.

Proof of Theorem 4. By the contraposition of Theorem 3 and the main theorem of Kambe ([4] Th. 2), it is clear.

Proof of Theorem 5. Since $\tau(L^n(3)) \oplus 1 = (n+1)\eta_{L^n}$ and $n+1 > n = \left\lceil \frac{2n+1}{2} \right\rceil$ $= [1/2 \dim \tau(L^n(3))]$, Theorem 4 implies that the tangent bundle $\tau$ is not extendible to $L^{\oplus (n+1,n)}(3)$. For every $m > 2n+1$, $\beta_s(2n+1, m) \leq n$ whenever $n \equiv 0 \mod 3$, $n \equiv 1 \mod 3$ $\beta_s(2n+1, m) < n$ whenever $n \equiv 2 \mod 3$. Then $\phi(2n+1, n)$ $= 2(2n+1)$.

This completes the proof of Theorem 5.
REMARK. The following table shows the value of $\phi(t, n)$ where $1 \leq t \leq 10$ and $1 \leq n \leq 16$.

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References
