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GALOIS THEORY FOR GRADED RINGS

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In this paper, we exhibit a Galois theory for graded rings in which the zero degree term is commutative and the automorphisms are homogeneous of zero degree.

If $R\subset S$ are rings, we will always suppose that R and S have the same unit. Futhermore all modules are unitary and ring homomorphisms carry the unit into the unit. In general, $G(S/R)$ will denote the group of all R-automorphisms of S but if S is a graded ring and R is an homogeneous subring, then $G(S/R)$ will be the group of all homogeneous R-automorphisms of zero degree of S. With $Z(S)$ we indicate the center of S.

Let $A=\bigoplus_{i\in I}A_i$ be a graded ring, in which I is a monoid that verifies a certain condition, $A_0\subset Z(A)$ and let B a subring of A. If A is weakly Galois over B in the sense of [7], that is, *A is separable over B and finitely generated and projective as a right B-module and there is a finite group $F\subset G(A/B)$ such that $A^F=B$*, then we show here that the theory of A over B can be reduced to the theory of A_0 over B_0. We obtain:

(a) A_0 is weakly Galois over B_0 and $A\simeq A_0\otimes_{B_0} B$.

(b) $G(A/B)\simeq G(A_0/B_0)$.

(c) For each finite subgroup H of $G(A/B)$, $A^H\simeq A_0^H\otimes_{B_0} B$.

(d) The fundamental theorems of Galois theory that we know in the commutative case are valid here ([1], [6], and [7]).

(e) If C is a subring of A containing B and separable over B, then C is an homogeneous subring and $C\simeq C_0\otimes_{B_0} B$, where $B_0\subset C_0\subset A_0$ and C_0 is B_0-separable.

(f) If σ is a B-automorphism of A, then σ is homogeneous of zero degree.

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1. On graded rings and graded modules

The definition of separable extension for non commutative rings is the same as in [3]. Here $(I,+)$ is an additive monoid, $A=\bigoplus_{i\in I}A_i$ is a graded ring over I and $M=\bigoplus_{i\in I}M_i$ is a graded module over A.

We say that I verifies (I_1) if:
For all i and j in I, $i+j=0 \Rightarrow i=j=0$.

If I is a totally ordered monoid we say that I verifies (I_2) (resp. (I_3)) if:
(I_2): *0 is the first element of I.*
(I_3): *I is cancellative, i.e., $i+j=i+k \Rightarrow j=k$, for i, j, k in I.*

If I is a totally ordered monoid that verifies (I_2) and (I_3), it shall be called admisible.

The conditions (I_2) and (I_3) are independent. If I verifies (I_2) then verifies (I_1) (the converse is not true). Therefore, if I is admisible then I verifies (I_1). It is clear that if I is an admisible monoid and $i_p\in I(p=1,\cdots,n)$, we have $\sum_{p=1}^{n} i_p\geqslant i_q$ for all q and $\sum_{p=1}^{n} i_p=i_q\Rightarrow i_p=0$ for all $p\neq q$.

The next proposition is clear:

Proposition 1.1. *Let $M=\bigoplus_{i\in I}M_i$ be a graded right module over the graded ring $A=\bigoplus_{i\in I}A_i$, and $B=\bigoplus_{i\in I}B_i$ an homogeneous subring of A, where I verifies (I_1). Then:*

(a) *M is finitely generated over $A \Rightarrow M_0$ is finitely generated over A_0.*
(b) *M is A-projective $\Rightarrow M_0$ is A_0-projective.*
(c) *A is B-separable $\Rightarrow A_0$ is B_0-separable.*

In [2] we developed a Galois theory of non commutative rings that verify the condition (H) (see section 2 in [2]). In graded rings this condition has the following setting:

Lemma 1.2. *Let $A=\bigoplus_{i\in I}A_i$ be a graded ring over an admisible monoid I and let us suppose that $A_0\subset Z(A)$. If $v\in A\otimes_Z A^0$ and $\mu(v)=\mu(v^2)$ then $\mu(v)=p_0\in A_0$ is an idempotent, where μ: $A\otimes_Z A^0\rightarrow A$ is the multiplication map.*

Proof. Let $v=\sum_{j=1}^{p} x_j\otimes y_j^0\in A\otimes_Z A^0$ be and let us suppose that $x_j=\sum_{i\in 1} a_j^i$, $y_h=\sum_{k\in I} b_h^k$ and $\mu(v)=\sum_{r\in I} p_r\in A$, are the decompositions in the homogeneous components, since,

$$\mu(v^2)=\sum_{j=1}^{p} x_j\left(\sum_{h=1}^{p} x_h y_h\right)y_j=\sum_{j=1}^{p} x_j\,\mu(v)y_j, \quad \text{we have,}$$

$$\sum_{s\in I} p_s=\sum_{j=1}^{p}\left(\sum_{i,j,k\in I} a_j^i\, p_r\, b_j^k\right) \quad (*).$$

As I is admisible and $A_0 \subset Z(A)$, from (*) we have $p_0 = p_0 \sum_{j=1}^{p} a_j^0\, b_j^0 = p_0^2$.

If the finite set $I' = \{r\colon r \in I,\ r \neq 0$ and $p_r \neq 0\}$ contains a first element m, from the (*) we obtain:

$p_m = \sum_{j=1}^{p} \sum_{i+j=m} a_j^i\, p_0\, b_j^k + \sum_{j=1}^{p} a_j^0\, p_m\, b_j^0 = 2p_0\, p_m$. Multiplying by p_0, we have $p_0 p_m = 0$ and then $p_m = 0$. Therefore, I' is empty and $p_r = 0$ for all r.

The proof of the following lemma is obtained by using the same method as the proof of the theorem 2.2 in [F].

Lemma 1.3. *Let $R \subset T \subset S$ be rings such that T is R-separable and let $\sum_i x_i \otimes y_i \in T \otimes_R T$ be the element that satisfies the separability conditions. If f: $T \to S$ is an R-ring homomorphism, then,*

(a) *$x \sum_i x_i f(y_i) = \sum_i x_i f(y_i) f(x)$, for all x in T.*

(b) *$\sum_i x_i f(y_i) = 1 \Leftrightarrow f = 1_T$ (i.e., f is the inclusion of T in S).*

(c) *If $e_f = \sum_i x_i \otimes (f(y_i))^0 \in S \otimes_Z S^0$, then $\mu(e_f) = \mu(e_f^2)$.*

Corollary 1.4. *Let $B \subset A$ an homogeneous subring, graded over an admisible monoid where $A_0 \subset Z(A)$ and $B \subset C$ a subring of A which is B-separable. If $\sum_i x_i \otimes y_i \in C \otimes_B C$ is the element that satisfies the separability conditions and f: $C \to A$ is a B-ring homomorphism then $\sum_i x_i f(x_i) = p_0 \in A_0$ is an idempotent.*

Proof. From the lemma 1.2 it follows trivially.

Proposition 1.5. *Let $B \subset C \subset A$ be homogeneous subrings of A, graded over an admisible monoid where $A_0 \subset Z(A)$ and C is B-separable. If f: $C \to A$ is a B-ring homomorphism homogeneous of zero degree, then $f/C_0 = 1_{C_0}$ if and only if $f = 1_C$.*

Proof. Let $\sum_i x_i \otimes y_i \in C \otimes_B C$ be the element that satisfies the separablilty conditions. From the former corollary, $\sum_i x_i f(y_i) = p_0 \in A_0$. We denote with x_i^0 and y_i^0 the homogeneous components of zero degree of x_i and y_i respectively and we have: $p_0 = \sum_i x_i^0 f(y_i^0)$, by equating of zero degrees in the last relation. If $f/C_0 = 1_{C_0}$, since that $y_i^0 \in C_0$ if follows that $p_0 = \sum_i x_i^0\, y_i^0 = 1$. Therefore, $\sum_i x_i f(y_i) = 1$ and from the lemma 1.3, $f = 1_C$.

We have proved the following theorem, that plays a fundamental rôle, as a straight forward consequence of the preceding proposition, with $G(A/B)$ we denote the group of all B-automorphisms homogeneous of zero degree of A.

Theorem 1.6. *Let $B \subset A$ be an homogeneous subring such that A is B-separable, graded over an admisible monoid where $A_0 \subset Z(A)$. If σ is a B-automor-*

phism homogeneous of degree zero of A, then $\sigma|A_0=1_{A_0}$ if and only if $\sigma=1_A$. Therefore, $G(A/B)\subset G(A_0/B_0)$ via the mapping $\sigma\to\sigma|A_0$.

2. Some remarks

From here on, with the exception of the cases speciallly appointed, we assume that $A=\bigoplus_{i\in I} A_i$ is a graded ring over an admisible monoid I, where $A_0\subset Z(A)$ and B is a subring of A, automorphism means automorphism homogeneous of zero degree and group of automorphisms means group of automorphisms homogeneous of zero degree. Finally, the group of all automorphisms means the group of all automorphisms homogeneous of zero degree. If $B=A^G$, $G\subset G(A/B)$, then B is a homogeneous subring of A. It is clear that if G is a finite group of automorphisms of A and $B=A^G$, A is strong Galois over B if and only if A_0 is strong Galois over B_0. If A is weakly Galois over B with all group G, then A_0 is strong Galois over B_0 with all group $G(A_0/B_0)\supset G_0$.

On the other hand, there is $c_0\in A_0$ such that $\mathrm{tr}(c_0)=1$ if and only if there is $c_0\in A_0$ such that $\mathrm{tr}(c_0)=1$.

If A is strong Galois over B and A has no idempotents except 0 and 1, $G(A/B)=G(A_0/B_0)$ and furthermore is equal to the group of all B-automorphisms (homogeneous and non homogeneous) of A, since we can to apply the theorem 5.3. in [2] (A verifies (H)). In this case, strong Galois is equivalent to weakly Galois and in the following we say *A is Galois over B* means *A is weakly Galois over B*.

We use frequently same property over the boolean spectrum of a ring R, developed in [7] for the commutative case. This property are valid here in an obvious way, since the idempotents are central idempotents. If A is Galois over B and $x\in \mathrm{Spect}\,\mathfrak{B}(B)$, then A_x is Galois over B_x in the graded sense.

3. Rings without non trivials idempotents

Proposition 3.1. *If A has no idempotents except* 0 *and* 1 *and A is Galois over B with group G, the usual correspondence in the Galois theory is a one-to-one correspondence between the subgroups of G and the subrings C of A such that $B\subset C$ and C is B-separable. If H is a subgroup of G and C is a B-separable subring of A, the following conditions are equivalent*:

(a) $C=A^H$.

(b) $C_0=A_0^H$ (i.e., C_0 *correspond to H in the* [1] *theory*).

Furthermore, all separable B-subring of A is an homogeneous subring.

Proof. Since A verifies (H) and $\mathrm{tr}(A)=B$, from the theorem 3.3 in [2] we have the correspondence between subgroups and subrings. As every subring C which is B-separable is equal to A^H for some subgroup H of G, C is homogeneous. It is clear that (a) implies (b). Conversely, if H is the subgroup such that $C_0=A_0^H$ (given by the theroy in [1]) and H' is the subgroup such that $C=A^{H'}$ (given by the theorem 3.3 in [2]), then $H'\subset H$. If $\sigma\in H$ then $\sigma/C_0 =1_{C_0}$. From proposition 1.5., $\sigma/C=1_C$ and follows that $\sigma\in H'$, which completes the proof.

Proposition 3.2. *Let A be strong Galois over B with group G. Then,*

(*a*) *For every subgroup H of G, $A^H\simeq A_0^H\otimes_{B_0} B$. In particular, $A \simeq A_0\otimes_{B_0} B$.*

Furthermore, if A has no idempotents except 0 *and* 1,

(b) *The condition* (*a*) *and* (*b*) *in the proposition* 5.1. *are equivalents to $C\simeq A_0^H\otimes_{B_0} B$.*

(c) *A subring $C=\bigoplus_{i\in I} C_i$ of A, is separable over B if and only if $C\simeq C_0\otimes_{B_0} B$, where C_0 is B_0-separable.*

Proof. From the theorem 5.1 in [4], follows trivially (*a*) and (*b*). If C_0 is B_0-separable and $C\simeq C_0\otimes_{B_0} B$, then C is B-separable. Conversely, if C is B-separable, $C=A_0^H\otimes_{B_0} B$ for some subgroup H of G with $A_0^H=C_0$, which completes the proof.

Proposition 3.3. *Let A_0 and B_0 be commutative rings, such that A_0 is strong Galois over B_0 with group G and let $B=B_0\oplus(\bigoplus_{i\in I} B_i)$ be any graded ring over an admisible monoid where $B_0\subset Z(B)_0$. Then $A=A_0\otimes_{B_0} B$ is a graded ring which is strong Galois over B with homogeneous group G and $A_0\subset Z(A)$.*

Proof. As A_0 and B are B_0-algebras and B_0 is a direct summand of B as B_0-modules, A is graded and $A_0\subset Z(A)$. To complete the proof it is enough to apply the theorem 5.2. In [4].

4. Rings with finitely many idempotents

Let G be the group of all B-automorphisms of A. Let us suppose that A has finitely many idempotents and that A is Galois over B with all group G. If B has no idempotents, these are in B_0 and then are central idempotents. Therefore, these induce a corresponding decomposition of B and A as a direct sum of rings, for which holds the same remark preceding to the proposition 1.2 in [6], Therefore, we can suppose that B has no idempotents except 0 and 1. We assume that B has this property.

Let G' be the group of all B_0-automorphisms of A_0. Then, A_0 is Galois over B_0 with all group G'. Since B_0 has no idempotents except 0 and 1, from the proposition 1.3 in [6], $A_0=\bigoplus_{i=1}^{n} A_0.e_i$, where $\{e_i\}$ is the finite set of the minimal idempotents of A_0 (and of A); each $A_0.e_i$ is Galois over B_0; G' is fiinte and is equal to the semidirect product of the symmetric group of order n and the product of the automorphism groups of the summands.

We emphasize, in this section, with the exception of the cases specially appoined, we suppose that A is Galois over B with all group G and B has no idempotents except 0 and 1 and we indicate with G' the group of all B_0-automorphisms of A_0 and with $\{e_i\}$ the set of minimal idempotents of A_0.

Proposition 4.1. *A is direct sum of the homogeneous subrings $A.e_i$ ($i=1,\cdots, n$). Each $A.e_i$ has no idempotents except* 0 *and* 1 *and is strong Galois over B. Finally $G=G'$ is semidirect product of the symmetric group of order n and the product of the automorphism groups of the summands $A.e_i$ over B.*

Proof. Since every idempotent in A is central, the same method as in the proof of the proposition 1.3. in [6], shows the first part and the decomposition of G in semidirect product. We also obtain that for every i, j, $A\cdot e_i\simeq A\cdot e_j$ and $G_i\simeq G_j$, where $G_i=G(A\cdot e_i/B)$. Since $A\cdot e_i$ verifies (H) and it is B-separable, if $(A\cdot e_i)^{G_i}=B$, the second part follows. In fact, we denote with B_i the image of B by the canonical map h_i: $B\rightarrow A\cdot e_i$. If $a\in(A\cdot e_i)^{G_i}$, choosing isomorphisms α_j: $A.e_i\rightarrow A\cdot e_j (j=1,\cdots,n)$ with $\alpha_i=id_{A\cdot e_i}$, we have that $c=\sum_{j=1}^{n}\alpha_j(a)\in A^G=B$ and $h_i(c)=a$. Therefore, $a\in B_i$ and this proof that $(A\cdot e_i)^{G_i}=B_i$.

Then $A\cdot e_i$ is strong Galois over B_i with group G_i. Then, B_i is *a* direct summand of $A\cdot e_i$ as right B_i-module (*i.e.* as right B-module) and therefore, B_i is B-projective. Then the following succession splits, $0\rightarrow \mathrm{Ker}(h_i)\rightarrow B\overset{h_i}{\rightarrow} B_i\rightarrow 0$.

We have, $Ker(h_i)=e.B$, where e is an idempotent in B. Since $e=0$, h_i is injective and $B_i\simeq B$.

Finally, if G' is the group of all B_0-automorphisms of $A_0\cdot e_i$, from the former remark in section 2, $G_i=G_i'$, for every i and then $G=G'$.

Using the same technique in [6] (see 4, section 3) we have:

Proposition 4.2. *For any subgroup H of G, A^H is B-separable.*

Proof. We define an equivalence relation in $\{1, 2,\cdots, n\}$ by $i\sim j$ if there is $\sigma\in H$ such that $\sigma(e_i)=e_j$. If J is an equivalence class and $e_J=\sum_{i\in J} e_i$, then e_j is a minimal idempotent of A^H and all minimal idempotents of A^H are of this form. Therefore, $A^H=\bigoplus_J A^H\cdot e_J$ and it is enough to prove that $A^H\cdot e_J$ is B-separable for every J.

Let $i_0\in J$ be and for each $i\in J$ we choose $\sigma_i\in H$ such that $\alpha_i=\sigma_i/A\cdot e_{i_0}$ is an isomorphism of $A\cdot e_{i_0}$ onto $A\cdot e_i$. We define θ: $A\cdot e_{i_0}\rightarrow A$ for $\theta(x)=\sum_{i\in J}\alpha_i(x)$. Then θ is a B-isomorphism onto a subring of A (except that θ maps the identity element in $A\cdot e_{i_0}$ to e_J, which is the identity element in $\mathrm{Im}(\theta)$ but not in A), such that $A^H\cdot e_J\subset \mathrm{Im}(\theta)$.

If H_{i_0} is the subgroup of G_{i_0} that we obtain for restriction to $A\cdot e_{i_0}$ of the elements $\sigma \in H$ such that $\sigma(e_{i_0})=e_{i_0}$, from the proposition 3.1., $(A.e_{i_0})^{H_{i_0}}$ is B-separable. But the image of $(A\cdot e_{i_0})^{H_{i_0}}$ for θ is exactly $A^H\cdot e_J$. Therefore, $A^H\cdot e_J$ is B-separable.

Proposition 4.3. *B is a direct summand of A as right B-module and the projection $\pi: A\to B$ is homogeneous of zero degree (this result is true if B has a finitely many idempotents). For each subgroup H of G, $A^H \simeq A_0^H \otimes_{B_0} B$. In particular, $A\simeq A_0\otimes_{B_0} B$.*

Proof. From the proposition 4.1 and from the proposition 3.2. follows that $A\simeq \bigoplus_{i=1}^{n} A\cdot e_i \simeq \left(\bigoplus_{i=1}^{n} A_0\cdot e_i\right)\otimes_{B_0} B \simeq A_0 \otimes_{B_0} B$, and this shows the first part.

Since $A_0 \supset B_0$ are commutative rings and A_0 is B_0-projective, B_0 is a direct summand of A_0 as B_0-module. If π': $A_0\to B_0$ is the projection, $\pi=\pi'\otimes 1$: $A_0\otimes_{B_0} B\to B_0\otimes_{B_0} B\cong B$ is an homogeneous of zero degree projection of A onto B. If B has a finitely many idempotents, this result follows by decomposition of B in a direct sum.

Let H be a subgroup of G and φ: $A_0^H\otimes_{B_0} B\to A^H$, such that $\varphi(a_0\otimes b)=a_0.b$. Since A^H is B-separable and A is finitely generated and projective as right B-module, A is A^H-separable and finitely generated and projective as right A^H-module.

Then A is Galois over A^H and follows that A^H is a direct summand of A as right A^H-module with an homogeneous projection of degree zero π_1: $A\to A^H$ and A_0^H is a direct summand of A_0 as A_0^H-module. The last conclusion shows that $A_0^H \otimes_{B_0} B$ is a direct summand of $A\cong A_0\otimes_{B_0} B$ as right B-module and follows that φ is injective, since it is the restriction of the last isomorphism to the direct summand. If $a\in A^H \subset A\cong A_0\otimes_{B_0} B$, then $a=\sum_i a_0^i.\, b_i$, where $a_0^i\in A_0$ and $b_i\in B$. Therefore, $a=\pi_1(a)=\sum_i \pi_1(a_0^i).b_i=\varphi(\sum_i \pi_1(a_0^i)\otimes b_i)\in \mathrm{Im}(\varphi)$, which completes the proof.

5. General case

Proposition 5.1. *If A is Galois over B with all group G, for every finite subgroup F of G, $A^F\simeq A_0^F\otimes_{B_0} B$. In particular, $A\cong A_0\otimes_{B_0} B$.*

Proof. Let φ: $A_0^F\otimes_{B_0} B\to A^F$, the map defined by $\varphi\,(a_0\otimes b)=a_0.b$. For every $x\in \mathrm{Spect}\ \mathscr{B}(B_0)=\mathrm{Spect}\ \mathscr{B}(B)$ (see section 2 in [7]), the theory in the former section applies to A_x over B_x. Then $(A_{0_x})^{F_x}\otimes_{B_{0_x}} B_x\simeq (A_x)^{F_x}$ and since F is finite follows that $(A_0^F)_x\otimes_{B_{0_x}} B_x\cong (A^F)_x$ and therefore $(A_0^F\otimes_{B_0} B)_x\simeq (A^F)_x$. Since this isomorphism is φ_x, for every $x\in \mathrm{Spect}\ \mathscr{B}(B)$, φ is an isomorphism.

Proposition 5.2. *If A is Galois over B with all group G, then $G\simeq G(A_0/B_0)$.*

Proof. From the theorem 1.6, the map $G\to G(A_0/B_0)$ a such that $\sigma\to\sigma/A_0$

is injective. Next if $\tau\in G(A_0/B_0)$ then $\tau\otimes 1$ is a B-automorphism of A such that $\tau\otimes 1/A_0=\tau$. Therefore, the map $G(A_0/B_0)\rightarrow G$, such that $\tau\rightarrow\tau\otimes 1$ is the inverse of the former map.

Proposition 5.3. *If A is Galois over B with all group G, every subring B-separable C is an homogeneous subring and there is a finite subgroup H of $G(A/B)$ such that $A^H=C$.*

Proof. Let φ: $A\rightarrow A_0\otimes_{B_0}B$ be the isomorphism of the proposition 5.1. Then $A\otimes_B A\simeq(A_0\otimes_{B_0}B)\otimes_B(A_0\otimes_{B_0}B)\simeq(A_0\otimes_{B_0}A_0)\otimes_{B_0}B\simeq A_0\otimes_{B_0}A$, where $A_0\otimes_{B_0}A$ is a graded ring in which, the elements homogeneous of degree i are given by the submodule $A_0\otimes_{B_0}A_i$. We denote with ψ: $A\otimes_B A\rightarrow A_0\otimes_{B_0}A$, the former isomorphism and then it is easy to check that if $a\otimes a'\in A\otimes_B A$, $\psi(a\otimes a')=\psi(a\otimes 1)(1\otimes a')$ and if $a_0\otimes a\in A_0\otimes_{B_0}A$, then $\psi^{-1}(a_0\otimes a)=a_0\otimes a\in A\otimes_B A$. Let C be a B-separable subring of A, x_i, $y_i(i=1,\cdots,m)$, the elements in C such that $e=\sum_{i=1}^{m}x_i\otimes y_i\in C\otimes_B C$ satisfies the separability conditions and i: $C\rightarrow A$ the inclusion. Then for every $x\in C$, $x(i\otimes i)(e)=(i\otimes i)(e)x$. Therefore, $\sum_{i,j}x_ix_j\otimes y_jy_i=\sum_j x_j\otimes y_j(\sum_i x_iy_i)=\sum_j x_j\otimes y_j$ in $A\otimes_B A$. In this relation we applied the isomorphism ψ. Then, using the lemma 1.2. and applying ψ^{-1}, we obtain

(1) $\sum_j x_j\otimes y_j=\sum_i u_i^0\otimes v_i^0\in A\otimes_B A$, where u_i^0 and v_i^0 are in A_0.

Since A is finitely generated and projective as right B-module and $A\simeq A_0\otimes_{B_0}B$, we choose $a_r^0\in A_0$, $\varphi_r\in\mathrm{Hom}_B(A^{\cdot}, B^{\cdot})$($A^{\cdot}$ is the structure of A as right B-module), $r=1,\cdots,n$, such that $a=\sum_r a_r^0\,\varphi_r(a)$, for every $a\in A$, where φ_r is homogeneous of zero degree for all r. Since C is separable over B, A is finitely generated and projective as right C-module and it is easy to see that the projective coordinate system is a_r^0, $\psi_r\in\mathrm{Hom}_c(A^{\cdot}, C^{\cdot})$, where $\psi_r(a)=\sum_j\varphi_r(a\cdot x_j)\cdot y_j$, for every a in A and for all r. From (1) follows that $\psi_r(a)=\sum_i\varphi_r(a\cdot u_i^0)\cdot v_i^0$ and then ψ_r is homogeneous of zero degree. Therefore, if $C_0=C\cap A_0$, for every $a_0\in A_0$ we have, $a_0=\sum_r a_r^0\,\Psi_r(a_0)$, where $\Psi_r/A_0\in\mathrm{Hom}_{C_0}(A_0, C_0)$.

Then A_0 is C_0-projective and finitely generated and follows that $A_0=C_0\oplus D_0$ as C_0-module. Let $c=\sum_{i\in I}c_i\in C$ be. If $c_0=c_0'+d_0$ where $c_0'\in C_0$ and $d_0\in D_0$ we have, $\psi_r(1)\cdot d_0=\psi_r(1)\cdot c_0-\psi_r(1)\cdot c_0'=$ zero degree$(\psi_r(1)c)-\psi_r(c_0')=$ zero degree $(\psi_r(c))-\psi_r(c_0')=\psi_r(c_0-c_0')\in C_0\cap D_0$. Therefore, $d_0=\sum_r a_r^0\,\psi_r(1)\cdot d_0=0$. From this follows that π: $C\rightarrow C_0$ defined by $\pi(\sum_{i\in I}c_i)=c_0$, for every $\sum_{i\in I}c_i\in C$, is a ring homomorphism onto C_0 and from the proposition 2.4 in [3], C_0 is separable over $\pi(B)=B_0$.

Then there is a finite subgroup F of G such that $A_0^F=C_0$. If $\sigma\in F$, from the corollary 1.4., $\sum_i x_i \sigma(y_i) = p_0 \in A_0$. Since $y_i^0 \in C_0$, by equating zero degree, $p_0 = \sum_i x_i^0 \sigma(y_i^0) = \sum_i x_i^0 y_i^0 = 1$ and from the lemma 1.3, $\sigma/C = 1_C$. Therefore $C \subset A^F = A_0^F \otimes_{B_0} B = C_0 \otimes_{B_0} B \subset C$ and follows that $C = A^F$ is an homogeneous subring.

Proposition 5.4. *If A is Galois over B with homogeneous group G, all B-automorphism of A (homogeneous and non homogeneous) is homogeneous of zero degree and is in G.*

Proof. If B has no idempotents except 0 and 1 and G' is the group of all B-automorphisms (homogeneous and non homogeneous) of A, with the same method in the proposition 4.1. and using the same notation we prove that G' is a semidirect product of the symmetric group of order n and the product of the automorphism (homogeneous and non homogeneous) groups of every $A \cdot e_i$ over B. From the remark in section 2, the automorphism groups of every $A \cdot e_i$ over B is equal to the group of automorphisms homogeneous of zero degree of $A \cdot e_i$ over B. Then, $G' = G$.

In the general case, if σ is a B-automorphism of A, σ_x is a B_x-automorphism of A_x, for every $x \in \text{Spect } \mathscr{B}(B)$. From the first part σ_x is homogeneous of zero degree and then, it is easy to see that σ is homogeneous of zero degree.

In [6] and [7] the correspondence of Galois theory is developed, between separable subrings of A and some subgroups of G. The concepts of fat subgroups and subgroups that verify (3.8a) and (3.8.b) in [6] and [7] respectively, are similar here since $G(A/B) = G(A_0/B_0)$ are the same as a subgroup of A or A_0. We say that these subgroups are special subgroups.

Theorem 5.5. *If A is Galois over B with group G, the usual correspondence in the Galois theory is a one-to-one correspondence between subrings of A containing B and are B-separables and special subgroups of G. If H is a special subgroup of G and C is a B-subring separable of A, the following conditions are equivalent:*

1°) $C = A^H$.

2°) $C_0 = A_0^H$ (*i.e., C_0 correspond to H in the Villamayor and Zelinsky theory*).

3°) $C = A_0^H \otimes_{B_0} B$.

In particular, a subring $C = {}_i\oplus_I C_i$ of A containing B is B-separable if and only if $C \simeq C_0 \otimes_{B_0} B$, where C_0 is B_0-separable.

Proof. If H is a special subgroup of G, there is a finite subgroup F of G such that $A^H = A^F$. Then $A^H \simeq A_0^F \otimes_{B_0} B = A_0^H \otimes_{B_0} B$ and since A_0^H is B_0-separable, A^H is B-separable. If C is a B-separable subring of A, from the proposition 5.4. $C = A^H$, for some special subgroup H of G. The correspondence is one-to-one since $A^H = A^{H'}$ where H and H' are special subgroups of

A, then $A_0^H = A_0^{H'}$ and follows that $H = H'$. The last part follows trivially from the $\{\sigma\colon \sigma \in G$ and $\sigma/C = 1_C\} = \{\sigma\colon \sigma \in G$ and $\sigma/C_0 = 1_{C_0}\}$, as follows from the proposition 1.5.

6. Some final results

Proposition 6.1. *If B has no idempotents except 0 and 1, A is separable over B and projective as right module over B and $A^G = B$, where G is the group of the all B-automorphisms of A if and only if there is a subgroup H of G such that A is strong Galois over B with group H.*

Proof. If A is separable over B and projective as right B-module and $A^G = B$, then A_0 is weakly Galois over B_0 with group $G(A_0/B_0)$ and $A_0 = \bigoplus_{i=1}^{n} A_0 \cdot e_i$, where $\{e_i\}$ is the set of the minimal idempotents in A_0 and $A_0 \cdot e_i$ is strong Galois over B_0. Then, it is easy to prove that $A \cdot e_i$ is Galois over B with group $G_i = G(A \cdot e_i/B)$. Therefore $A \cdot e_i$ is finitely generated as B-module, and follows that A is finitely generated as B-module. Now, as in the first part of the proof of (3.15) in [7], we have a subgroup H of G such that $A^H = B$ and $o(H) = n \cdot o(G_i)$. Then, as in the theorem 2 in [5] we can obtain that A is strong Galois over B with group H. The converse is trivial.

Corollary 6.2. *If B has no idempotents except 0 and 1, A is weakly Galois over B with all group G if and only if there is a subgroup H of G such that A is strong Galois over B with group H.*

The former proposition shows that we can omit the assumption *A is finitely generated over B,* as in [6]. If B has idempotents, this results follows by localization in each $x \in \operatorname{Spect} \mathscr{B}(B)$. Besides, if B has no idempotents except 0 and 1 and A is Galois over B, A is strong Galois over B with group H and follows that $tr_H\colon A \to B$, defined by $tr_H(a) = \sum_{\sigma \in H} \sigma(a)$, is surjective. Then A is faithfully projective as right B-module. In general, we have:

Proposition 6.3. *If A is Galois over B with group G, then A is faithfully projective as right B-module.*

Proof. If $f\colon M \to N$ is a left B-homomorphism such that $l_A \otimes f\colon A \otimes_B M \to A \otimes_B N$ is injective, since $A \simeq A_0 \otimes_{B_0} B$ we have, $l_{A_0} \otimes f\colon A_0 \otimes_{B_0} M \to A_0 \otimes_{B_0} N$ is injective. Since A_0 is faithfully projective over B_0, it follows that f is injective.

Theorem 6.4. *Let A_0 and B_0 be commutative rings such that A_0 is Galois over B_0 with group G and $B = B_0 \oplus (\bigoplus_{i \neq 0} B_i)$ any graded ring over an admisible monoid where $B_0 \subset Z(B)$. Then $A = A_0 \otimes_{B_0} B$ is a graded ring which is Galois over B with homogeneous group G and $A_0 \subset Z(A)$.*

Proof. If B_0 has no idempotents except 0 and 1, there is a subgroup H of G such that A_0 is strong Galois over B_0 with group H. Then, from the proposition 3.3. $A = A_0 \otimes_{B_0} B$ is strong Galois over B with homogeneous group G and therefore follows our result. In general, $A = A_0 \otimes_{B_0} B$ is B-separable and finitely generated and projective as right B-module. If H is a finite subgroup of G such that $A_0^H = B_0$, for each $x \in \text{Spect}\, \mathscr{B}(B_0)$, $(A_{0_x})^{H_x} = B_{0_x}$. From the first part $A_{0_x} \otimes_{B_{0_x}} B_x$ is weakly Galois over B_x and then $(A^H)_x = (A_x)^{H_x} = (A_{0_x} \otimes_{B_{0_x}} B_x)^{H^x} = (A_{0_x})^{H_x} \otimes_{B_{0_x}} B_x = B_x$. Therefore $A^H = B$, which completes the proof.

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