Shibata, K.
Osaka J. Math.
8 (1971), 219-232

# ON BOARDMAN'S GENERATING SETS OF THE UNORIENTED BORDISM RING

KATSUYUKI SHIBATA

(Received September 24, 1970)

## Introduction

For a pointed finite $CW$ pair $(X, A)$, define as usual the $k$-dimensional unoriented cobordism group $\mathfrak{N}^k(X, A)$ of $(X, A)$ by

$$\mathfrak{N}^k(X, A) = \varinjlim_n [S^{n-k}(X/A), MO(n)],$$

and denote $\sum_{-\infty<k<\infty} \mathfrak{N}^k(X, A)$ by $\mathfrak{N}^*(X, A)$.

We identify the coeficient ring $\mathfrak{N}^*$ with the unoriented bordism ring $\mathfrak{N}_*$ by the Atiyah-Poincaré duality [2]

$$D: \mathfrak{N}_k \rightarrow \mathfrak{N}^{-k}.$$

Let $P_n$ be the $n$-dimensional real projective space and $\eta_n$ be the canonical line bundle over $P_n$. Define

$$\mathfrak{N}^*(BO(1)) = \varprojlim_n \mathfrak{N}^*(P_n) \cong \mathfrak{N}_*[[W_1]],$$

where $W_1 = \varprojlim_n W_1(\eta_n)$ is the cobordism first Stiefel-Whitney class [4]. On account of the Kunneth formula, the homomorphism

$$\mu_{m,n}^*: \mathfrak{N}^*(P_{m+n}) \rightarrow \mathfrak{N}^*(P_m \times P_n)$$

induced by a continuous map $\mu_{m,n}$ satisfying $\mu_{m,n}^* \eta_{m+n} \cong \pi_1^* \eta_m \otimes \pi_2^* \eta_n$ gives rise to the comultiplication

$$\mu^*: \mathfrak{N}^*(BO(1)) \rightarrow \mathfrak{N}^*(BO(1)) \underset{\mathfrak{N}_*}{\otimes} \mathfrak{N}^*(BO(1)).$$

Let

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots \quad (z_i \in \mathfrak{N}_i)$$

be a primitive element in $\mathfrak{N}^*(BO(1))$ with respect to this comultiplication. Such elements exist ([3]). Fix once and for all a primitive element $P$ of such kind.

Following Novikov [8, appendix II], we define in section 1 a cobordism stable operation $\Phi_P$ which is a multiplicative projection characterised by the formula

$$\Phi_P(W_1) = P .$$

The restriction of the natural transformation

$$\mu \,|\, \text{Image } \Phi_P : \text{Image } \Phi_P \to H^*(X, A ; Z_2)$$

is a natural ring isomorphism in the category of finite $CW$ pairs. And this induces a natural $\mathfrak{N}_*$-algebra isomorphism

$$\mathfrak{N}^*(X, A) \cong \mathfrak{N}_* \hat{\otimes} H^*(X, A ; Z_2) .$$

Conversely, any such natural isomorphism, commuting with suspensions, is induced by $\Phi_P$ for some choice of a primitive element $P$.

In section 2, we study the relation between the operations $S_\omega$ and $\bar{S}_\omega$ defined in [8]. The result is applied in section 3 to prove that the coefficient $z_{2k}$ of a primitive element $P$ is the bordism class $[P_{2k}]$ of the real projective space for each $k \geqq 0$.

And the coefficient $z_{4k+1}$ is shown to be the class $[P(1, 2k)]$ of Dold manifold [5] in section 4.

The coefficients $z_i$ of dimensions $i$ other than $2k$ and $4k+1$ are expressed as very complicated polynomials in the generators of Dold [5] or of Milnor [7].

The present paper is motivated by the following classification theorem stated in the proof of Theorem 8.1 in [3].

**Theorem. P.** (Boardman [3])

*For an arbitrary family of decomposable elements $\{y_{2^i-1} ; y_{2^i-1} \in \mathfrak{N}_{2^i-1}, i \geqq 1\}$, there exists one and the only one primitive element*

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots .$$

in $\mathfrak{N}^*(BO(1))$, *satisfying*

$$z_{2^i-1} = y_{2^i-1} \quad (i \geqq 1) .$$

*The coefficients $z_{k-1}$ with $k$ not a power of 2 are a set of polynomial generators for $\mathfrak{N}_*$.*

*Moreover, if $z_{2^i-1} = z'_{2^i-1}$ for $1 \leqq i \leqq n$ for primitive elements*

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots$$

*and*

$$P' = W_1 + z'_2 W_1^3 + z'_4 W_1^5 + z'_5 W_1^6 + z'_6 W_1^7 + z'_7 W_1^8 + \cdots$$

*then* $z_{k-1}=z'_{k-1}$ *for k not a multiple of* $2^{n+1}$.

The author wishes to thank Professors M. Nakaoka and F. Uchida for their advices and encouragement.

## 1. Operation $\Phi_P$

Let $\mathcal{A}^*(0)=\sum_{-\infty<i<\infty}\mathcal{A}^i(0)$ denote the ring of stable operations in the unoriented cobordism theory. There is an isomorphism of $\mathfrak{N}_*$-modules ([6], [8])

$$\Psi : \mathcal{A}^*(0) \to \mathfrak{N}_* \hat{\otimes} Z_2[[W_1, W_2, \cdots, W_k, \cdots]],$$

where $\mathfrak{N}_*$ is identified with $\mathfrak{N}^*$ by the duality and $\hat{\otimes}$ denotes the complete tensor product.

For a partition $\omega=(i_1, i_2, \cdots, i_r)$, denote $W_\omega$ the symmetrized monomial of the $W_k$ and the operation $S_\omega\in\mathcal{A}^*(0)$ is defined by $S_\omega=\Psi^{-1}(W_\omega)$.

For a primitive element

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

in $\mathfrak{N}^*(BO(1))$ and for a partition $\omega=(i_1, i_2, \cdots, i_r)$, we denote the product $z_{i_1}\cdot z_{i_2}\cdots z_{i_r}$ as $z_\omega^{(P)}$.

Following the line of Novikov [8; appendix II], we define an operation $\Phi_P\in\mathcal{A}^0(0)$ by

$$\Phi_P = \sum_\omega z_\omega^{(P)} S_\omega ,$$

where the summation runs through all the partitions.

**Lemma 1.1.**

(1) $\Phi_P(x \cdot y) = \Phi_P(x)\cdot\Phi_P(y)$.

(2) $\Phi_P(z_0) = z_0$ *for* $z_0\in\mathfrak{N}_0$ *and*

$\Phi_P(y) = 0$ *for* $y\in\mathfrak{N}_i$ $(i>0)$.

(3) $(\Phi_P)^2 = \Phi_P$.

Proof.

(1). By the definition of $\Phi_P$ and from the Cartan formula for $S_\omega$ ([6], [8]), part (1) is easily derived.

(2). It is obvious by definition that $\Phi_P(z_0)=z_0$ for $z_0\in\mathfrak{N}_0$.

It is known that $S_\omega(W_1)=W_1^{k+1}$ if $\omega=(k)$ for some $k\geqq 0$ and that $S_\omega(W_1)=0$ otherwise ([6], [8]). Thus $\Phi_P(W_1)=P$. By the naturality of $\Phi_P$, $(\Phi_P)^2(W_1)=\Phi_P(P)$ is also a primitive element with the leading term $W_1$. So it follows from Theorem $P$ in the introduction together with the fact that $\mathfrak{N}_1\cong\mathfrak{N}_3\cong\{0\}$ that

$$(\Phi_P)^2(W_1)-\Phi_P(W_1)=\sum_{j\geqq 1} y_{8j-1}W_1^{8j}$$

for some decomposable elements $y_{8j-1}\in\mathfrak{N}_{8j-1}$.

On the other hand,

$$\begin{aligned}(\Phi_P)^2(W_1)-\Phi_P(W_1) &= \Phi_P(W_1+\sum_{k\geqq 3} z_{k-1}W_1{}^k)-\Phi_P(W_1)\\ &= \sum_{k\geqq 3}\Phi_P(z_{k-1})(W_1+\sum_{l\geqq 3} z_{l-1}W_1{}^l)^k\,.\end{aligned}$$

Comparing both formulas, we see that $\Phi_P(z_{k-1})=0$ for $k\leqq 7$. So $\Phi_P(z_{8-1})=0$ since $z_7$ is decomposable. So $y_7=0$ and it follows Theorem $P$ that $y_{16j+7}=0$ for all $j\geqq 0$. Repeting this procedure, we can inductively deduce that $\Phi_P(z_{k-1})=0$ for all $k\geqq 3$. At the same time we have proved that $(\Phi_P)^2(W_1)=\Phi_P(W_1)$.

Now $(\Phi_P)^2$ is also a multiplicative operation. As in the weakly complex case ([8]), a multiplicative operation of the unoriented cobordism theory is easily seen to be uniquely determined by its value on $W_1$. Therefore $(\Phi_P)^2=\Phi_P$. This completes the proof of Lemma 1.1.

**Notation.** For a partition $\omega=(i_1, i_2, \cdots, i_r)$, let $||\omega||=i_1+i_2+\cdots+i_r$ be its degree and $|\omega|=r$ its length. And we call $\omega$ *non-dyadic* if none of the component $i_k$ of $\omega$ is of the form $2^m-1$.

**Theorem 1.2.** *On the category of finite pointed CW pairs and continuous maps, there is a natural direct sum splitting as a graded $Z_2$-vector space*

$$\mathfrak{N}^*(X, A)=\bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))\,,$$

*where* (1) *the restriction*

$$\mu\,|\,\text{Image } \Phi_P : \Phi_P(\mathfrak{N}^*(X, A))\to H^*(X, A\,;\,Z_2)$$

*is a natural $Z_2$-algebra isomorphism, and* (2) *the scalar multiplication*

$$z_\omega^{(P)}\cup : \Phi_P(\mathfrak{N}^*(X, A))\to z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$$

*is a graded $Z_2$-module isomorphism of degree $-||\omega||$ if $\omega$ is non-dyadic.*

*Therefore we obtain a natural equivalence of graded $\mathfrak{N}_*$-algebras*

$$\mathfrak{N}^*(X, A)\xrightarrow[\cong]{}\mathfrak{N}^*\mathbin{\hat{\otimes}} H^*(X, A\,;\,Z_2)$$

*which commutes with suspension.* (*Suspension $S$ and a bordism element $x$ act on the right by $S(y\hat{\otimes}a)=y\hat{\otimes}S(a)$ and $x(y\hat{\otimes}a)=x\cdot y\hat{\otimes}a$, respectively.*)

*Moreover, the converse holds; such an equivalence is induced by* $\bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P$ *for some choice of a primitive element $P$.*

For the proof of the above theorem, we need the following operations which are just the unoriented analogue of those defined in [8].

**Lemma 1.3.**

*For an indecomposable element* $y_i \in \mathfrak{N}_i$, *define an operation* $\Delta_{y_i} = \sum_{k\geqq 1} y_i^{k-1} S_{(i)^k}$. ($(i)^k = (i, i, \cdots, i)$ ; *the k copies of i*)

*Then*

$$\Delta_{y_i}(a \cdot b) = \Delta_{y_i}(a) \cdot b + a \cdot \Delta_{y_i}(b) + y_i \cdot \Delta_{y_i}(a) \cdot \Delta_{y_i}(b)$$

*and, in particular,*

$$\Delta_{y_i}(y_i \cdot a) = a .$$

The proof of the lemma is straightforward from the definition of $\Delta_{y_i}$ and the fact that $S_{(i)}(y_i) = 1 \in Z_2$.

Proof of Theorem 1.2.

First we prove property (1). By (2) of Lemma 1.1, property (1) holds for $(X, A) = (S^0, P)$. Since $\Phi_P$ commutes with suspensions, (1) also holds for $(X, A) = (S^n, P)$ for $n \geqq 1$. Since $\Phi_P$ is a projection, $\Phi_P(\mathfrak{N}^*( \ , \ ))$ is also a cohomology theory. So the general cases are proved by induction on the number of cells in $X - A$, using the five lemma.

Next we prove property (2). The multiplication

$$z_\omega^{(P)} \cup : \Phi_P(\mathfrak{N}^*(X, A)) \to z_\omega^{(P)} \Phi_P(\mathfrak{N}^*(X, A))$$

is obviously a graded $Z_2$-module epimorphism of degree $-||\omega||$.

Suppose $z_\omega^{(P)} \cdot a = 0$ for $a \in \Phi_P(\mathfrak{N}^*(X, A))$ and for a non-dyadic $\omega$. Order the components of $\omega = (i_1, i_2, \cdots, i_r)$ as $i_1 \leqq i_2 \leqq \cdots \leqq i_r$ and define the operation $\Delta_{z\omega}^{(P)}$ by

$$\Delta_{z\omega}^{(P)} = \Delta_{z_{i_1}} \circ \Delta_{z_{i_2}} \circ \cdots \circ \Delta_{z_{i_r}} .$$

Then $a = \Delta_{z\omega}^{(P)}(z_\omega^{(P)} \cdot a) = \Delta_{z\omega}^{(P)}(0) = 0$ by Lemma 1.3. This proves property (2).

Totally order the set of all non-dyadic partitions by $\omega' < \omega$ if (*a*) $||\omega'|| < ||\omega||$ or (*b*) $||\omega'|| = ||\omega||$ and $i_r = j_s, \cdots, i_{r-m+1} = j_{s-m+1}, i_{r-m} > j_{s-m}$ for some $m \geqq 0$, where $\omega' = (i_1, i_2, \cdots, i_r)$ and $\omega = (j_1, j_2, \cdots, j_s)$ with $i_1 \leqq i_2 \leqq \cdots \leqq i_r$ and $j_1 \leqq j_2 \leqq \cdots \leqq j_s$.

We show that

$$\Phi_P \Delta_{z\omega'}^{(P)}(z_\omega^{(P)} \Phi_P(y)) = 0$$

for any homogeneous element $y$ if $\omega' < \omega$. In case $||\omega'|| < ||\omega||$, Lemma 1.3 implies that

$$\Phi_P \Delta_{z\omega'}^{(P)}(z_\omega^{(P)} \Phi_P(y)) = \Phi_P(\sum_i u_i \cdot y_i)$$

for some elements $u_i \in \mathfrak{N}_*$ and $y_i \in \Phi_P(\mathfrak{N}^*(X, A))$ with $\dim u_i \geqq ||\omega|| - ||\omega'|| > 0$. Thus, by Lemma 1.1 (1), (2),

$$\Phi_P(\sum_i u_i y_i) = \sum_i \Phi_P(u_i)\Phi_P(y_i) = 0\,.$$

In case $||\omega'||=||\omega||$ and $i_r=j_s, \cdots, i_{r-m}>j_{s-m}$,

$$\begin{aligned}&\Phi_P\Delta_{z\omega'}^{(P)}(z_\omega^{(P)}\Phi_P(y))\\ &= \Phi_P\Delta_{z(i_1,\cdots,i_{r-m-1})}^{(P)}(z_{j_1}\cdots z_{j_{s-m}}\Delta_{z i_{r-m}}\Phi_P(y)) = 0\,.\end{aligned}$$

The last equality follows from the preceding case.

Let $\sum_{\omega'<\omega} z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$ be the graded vector space spanned by all $z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$ with $\omega'<\omega$.

It follows from the above fact that

$$\sum_{\omega'<\omega} z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*X, A))\frown z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = 0$$

for each $\omega$, so that there is a direct sum splitting

$$\sum_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = \bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))\,.$$

Since it can be proved similarly as above that Image $(\Phi_P\circ\Delta_{z\omega}^{(P)})$ $=$Image $\Phi_P$ for each non-dyadic $\omega$, we have proved that there is a natural linear endomorphism of degree zero

$$\sum_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P\Delta_{z\omega}^{(P)}: \mathfrak{N}^*(X, A) \to \bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))\subset\mathfrak{N}^*(X, A)\,.$$

It is clearly an automorphism for $(X, A)=(S^0, P)$ and therefore an automorphism for every finite $CW$ pair by the effect of suspensions and of the five lemma. Thus

$$\bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = \mathfrak{N}^*(X, A)\,.$$

Since $z_\omega^{(P)}\Phi_P(y)\cdot z_{\omega'}^{(P)}\Phi_P(y') = z_{\omega\omega'}^{(P)}\Phi_P(y\cdot y')$, we have obtained a natural equivalence of graded $\mathfrak{N}_*$-algebras

$$\Theta_P: \mathfrak{N}^*(X, A) \cong \mathfrak{N}^* \hat{\otimes} H^*(X, A\,; Z_2)$$

which commutes with suspension.

Conversely, each such equivalence $\Theta$ induces a natural monomorphism of a graded $Z_2$-algebra

$$\lambda = \Theta^{-1}|H^*(X, A\,; Z_2): H^*(X, A\,; Z_2) \to \mathfrak{N}^*(X, A)\,.$$

Then the composition $\lambda\circ\mu$ is a stable miltiplicative operation in $\mathcal{A}^*(0)$ and $\lambda\circ\mu(W_1)=\lambda(w_1)=P$ is a primitive element in $\mathfrak{N}^*(BO(1))$. And the element $P$ has the leading term $W_1$ since

$$\Theta: \mathfrak{N}_*[[W_1]] \to \mathfrak{N}_* \hat{\otimes} Z_2[[w_1]]$$

is an $\mathfrak{N}_*$-algebra isomorphism. Therefore

$$\Theta = \bigoplus_{\omega;\,\text{non-dyadic}} \{1 \hat{\otimes} (\mu \,|\, \text{Image } \Phi_P)\}:$$
$$\mathfrak{N}^*(X, A) = \bigoplus_{\omega;\,\text{non-dyadic}} z_\omega^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) \to \hat{\bigoplus_{\omega;\,\text{non-dyadic}}} \{z_\omega^{(P)} \hat{\otimes} H^*(X, A; Z_2)\}$$

This completes the proof of Theorem 1.2.

## 2. Operations $\bar{S}_\omega$

Let $\bar{W}_\omega$ denote the symmetrized monomial of the cobordism normal characteristic classes $\bar{W}_k$. ($\bar{W}_\omega(\xi) = W_\omega(-\xi)$ for every stable vector bundle $\xi$.) The operation $\bar{S}_\omega$ is defined in [8] by $\bar{S}_\omega = \Psi^{-1}(\bar{W}_\omega)$, where $\Psi$ is the additive isomorphism mentioned in section 1.

**Notation 2.1.** (Landweber [6])

For a partition $\omega = (i_1, \cdots, i_r)$ let $r_\omega(i)$ denote the occurrences of the integer $i$ in $\omega$. And define

$$\binom{n}{\omega} = \begin{cases} 0 & \text{if} \quad n < |\omega| = r \\ \dfrac{n!}{r_\omega(1)!\, r_\omega(2)! \cdots (n - |\omega|)!} & \text{if} \quad n \geqq |\omega|. \end{cases}$$

The modulo 2 reduction of $\binom{n}{\omega}$ is denoted by $\binom{n}{\omega}_2$.

Similarly to the weakly complex case [8], we can easily determine the value $\bar{S}_\omega[P_k]$.

**Lemma 2.2.**

(1) $\bar{S}_\omega[P_k] = \binom{k+1}{\omega}_2 [P_{k-||\omega||}]$.

(2) $S_\omega[P_k] = \binom{2^p-k-1}{\omega}_2 [P_{k-||\omega||}]$ for $p$ such that $2^p > k+1$.

Proof. By the geometric interpretation of the action of $\mathcal{A}^*(0)$ on $\mathfrak{N}_*$ given in [6], [8], $\bar{S}_\omega[P_k] = \varepsilon W_\omega(\tau_{P_k}) = \varepsilon\binom{k+1}{\omega}_2 W_1^{||\omega||} = \binom{k+1}{\omega}_2 [P_{k-||\omega||}]$. Part (2) is proved similarly. Now we give some relations between $S_\omega$ and $\bar{S}_\omega$.

**Lemma 2.3.**

(1) *If the occurrence* $r_\omega(i) \leqq 1$ *in* $\omega$ *for all* $i$, then $S_\omega = \bar{S}_\omega$.

(2) $S_{(i)^k}=\sum_{\|\omega\|=k}\bar{S}_{i*\omega}$ *and dually*

$\bar{S}_{(i)^k}=\sum_{\|\omega\|=k}S_{i*\omega}$,

*where* $i*\omega$ *is meant a partition* $(i\cdot j_1, i\cdot j_2, \cdots, i\cdot j_r)$ *for* $\omega=(j_1, j_2, \cdots, j_r)$.

After Landweber [6] we denote the partition $(i)^k$ by $k\Delta_i$ and the totality of linear combinations of the $S_\omega$ by $A^*(0)$. $A^*(0)$ is proved a Hopf algebra over $Z_2$ ([6], [8]).

**Theorem 2.4.** (Landweber [6])
*The set* $\{S_{2^k\Delta_1}, S_{2^k\Delta_2}; k\geqq 0\}$ *provides a minimal set of generators of* $A^*(0)$.

**Corollary 2.5.**
*The set* $\{\bar{S}_{2^k\Delta_1}, \bar{S}_{2^k\Delta_2}; k\geqq 0\}$ *provides a minimal set of generators of* $A^*(0)$.

Proof of Lemma 2.3.
By the Whitney product formula, it follows that $\sum_{\omega=\omega_1\omega_2}W_{\omega_1}\cdot\bar{W}_{\omega_2}=0$ if $\omega\neq(0)$. Therefore $W_{(i)}=\bar{W}_{(i)}$ for all $i\geqq 1$ and we see by induction on the lengths of partitions that $W_\omega=\bar{W}_\omega$ if $r_\omega(i)\leqq 1$ for all $i$. Part (1) follows from this and from the definition of $S_\omega$ and $\bar{S}_\omega$.

Put

$$\sum_{0\leqq i\leqq s}\bar{W}_i x^i=\prod_{1\leqq j\leqq s}(1+u_j x)$$

for a sufficiently large $s$.

Then part (2) of the lemma is proved by induction on $k$ as follows;

$$\begin{aligned}
W_{(i)^k} &= \sum_{0\leqq l\leqq k-1}W_{(i)^l}\bar{W}_{(i)^{k-l}}=\sum_{0\leqq l\leqq k-1}\Big(\sum_{\|\omega\|=l}\bar{W}_{i*\omega}\Big)\cdot\bar{W}_{(i)^{k-l}}\\
&=\sum_{0\leqq l\leqq k-1}\Big\{\sum_{j_1+\cdots+j_m=l}\big(\sum(u_1^i)^{j_1}\cdots(u_m^i)^{j_m}\big)\Big\}\Big\{\sum(u_1^i)\cdots(u_{k-l}^i)\Big\}\\
&=\sum_{i_1+\cdots+i_n=k}\big(\sum(u_1^i)^{i_1}\cdots(u_n^i)^{i_n}\big)\Big(\sum_{0\leqq l\leqq k-1}\binom{n}{k-l}_2\Big)\\
&=\sum_{\|\omega\|=k}\bar{W}_{i*\omega}\binom{|\omega|}{0}_2=\sum_{\|\omega\|=k}\bar{W}_{i*\omega}\,.
\end{aligned}$$

Part (2) follows from this.

Proof of Corollary 2.5.
It follows from Lemma 2.3 and Theorem 2.4 that

$\bar{S}_{\Delta_1}=S_{\Delta_1}$,
$\bar{S}_{2^k\Delta_1}=S_{2^k\Delta_1}+S_{2^{k-1}\Delta_2}+$decomposables in $A^*(0)$, and
$\bar{S}_{2^k\Delta_2}=S_{2^k\Delta_2}+$decomposables in $A^*(0)$.

Thus the corollary follows from Theorem 2.4.

## 3. Even dimensional coefficients

Following suit of Novikov [8, appendix I], we obtain the following. We omit the proof.

**Lemma 3.1.**

*For a partition $\omega$ and for a positive integer $k=2^p(2q+1)$ ($p\geqq 0$, $q\geqq 1$), the following formula holds if $||\omega||\geqq 2^p$;*

$$\sum_{\omega=\omega_1\omega_2} S_{\omega_1}(z_{k-1-||\omega_2||})\binom{k-||\omega_2||}{\omega_2}_2 = 0\,,$$

*where the $z_i$ denote the coefficients of a fixed primitive element $P$ as in the introduction.*

Now we prove the following theorem.

**Theorem 3.2.**

*The coefficient $z_{2k}$ of a primitive element*

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

*in $\mathfrak{N}^*(BO(1))$ is equal to the bordism class $[P_{2k}]$ for all $k\geqq 1$.*

Proof. For $k-1$, the theorem is clear since $z_2$ is indecomposable from Theorem $P$ in the introduction.

Assume that the theorem holds up to dimension $2(k-1)\geqq 2$.

In order to show that $S_\omega(z_{2k}+[P_{2k}])=\bar{W}_\omega(z_{2k}+[P_{2k}])=0$ for all $\omega$ with $||\omega||=2k$, it saffices from Theorem 2.4 to prove

$$S_{2^s\Delta_i}(z_{2k}+[P_{2k}]) = 0 \quad (i=1, 2)\,.$$

To prove this, we see from Lemma 3.1 and the induction assumption that it is sufficient to show

$$\sum_{m+n=2s} S_{m\Delta_i}[P_{2k-ni}]\binom{2k+1-\mathrm{ni}}{n}_2 = 0 \quad (i=1, 2)\,.$$

This is obvious in case $2^si>2k$ or $s=0$ since

$$S_{m\Delta_i}[P_{2k-ni}] = \left\{\sum_{||\omega||=m}\binom{2k+1-ni}{\omega}_2\right\}[P_{2k-2^si}]$$

by Lemmas 2.2 (1) and 2.3 (2).

For the remaining cases, it suffices to prove the following lemma.

**Lemma 3.3.**

(1) $$\sum_{m+n=s}\left(\sum_{||\omega||=m}\binom{k-n}{\omega}\right)\binom{k-n}{n} \equiv 0 \quad (mod\ 2) \quad for \quad k\geqq s\geqq 2\,.$$

(2) $\sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-2n}{\omega}\right)\binom{k-2n}{n} \equiv 0 \quad (mod\ 2) \quad for \quad k\geqq 2s\geqq 2\,.$

Proof.

(1) Put

$$A(k,\ s)=\sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-n}{\omega}\right)\binom{k-n}{n} \quad (k\geqq 0,\ s\geqq 0)\,, \quad \text{and}$$

$$B(k,\ s)=\sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-2n}{\omega}\right)\binom{k-2n}{n} \quad (k\geqq 0,\ s\geqq 0)\,.$$

Then it holds in general that

$$\binom{k-n}{n}=\binom{k-n-1}{n}+\binom{k-n-1}{n-1} \quad \text{and}$$

$$\sum_{\|\omega\|=m}\binom{k-n}{\omega}=\sum_{0\leqq\|\omega\|\leqq m}\binom{k-n-1}{\omega}\,.$$

So we obtain that

(*) $\quad A(k,\ s)=\sum_{0\leqq s'\leqq s}A(k-1,\ s')+\sum_{0\leqq s''\leqq s-1}A(k-2,\ s'')$ and

(**) $\quad B(k,\ s)=\sum_{0\leqq s'\leqq s}B(k-1,\ s')+\sum_{0\leqq s''\leqq s-1}B(k-3,\ s'')$.

Part (1) clearly holds when $k=s=2$.

Assume, by induction, that (1) holds for such $(k,\ s)$ that $k_0>k\geqq 2$ and $k\geqq s\geqq 2$.

Thus, for $(k_0,\ s_0)$ with $k_0>s_0\geqq 2$,

$$A(k_0,\ s_0)\equiv\sum_{s'=0,1}A(k_0-1,\ s')+\sum_{s''=0,1}A(k_0-2,\ s'')\equiv 0 \quad (\text{mod } 2)$$

by the induction hypothesis and by the fact that $A(k,\ s)\equiv 1$ for $k\geqq s$ and $s=0,\ 1$.

And for $(k_0,\ k_0)$, the iterated application of (*) shows that

$$\begin{aligned} A(k_0,\ k_0) &\equiv A(k_0-1,\ k_0)+A(k_0-2,\ k_0-1) \\ &\equiv A(1,\ k_0)+\sum_{0\leqq s''\leqq k_0-1}A(0,\ s'')\equiv 0 \quad (\text{mod } 2)\,. \end{aligned}$$

Part (2) of the lemma is proved similarly, using the formula (**) repeatedly. This completes the proof of Lemma 3.3 and Theorem 3.2.

REMARK 3.4. Theorem 3.2 has been proved independently by F. Uchida [9] by a geometric method.

## 4. The coefficients of dimensions $4k+1$

A. Dold has defined in [5] manifolds $P(m,n)$ which are the identification spaces of $S^m \times CP_n$ with $(x, z)=(-x, \bar{z})$. He proved that, for $2^p(2q+1)-1$ $(p\geqq 1, q\geqq 1)$, the bordism class $[P(2^p-1, 2^pq)]$ provides a polynomial generator of $\mathfrak{N}_*$ in the corresponding dimension.

**Theorem 4.1.**
*The coefficient* $z_{4k+1}$ *of a primitive element*

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

*in* $\mathfrak{N}^*(BO(1))$ *is equal to the bordism class* $[P(1, 2k)]$ *for all* $k\geqq 1$.

For the proof of this theorem, we need the following notations.

**Notation 4.2.**

(1) Let $c_p(m)$ denote the coefficient of $2^p$ in the dyadic expansion of the integer $m$;

$$m = c_0(m)+c_1(m)\cdot 2+c_2(m)\cdot 2^2+\cdots, \ c_i(m) = 0, 1 .$$

(2) For a partition $\omega$, we denote by $\omega(c_p)$ the partition determined by $r_{\omega(c_p)}(i)=c_p(r_\omega(i))$ for all $i\geqq 1$. Thus $\omega=\prod_{0\leqq p}(\omega(c_p))^{2^p}$. For brevity, $\prod_{2\leqq p}(\omega(c_p))^{2^{p-2}}$ and $\omega(c_1)^2\cdot\omega(c_0)$ are denoted as $\bar{\omega}$ and $\bar{\bar{\omega}}$, respectively; $\omega=(\bar{\omega})^4\bar{\bar{\omega}}$.

**Lemma 4.3.**

$$\binom{n}{\omega}_2=\prod_{0\leqq p}\binom{c_p(n)}{\omega(c_p)}_2. \quad \text{Thus } \binom{n}{\omega}_2=\binom{\frac{n-c_1(n)\cdot 2-c_0(n)}{4}}{\bar{\omega}}_2\binom{c_1(n)\cdot 2+c_0(n)}{\bar{\bar{\omega}}}_2.$$

Proof. By definition,

$$\binom{n}{\omega}_2=\binom{n}{r_\omega(1)}_2\binom{n-r_\omega(1)}{r_\omega(2)}_2\cdots\binom{n-\sum_{1\leqq i\leqq k-1}r_\omega(i)}{r_\omega(k)}_2\cdots.$$

Then, by Lucus' theorem [1],

$$\prod_{1\leqq k}\binom{n-\sum_{1\leqq i\leqq k-1}r_\omega(i)}{r_\omega(k)}_2=\prod_{1\leqq k}\left(\prod_{0\leqq p}\binom{c_p(n-\sum_{1\leqq i\leqq k-1}r_\omega(i))}{c_p(r_\omega(k))}_2\right)$$

$$=\prod_{0\leqq p}\left(\prod_{1\leqq k}\binom{c_p(\mathrm{n})-\sum_{1\leqq i\leqq k-1}c_p(r_\omega(i))}{c_p(r_\omega(k))}_2\right)=\prod_{0\leqq p}\binom{c_p(n)}{\omega(c_p)}_2.$$

This completes the proof.

Now we calculate all the normal Stiefel-Whitney numbers of $P(1, 2\mathrm{k})$. It is easily seen that the cobordism Stiefel-Whitney numbers of manifolds agree with the cohomological ones ([6], [8]). So, by abuse of a notation, we denote both Stiefel-Whitney numbers by $W_\omega$ (and the normal ones by $\bar{W}_\omega$).

**Lemma 4.4.**

$$\bar{W}_\omega[P(1, 2k)] = \begin{cases} 0 \quad \textit{if} \quad |\bar{\bar{\omega}}| \geqq 3 \quad \textit{and} \quad \bar{\bar{\omega}} \neq 3\Delta_1 \quad \textit{or} \quad \bar{\bar{\omega}} = (1)\,, \\ \binom{2^p-1-k}{\bar{\omega}}_2 \quad \text{if} \quad \bar{\bar{\omega}} = 3\Delta_1 \quad \text{or} \\ \qquad\qquad 2 \geqq |\bar{\bar{\omega}}| \geqq 1 \quad \text{and} \quad \bar{\bar{\omega}} \neq (1)\,, \end{cases}$$

where $p$ is any integer with $2^p > k+1$.

Proof.

According to Dold [5].

$$H^*(P(1, 2k)\,;\, Z_2) \simeq H^*(P_1 \times CP_{2k}\,;\, Z_2)$$

as a ring. Let $c$ and $d$ denote the 1- and 2-dimensional generators of $H^*(P(1, 2k)\,;\, Z_2)$. The total Whitney class is given in [5] by

$$w_* P(1, 2k) = (1+c)(1+c+d)^{2k+1}\,,$$

and thus

$$\bar{w}_* P(1, 2k) = (1+c)(1+t)^{4(2^p-k-1)}(1+t_1)(1+t_2)\,,$$

where $p$ is any integer with $2^p > k+1$ and $t^2 = t_1 \cdot t_2 = d$ and $t_1 + t_2 = c$.

By formula (26) in [5],

$$t_1^{2i} + t_2^{2i} = 0 \quad \text{and} \quad t_1^{2i+1} + t_2^{2i+1} = cd^i\,.$$

The lemma follows from these facts and the preceding lemma.

Proof of Theorem 4.1.

Theorem $P$ in the introduction asserts that $z_{4+1} = [P(1, 2)]$. Assume, by induction, that $z_{4k'+1} = [P(1, 2k')]$ for $k' \leqq k-1$.

By Lemma 3.1 and Theorem 3.2, together with Lemma 2.2 (2), 4.3 and 4.4,

$$S_\omega(z_{4k+1}) = \sum_{\substack{\omega=\omega_1\omega_2 \\ ||\omega_2|| = 4m \neq 0}} S_{\omega_1}(z_{4(k-m)+1}) \binom{k-m}{\bar{\omega}}_2 \binom{2}{\bar{\bar{\omega}}_2}_2$$
$$+ \sum_{\substack{\omega=\omega_1\omega_2 \\ ||\omega_2|| = 2n+1}} \binom{2^p-1-4||\bar{\omega}_1|| - ||\bar{\bar{\omega}}_1||}{(\bar{\omega}_1)^4 \bar{\bar{\omega}}_1}_2 \binom{4||\bar{\omega}_1|| + ||\bar{\bar{\omega}}_1|| + 1}{(\bar{\omega}_2)^4 \bar{\bar{\omega}}_2}_2$$

for $\omega$ such that $||\omega|| = 4k+1$. (The terms with $||\omega_2|| \underset{(4)}{\equiv} 2$ vanish by Lemma 4.3.)

Therefore, by the induction hypothesis and by Lemma 4.3, together with the fact that $|\bar{\bar{\omega}}_1| + |\bar{\bar{\omega}}_2| = |\bar{\bar{\omega}}| + 4l$ $(l \geqq 0)$, it can be shown that

$$S_\omega(z_{4k+1}) = \sum 0 + \sum 0 = 0 \quad \text{if} \quad |\bar{\bar{\omega}}| \geqq 5\,.$$

In case $\bar{\bar{\omega}}=(2i,\ 2i,\ 4j,\ 4(k-||\varpi||-i-j)+1)$,

$$S_\omega(z_{4k+1})=\sum_{\varpi=\varpi_1\varpi_2}\left\{\binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2\binom{k-||\varpi_2||-i}{\varpi_2}_2\right.$$
$$\left.+\binom{2^{p'}-1-(||\varpi||+i+j-||\varpi_2||)}{\varpi_1}_2\binom{||\varpi||+i+j-||\varpi_2||}{\varpi_2}_2\right\}$$

by the induction hypothesis and by Lemma 4.3.

Suppose $\binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2\binom{k-||\varpi_2||-i}{\varpi_2}_2=1$ for some separation $\varpi_1\varpi_2$ of $\varpi$.

Since $c_p(2^{p'}-1-(k-||\varpi_2||-i))\neq c_p(k-||\varpi_2||-i)$ for each $p$, there is at most one $i\geqq 1$ such that $c_p(r_{\bar{\omega}}(i))\neq 0$. Let $r$ be the number of such odd integers $2i+1\geqq 1$ that satisfy $r_{\bar{\omega}}(2i+1)>0$. Then, by Lemma 4.3, the numbers of such separations $\varpi_1\varpi_2=\varpi$ and $\varpi_1{}'\varpi_2{}'=\varpi$ that satisfy

$$\binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2\binom{k-||\varpi_2||-i}{\varpi_2}_2=1 \quad\text{and}$$
$$\binom{2^{p'}-1-(||\varpi||+j+k-||\varpi_2{}'||)}{\varpi_1{}'}_2\binom{||\varpi||+j+k-||\varpi_2{}'||}{\varpi_2{}'}_2=1,$$

respectively, are both $2^r$.

The situation is the same if we suppose

$$\binom{2^{p'}-1-(||\varpi||+j+k-||\varpi_2||)}{\varpi_1}_2\binom{||\varpi||+j+k-||\varpi_2||}{\varpi_2}_2=1$$

for some separation $\varpi_1\cdot\varpi_2=\varpi$.

Therefore $S_\omega(z_{4k+1})=0+0=0$ or $=1+1=0$ if $\bar{\bar{\omega}}=(2j,\ 2j,\ 4k,\ 4(s-||\varpi||-j-k)+1)$.

We can prove analogously in other cases when $\bar{\bar{\omega}}=(1)$ or $|\bar{\omega}|\geqq 3$ and $\bar{\omega}\neq 3\Delta_1$ that $S_\omega(z_{4k+1})=0$.

When $|\bar{\omega}|=2$, from dimensional reasons, $\bar{\bar{\omega}}=(2j,\ 4(s-||\varpi||)-2j+1)$ for some $j\geqq 1$. In this case

$$S_\omega(z_{4k+1})=\sum_{\substack{\varpi=\varpi_1\varpi_2\\ \varpi_2\neq(0)}}\binom{2^p-1-(k-||\varpi_2||)}{\varpi_1}_2\binom{k-||\varpi_2||}{\varpi_2}_2$$
$$+\sum_{\varpi=\varpi_1\varpi_2}\binom{2^p-1-(2||\varpi||-2||\varpi_2||+j)}{\varpi_1}_2\binom{2||\varpi||-2||\varpi_2||+j}{\varpi_2}_2$$
$$=\sum_{\varpi=\varpi\cdot(0)}\binom{2^p-1-(k-||\varpi_2||)}{\varpi_1}_2\binom{k-||\varpi_2||}{\varpi_2}_2$$
$$=\binom{2^p-1-k}{\varpi}_2$$

as required.

When $\bar{\bar{\omega}}=3\Delta_1$ or $\bar{\bar{\omega}}=\Delta_{4n+1}$ ($n\geqq 1$), analogous arguments show that $S_\omega(z_{4k+1}) =\binom{2^p-1-k}{\bar{\omega}}$.

Comparing these facts with Lemma 4.4, we deduce that $S_\omega(z_{4k+1}) =\bar{W}_\omega(z_{4k+1})=\bar{W}_\omega(P(1, 2k))$ for all $\omega$ with $||\omega||=4k+1$. This completes the proof of Theorem 4.1.

OSAKA UNIVERSITY

## References

[1] J. Adem: *The relations on Steenrod powers of cohomology classes*, Algebraic Geometry and Topology, Princeton University Press, 1957, 191-238.

[2] M.F. Atiyah: *Bordism and cobordism*, Proc. Cambridge Philos. Soc. **57** (1961), 200-208.

[3] J.M. Boardman: Stable Homotopy Theory, Ch. IV, (mimeographed) Warwick Univ., 1967.

[4] P.E. Conner - E.E. Floyd: The Relation of Cobordism to K-theories, Springer-Verlag, 1966.

[5] A. Dold: *Erzeugende der Thomschen Algebra* $\mathfrak{N}$, Math. Z. **65** (1956), 25-35.

[6] P.S. Landweber: *Cobordism operations and Hopf algebras*, Trans. Amer. Math. Soc. **129** (1967), 94-110.

[7] J. Milnor: *On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology **3** (1965), 223-230.

[8] S.P. Novikov: *The method of algebraic topology from the view point of cobordism* (Russian), Izv. Akad. Nauk SSSR **31** (1967), 855-951.

[9] F. Uchida: *Bordism algebra of involutions*, Proc. Japan Acad. **46** (1970), 615-619.