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ALGEBRAIC STEENROD OPERATIONS IN THE SPECTRAL SEQUENCE ASSOCIATED WITH A PAIR OF HOPF ALGEBRAS

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Araki [3], [4] and Vazquez [10] investigated behaviors of Steenrod reduced powers in the spectral sequence associated with a fibre space in the sense of Serre. The main purpose of this paper is to establish an algebraic analogy to their works. For example, works of Adams [1], [2] and others, [6], [11], [12], implicitly contain a useful, direct application of our results.

1. Steenrod operations in the spectral sequence associated with an algebraic system $\mathfrak{S}$

DEFINITION 1. By a graded differential algebra $\mathfrak{S}=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$, we mean

1) a graded cochain complex C over the field Z_2:

$$C\colon C^0\to C^1\to\cdots\to C^n\xrightarrow{\delta^n} C^{n+1}\to\cdots, \text{ where } \delta^n\colon C^n\to C^{n+1}$$

is a morphism of graded vector spaces over Z_2,

2) for each integer p, F^pC is a subcomplex of C such that
   i) $F^{p+1}C$ is a subcomplex of F^pC (in notation: $F^pC\supset F^{p+1}C$)
   ii) $F^pC=C$ if $p\leq 0$, and iii) $F^pC^n=0$ if $p>n$,

3) for each integer i there exists a Z_2-linear map $\underset{i}{\cup}\colon C\otimes C\to C$ such that if $x\in F^pC^{m,s}$ and $y\in F^qC^{n,t}$, then $x\underset{i}{\cup}y\in F^{\alpha}C^{m+n-i,s+t}$ for $\alpha=\mathrm{Max}\{p+q-i, p, q\}$, where $x\underset{i}{\cup}y=\underset{i}{\cup}(x\otimes y)$, $x\cup y=x\underset{0}{\cup}y$ in notations, and s,t stand for gradings. $\underset{i}{\cup}$ satisfies the following conditions:

   i) $\underset{i}{\cup}$ is trivial if $i<0$, ii) For $x\in F^qC^m$ and $y\in F^qC^n$, $x\underset{i}{\cup}y=0$ if $i>m$ or n,

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iii) $x \cup (y \cup z) = (x \cup y) \cup z$, iv), $1 \cup x = x \cup 1 = x$ for some $1 \in C^{0,0}$, and
v) $\delta(x \underset{i}{\cup} y) = x \underset{i-1}{\cup} y + y \underset{i-1}{\cup} x + \delta x \underset{i}{\cup} y + x \underset{i}{\cup} \delta y$.

Associating $\mathfrak{S}$ with an exact couple $\langle D, E, i, j, k \rangle$ by defining $D_1^{p,q} = H^{p+q}(F^pC)$, $E_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C)$, and i, j, k as usual, we have a spectral sequence $\{E_\gamma, d_\gamma \mid \gamma \geq 1\}$. Let us define Steenrod operations in the spectral sequence as Araki [4] and Vazquez [10] did. Define a map $\theta_i: C \to C$ by $\theta_i(x) = x \underset{i}{\cup} x + x \underset{i+1}{\cup} \delta x$, then we have

Proposition 1. *θ_i induces Steenrod operations ${}_BSt_i$, ${}_FSt_i$ in the spectral sequence associated with the algebraic system $\mathfrak{S}$ such that*

$$ {}_BSt_i \colon E_\gamma^{p,q} \to E_{2\gamma-2}^{2p-i,2q} \text{ for } \infty \geq \gamma \geq 2\,, $$

and

$$ {}_FSt_i \colon E_\gamma^{p,q} \to E_\gamma^{p,2q+p-i} \text{ for } \infty \geq \gamma \geq 1\,. $$

They are all Z_2-homomorphisms.

Proof. It is straightforward by definition that if we denote, as usual, $Z_\gamma^{p,q} = \{x \in F^pC^{p+q} \mid \delta x \in F^{p+\gamma}C^{p+q+1}\}$, $B_\gamma^{p,q} = \{x \in F^pC^{p+q} \mid {}^\exists y \in F^{p-\gamma}C^{p+q-1},\ \delta y = x\}$, $Z_\infty^{p,q} = \{x \in F^pC^{p+q} \mid \delta x = 0\}$, and $B_\infty^{p,q} = \{x \in F^pC^{p+q} \mid {}^\exists y \in C^{p+q-1},\ \delta y = x\}$, then $\theta_i(Z_\gamma^{p,q}) \subset Z_{2\gamma-1}^{2p-i,2q} \cap Z_\gamma^{p,2q+p-i} \subset Z_{2\gamma-2}^{2p-i,2q} \cap Z_\gamma^{p,2q+p-i}$, $\theta_i(B_{\gamma-1}^{p,q}) \subset B_{2\gamma-3}^{2p-i,2q} \cap B_{\gamma-1}^{p,2q+p-i}$, $\theta_i(Z_\infty^{p,q}) \subset Z_\infty^{2p-i,2q} \cap Z_\infty^{p,2q+p-i}$, and $\theta_i(B_\infty^{p,q}) \subset B_\infty^{2p-i,2q} \cap B_\infty^{p,2q+p-i}$. Note that the restriction on $\gamma \geq 2$ comes from the following observation. If $x \in B_{\gamma-1}^{p,q}$, then

$$ \theta_i(x) = \delta y \underset{i}{\cup} \delta y = \delta(y \underset{i}{\cup} x + y \underset{i-1}{\cup} y)\,, $$

where $\delta y = x$ with $y \in F^{p-\gamma+1}C^{p+q-1}$. Since $y \underset{i}{\cup} x + y \underset{i-1}{\cup} y \in F^{2p-i-(2\gamma-3)}C^{2p+2q-i-1}$ if $\gamma \geq 2$, $\theta_i(x) \in B_{2\gamma-3}^{2p-i,2q}$ for $\gamma \geq 2$. Hence, θ_i induces ${}_BSt_i$ and ${}_FSt_i$ as stated in Proposition 1. For $x_1, x_2 \in Z_\gamma^{p,q}$

$$ \theta_i(x_1 + x_2) = \theta_i(x_1) + \theta_i(x_2) + \delta(x_1 \underset{i+1}{\cup} x_2) + x_2 \underset{i+1}{\cup} \delta x_1 + \delta x_1 \underset{i+1}{\cup} x_2 $$

from the bilinearity of $\underset{i}{\cup}$. Since

$$ \delta(x_1 \underset{i+1}{\cup} x_2) \in B_1^{2p-i,2q} \cap B_0^{p,2q+p-i} \subset B_{2\gamma-3}^{2p-i,2q} \cap B_{\gamma-1}^{p,2q+p-i}, $$

and

$$ x_2 \underset{i+1}{\cup} \delta x_1 + \delta x_1 \underset{i+1}{\cup} x_2 \in Z_{2\gamma-3}^{2p-i+1,2q-1} \cap Z_{\gamma-1}^{p+1,2q+p-i-1}\,, $$

${}_BSt_i$ and ${}_FSt_i$ are Z_2-homomorphisms.

For completeness sake let us show some properties of Steenrod operations which are useful for their applications. (For example, for computation of cohomology of the Steenrod algebra.) Let $E_{\gamma,s}^{a,b}$ be the subvector space of $E_\gamma^{a,b}$ spanned by $(d_s, \cdots, d_{\gamma+1}, d_\gamma)$-cocycles and let $\kappa_{s+1}^\gamma\colon E_{\gamma,s}^{a,b}\to E_{s+1}^{a,b}$ be the natural epimorphism. An element in $E_{2,b}^{a,b}$ will be said to be g-transgressive.

Proposition 2. *${}_FSt_i\colon E_\gamma^{p,q}\to E_\gamma^{p,2q+p-i}$ is trivial if $i<p$ or $i>p+q$,*

$${}_BSt_i\colon E_\gamma^{p,q}\to E_{2\gamma-2}^{2p-i,2q} \text{ is trivial if } i>p \text{ or } i<0,$$

and

$${}_BSt_p=\kappa_{2\gamma-2}^\gamma\ {}_FSt_p\,.$$

Proof. If $p>i$ and $x\in Z_\gamma^{p,q}$, then $\theta_i(x)\in F^{2p-i}C^{2p+2q-i}\subset F^{p+1}C^{2p+2q-i}$ and $\delta(\theta_i(x))=\delta x\underset{i+1}{\cup}\delta x\in F^{p+\gamma}C^{2p+2q-i+1}$. Hence, $\theta_i(x)\in Z_{\gamma-1}^{p+1,2q+p-i-1}$, so that by definition the triviality of ${}_FSt_i$ is proved if $p>i$. The rest of the proof is immediate, and hence, is omitted.

Proposition 3. *If $\alpha\in E_{\gamma,c}^{p,q}$, then ${}_FSt_i(\alpha)\in E_{\gamma,d}^{p,2q+p-i}$, where*

$$d=Max\{p+2c-i,\ c\}$$

and ${}_BSt_i(\alpha)\in E_{\gamma,2c}^{2p-i,2q}$.

Proof. Recall that

$$E_{\gamma,c}^{p,q}=Z_{c+1}^{p,q}+Z_{\gamma-1}^{p+1,q-1}/Z_{\gamma-1}^{p+1,q-1}+B_{\gamma-1}^{p,q}\ \subset E_\gamma^{p,q}.$$

If x is a representative of α, then $\theta_i(x)\in F^pC^{2p+2q-i}\cap F^{2p-i}C^{2p+2q-i}$ and $\delta(\theta_i(x))\in F^{2p+2c-i+1}C^{2p+2q-i+1}\cap F^{p+c+1}C^{2p+2q-i+1}$. If $i\geq p$, then $\theta_i(x)\in Z_{d+1}^{p,2q+p-i}$ where $d=\mathrm{Max}\{p+2c-i,\ c\}$, while if $p\geq i$, then $\theta_i(x)\in Z_{2c+1}^{2p-i,2q}$. Hence, the proof is completed.

Proposition 4. *If $\alpha\in E_2^{p,q}$ is g-transgressive, then ${}_FSt_i(\alpha)\in E_2^{p,2q+p-i}$ is also g-transgressive. Moreover we have*

$$(1)\qquad \kappa_{2q}^\lambda\ {}_BSt_{i+1}\,d_{q+1}\,\kappa_{q+1}^2(\alpha)=d_\lambda\kappa_\lambda^2\ {}_FSt_i(\alpha)\,,$$

where $\lambda=2q+(p-i)+1$ and $\kappa_{2q}^\lambda=\kappa_\lambda^{2q}$ if $\lambda>2q$.

Proof. It is obvious from Proposition 3 that ${}_FSt_i(\alpha)$ is g-transgressive. If x is a representative of α, then both sides of (1) is represented by $\delta x\underset{i+1}{\cup}\delta x$. Hence, the proof is completed.

2. Comparison theorem in homological algebra

To prepare for later sections the algebraic Steenrod operations are introduced by the iterated use of a comparison theorem in relative homological algebra [5] (For the theorem in a more general and rigorous setting, see [8]), and the explicit formulas of chain homotopies [1], [11] involved in the theorem are presented in this section.

Let α: $A \to B$ be a morphism of graded augmented algebras A and B over a commutative ring R with unity, and let M and N be left graded modules over algebras A and B respectively. A morphism of graded R-modules f: $M \to N$ is called a α-homomorphism iff $f(ax)=\alpha(a)f(x)$ for $a \in A$ and $x \in M$.

Proposition 5. *Let* ε: $\mathfrak{X} \to M$ *be a* R*-split exact resolution of* M *in the category* ${}_A\mathfrak{M}$ *of left* A*-modules and let* η: $\mathfrak{Y} \to N$ *be a* R*-split exact resolution of* N *in the category* ${}_B\mathfrak{M}$. *Then, for any* α*-homomorphism* f: $M \to N$ *there exists a* α*-chain map extension* F: $\mathfrak{X} \to \mathfrak{Y}$ *of* f *in the sense that*
1) *for each* $n \geq 0$, F_n: $X_n \to Y_n$ *is a* α*-homomorphism, and*
2) $d_n F_n = F_{n-1} \partial_n$ *for* $n \geq 1$ *and* $f\varepsilon = \eta F_0$, *where*

$$\begin{array}{ccccccccccccc}
M & \overset{\varepsilon}{\underset{s_{-1}}{\rightleftarrows}} & X_0 & \overset{\partial_1}{\underset{s_0}{\rightleftarrows}} & X_1 & \rightleftarrows & \cdots & \overset{\partial_n}{\rightleftarrows} & X_n & \underset{s_n}{\rightleftarrows} & \cdots & & :\mathfrak{X} \\
\downarrow f & & \downarrow F_0 & & \downarrow F_1 & & & & \downarrow F_n & & & & \downarrow F \\
N & \overset{\eta}{\underset{t_{-1}}{\rightleftarrows}} & Y_0 & \overset{d_1}{\underset{t_0}{\rightleftarrows}} & Y_1 & \rightleftarrows & \cdots & \overset{d_n}{\rightleftarrows} & Y_n & \underset{t_n}{\rightleftarrows} & \cdots & & :\mathfrak{Y}
\end{array}$$

3) *If* F, F' *are* α*-chain map extensions of* f, *then there exists a* α*-chain homotopy* h: $\mathfrak{X} \to \mathfrak{Y}$ *connecting* F *with* F'.

Proof. First let us observe that the proposition is the usual comparison theorem in case when $A=B$ and α is the identity map. The following remarks enable us to reduce the proposition to the classical theorem; 1) any B-module Z can be considered as an A-module by definition $az=\alpha(a)z$ for $a \in A$ and $z \in Z$, 2) any morphism g: $Z \to Z'$ in ${}_B\mathfrak{M}$ can be regarded as a morphism in ${}_A\mathfrak{M}$ by considering Z, Z' as A-modules because

$$g(az) = g(\alpha(a)z) = \alpha(a)g(z) = ag(z)\,,$$

3) a R-homomorphism k: $X \to Y$ is a α-homomorphism iff k is a morphism in ${}_A\mathfrak{M}$ considering Y as an A-module. For $k(ax)=\alpha(a)k(x)=ak(x)$. From 1) and 2), η: $\mathfrak{Y} \to N$ can be considered as a R-split exact complex of N in ${}_A\mathfrak{M}$, and from 3) f: $M \to N$ is a morphism in ${}_A\mathfrak{M}$. It follows from the usual comparison theorem that there exists a chain map extension F of f in ${}_A\mathfrak{M}$. From 3) F is a α-homomorphism. It is immediate to see the rest of the proof. This proves the proposition.

Let us apply the proposition to the following case. Let A be a co-commutative Hopf algebra over Z_2 and let α: $A \to A \otimes A$ be the cocommutative comultiplication Δ. Since $M=Z_2$ and $N=Z_2\otimes Z_2\cong Z_2$ can be considered by augmentations as a left A-module and a left $A\otimes A$-module respectively, the α-map $f\colon Z_2\to Z_2\otimes Z_2$ defined by $f(1)=1\otimes 1$, can be extended to a Δ-chain map $h^0\colon \mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ by the direct application of the proposition, where $\mathfrak{X}$ is a Z_2-split exact resolution of Z_2. If $\rho\colon \mathfrak{X}\otimes\mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ is the twisting chain map, then ρh^0 is again a Δ-chain map extension of f, because Δ is cocommutative. Hence, there exists a Δ-chain homotopy h^1 connecting h^0 with ρh^0. Since ρh^1 is a Δ-chain homotopy and since $h^1+\rho h^1$ is a Δ-chain map extension of the trivial Δ-homomorphism $0\colon Z_2\to\mathcal{I}md_1$, there exists a Δ-chain homotopy h^2 connecting h^1 and ρh^1. By the iterated use of the same arguments we have a sequence of Δ-chain homotopies $\{h^0, h^1, \cdots, h^i, \cdots\}$. Hence, we have

Proposition 6. *Let A be a cocommutative Hopf algebra over Z_2 and let $\Delta\colon A\to A\otimes A$ be the comultiplication. If $\varepsilon\colon \mathfrak{X}\to Z_2$ is a Z_2-split exact resolution of the A-module Z_2, then there exists a sequence of Δ-homomorphisms $h^i\colon \mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ for $i=0, 1,\cdots, n, \cdots$ such that* 1) *h^0 is a grade preserving Δ-chain map and* 2) *for $i>0$ h^i is a Δ-chain homotopy connecting h^{i-1} with ρh^{i-1} which raises the homological dimensions by i and preserves the grading, where $\rho\colon \mathfrak{X}\otimes\mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ is the twisting chain map.*

Consider a diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathfrak{X}, Z_2)\otimes\mathrm{Hom}_A(\mathfrak{X}, Z_2) & \overset{\underset{i}{\cup}}{\cdots\cdots>} & \mathrm{Hom}_A(\mathfrak{X}, Z_2) \\ \chi \searrow & & \nearrow h^{i\#} \\ & \mathrm{Hom}_{A\otimes A}(\mathfrak{X}\otimes\mathfrak{X}, Z_2) & \end{array}$$

where χ is the Z_2-chain map defined by $\chi(f\otimes g)(x\otimes y)=f(x)g(y)$ for $f, g\in \mathrm{Hom}_A(\mathfrak{X}, Z_2)$ and for $x, y\in\mathfrak{X}$.

DEFINITION 2. The cup-i-product $\underset{i}{\cup}$ in the cochain complex $C=\mathrm{Hom}_A(\mathfrak{X}, Z_2)$ is defined by $h^{i\#}\cdot\chi$.

Denoting $\mathrm{Hom}_A^s(X_p, Z_2)$ by $C^{p,s}$ for each homological dimension $p\geq 0$ and the grading $s\geq 0$, we have the cochain complex

$$C^{*s} = \{C^{p,s} \text{ for } p = 0, 1, \cdots, n, \cdots\}$$

such that $C=\{C^{*s}\mid s=0, 1, \cdots\}$. Then $f\underset{i}{\cup} g=\underset{i}{\cup}(f\otimes g)\in C^{p+q-i,s+t}$ for $f\in C^{p,s}$ and $g\in C^{q,t}$. It is immediate to see by definition the coboundary formula

$$\delta(f \underset{i}{\cup} g) = f \underset{i-1}{\cup} g + g \underset{i-1}{\cup} f + \delta f \underset{i}{\cup} g + f \underset{i}{\cup} \delta g\,.$$

For

$$\begin{aligned}\delta(f \underset{i}{\cup} g) &= \partial^{\#} h^{i\#} \chi(f \otimes g) = \chi(f \otimes g) h^i \partial \\ &= \chi(f \otimes g)(h^{i-1} + \rho h^{i-1} + d h^i) \\ &= h^{i-1\#}\chi(f\otimes g) + h^{i-1\#}\rho^{\#}\chi(f\otimes g) + h^{i\#}d^{\#}\chi(f\otimes g) \\ &= h^{i-1\#}\chi(f\otimes g) + h^{i-1\#}\chi(g\otimes f) + h^{i\#}\chi(\delta\otimes 1 + 1\otimes\delta)(f\otimes g) \\ &= f \underset{i-1}{\cup} g + g \underset{i-1}{\cup} f + \delta f \underset{i}{\cup} g + f \underset{i}{\cup} \delta g\,.\end{aligned}$$

DEFINITION 3. Algebraic Steenrod operation ${}_ASq_i$: $H^{p,s}(A) \rightarrow H^{2p-i,2s}(A)$ is defined by ${}_ASq_i(\xi) = \overline{f \underset{i}{\cup} f}$, where $\xi \in H^{p,s}(A)$ is represented by $f \in C^{p,s}$ with $\delta f = 0$, and the bar over $f \underset{i}{\cup} f$ stands for the cohomology class.

Adams [1] and others (for example, see [11]) computed explicitely a Δ-homomorphism h^i in case when $\mathfrak{X}$ is the bar resolution $B(A)$. If

$$\Delta(a) = \sum a' \otimes a''$$

for $a \in A$, then we have

$$\begin{aligned} &h_n^0([a_1|a_2|\cdots|a_n]) \\ &\quad = 1\otimes[a_1|\cdots|a_n] + \sum_{1\leq\rho\leq n}[a_1'|\cdots|a_\rho']\otimes a_1''\cdots a_\rho''[a_{\rho+1}|\cdots|a_n]\,,\end{aligned}$$

for odd i,

$$\begin{aligned} &h_n^i([a_1|\cdots|a_n]) \\ &= \sum_{0\leq\rho_0<\rho_1<\cdots\rho_i\leq n}[a_1'|\cdots|a_{\rho_0}'|a_{\rho_0+1}'\cdots a_{\rho_1}'|a_{\rho_1+1}'|\cdots|a_{\rho_2}'|\cdots|a_{\rho_{i-1}+1}'\cdots a_{\rho_i}'|a_{\rho_i+1}|\cdots|a_n] \\ &\quad \otimes a_1''\cdots a_{\rho_0}''[a_{\rho_0+1}''|\cdots|a_{\rho_1}''|a_{\rho_1+1}''\cdots a_{\rho_2}''|\cdots|a_{\rho_{i-1}+1}''|\cdots|a_{\rho_i}'']\,,\end{aligned}$$

for even i,

$$\begin{aligned} h_n^i([a_1|\cdots|a_n]) &= \sum_{0\leq\rho_0<\rho_1<\cdots<\rho_i\leq n}[a_1'|\cdots|a_{\rho_0}'|a_{\rho_0+1}'\cdots a_{\rho_1}'|\cdots|a_{\rho_{i-1}+1}'|\cdots|a_{\rho_i}'] \\ &\quad \otimes a_1''\cdots a_{\rho_0}''[a_{\rho_0+1}''|\cdots|a_{\rho_1}''|\cdots|a_{\rho_{i-1}+1}''\cdots a_{\rho_i}''|a_{\rho_i+1}|\cdots|a_n]\,.\end{aligned}$$

Let us sketch the method of computation for completeness sake. Let S be the contracting homotopy for $B(A)$, then $t = S\otimes 1 + \varepsilon\otimes S$ is a contracting homotopy for $B(A)\otimes B(A)$. Define $h_0^0 = \Delta$, then $h_1^0 S_0 = t_0 h_0^0$ determine h_1^0 by $h_1^0(ax) = \Delta(a) h_1^0(x)$. Inductively h^0 is obtained easily. Define $h_0^1 = t_0(h_0^0 + \rho h_0^0)$, then h_1^1 is calculated by $h_1^1 S_0 = t_1(h_1^0 + \rho h_1^0) S_0 + t_1 h_0^1$ and $h_1^1(ax) = \Delta(a) h_1^1(x)$. Repeat this process, we get the above formula.

3. $\mathfrak{S}(\Gamma, \Lambda)$ associated with a pair of Hopf algebras (Γ, Λ)

Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that the subhopf algebra Λ is central in Γ. Then we have a sequence of Hopf algebras

$$\Lambda \xrightarrow{i} \Gamma \xrightarrow{\pi} \Omega = \Gamma / \; I(\Lambda)\cdot\Gamma$$

where the inclusion i and the projection π are morphisms of Hopf algebras (see [1]). In this setting we are going to associate with a pair of Hopf algebras (Γ, Λ) a graded differential algebra $\mathfrak{S}(\Gamma, \Lambda)=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$ so that behaviors of algebraic Steenrod operations can be discussed in the spectral sequence $\{E_\gamma, d_\gamma\}$ associated with $\mathfrak{S}(\Gamma, \Lambda)$.

Recall the filtration in the bar construction $B(\Gamma)$ which Adams introduced in [1]. For each integer p define a subcomplex $F_pB(\Gamma)$ of $B(\Gamma)$ such that $F_pB(\Gamma)_n$ is the Γ-submodule of $B(\Gamma)_n=\Gamma\otimes I(\Gamma)^n$ generated by elements of the form $\gamma[\gamma_1|\cdots|\gamma_n]$ with the property that $\gamma_s\in I(\Lambda)$ for at least $(n-p)$ values of s. Then it is immediate to see that F is the canonical increasing filtration in $B(\Gamma)$. Define the product filtration $\overset{\times}{F}$ in $B(\Gamma)\otimes B(\Gamma)$ by

$$\overset{\times}{F}_p(B(\Gamma)\otimes B(\Gamma)) = \underset{p\geq s\geq 0}{\cup} F_{p-s}B(\Gamma)\otimes F_sB(\Gamma)\,.$$

Then $(B(\Gamma)\otimes B(\Gamma), \overset{\times}{F})$ is a resolution of $\Gamma\otimes\Gamma$-module Z_2 with the increasing filtration $\overset{\times}{F}$. Let $\Delta\colon \Gamma\to\Gamma\otimes\Gamma$ be the cocommutative diagonal and let ρ be the twisting chain map of $B(\Gamma)\otimes B(\Gamma)$. Then we have

Theorem 1. *There exists a sequence of Δ-homomorphisms*

$$h^i\colon B(\Gamma)\to B(\Gamma)\otimes B(\Gamma)$$

for $i=0, 1, \cdots, n, \cdots$ *such that* 1) h^0 *is a Δ-chain map which preserves grading and filtration,* 2) h^i *is a Δ-chain homotopy connecting h^{i-1} and ρh^{i-1} which preserves grading, raises homological dimension by i, and satisfies the filtration condition*

$$h^i(F_pB(\Gamma))\subset \overset{\times}{F}_\alpha(B(\Gamma)\otimes B(\Gamma))$$

for $\alpha=Min\{2p, p+i\}$.

Proof. In virtue of Proposition 6 it remains only to prove that h_n^i shown in §2 satisfies the filtration condition. By denoting

$$\Delta(\gamma) = \sum \gamma'\otimes\gamma''$$

the three formulas $h_n^i([\gamma_1|\cdots|\gamma_n])$ show that for each j with $n\geq j\geq 1$ exactly one of the three elements γ_j, γ_j', and γ_j'' appears solely between bars. For example, if i is odd, each of the elements $\gamma'_1, \cdots, \gamma'_{\rho_0}, \gamma''_{\rho_0+1}, \cdots, \gamma''_{\rho_1}, \gamma'_{\rho_1+1} \cdots, \gamma'_{\rho_2}, \cdots, \gamma''_{\rho_{i-1}+1}, \cdots, \gamma''_{\rho_i}, \gamma_{\rho_i+1}, \cdots, \gamma_n$ appears solely in $|\quad|$. It follows that if $[\gamma_1|\cdots|\gamma_n] \in F_p B(\Gamma)_n$, each term of the sum on the right hand sides of the formulas contains at least $(n-p)$ elements in $I(\Lambda)$. By definition of the product filtration $\overset{\times}{F}$ we obtain

$$h^i(F_p B(\Gamma)) \subset \overset{\times}{F}_{p+i}(B(\Gamma) \otimes B(\Gamma)) .$$

If $p \geq i$, then the proof is complete, because Min $\{p+i, 2p\} = p+i$. If $i \geq p$, it is seen that among i products

$$\gamma'_{\rho_0+1} \cdots \gamma'_{\rho_1}, \gamma''_{\rho_1+1} \cdots \gamma''_{\rho_2}, \cdots, \gamma'_{\rho_{i-1}+1} \cdots \gamma'_{\rho_i}$$

(or $\gamma''_{\rho_{i-1}+1} \cdots \gamma''_{\rho_i}$ if i is even) there exist at least $(i-p)$ products contained in $I(\Lambda)$. Otherwise, at least $(p+1)$ products are not contained in $I(\Lambda)$. Then γ_s are not in $I(\Lambda)$ for at least $(p+1)$ values of s. This is a contradiction. It follows that each term of the sum for $h^i_n([\gamma_1|\cdots|\gamma_n])$ has at least

$$(n-p)+(i-p) = n+i-2p$$

elements in $I(\Lambda)$. Therefore,

$$h^i(F_p B(\Gamma)) \subset \overset{\times}{F}_{2p}(B(\Gamma) \otimes B(\Gamma))$$

if $i \geq p$, where Min $\{p+i, 2p\} = 2p$. This completes the proof.

Now let us dualize what we have obtained in this section. Let (C, δ) be the cochain complex $\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$ over Z_2. For each integer p define a subcomplex $F^p(C)$ by the image of

$$\mathrm{Hom}_\Gamma (B(\Gamma)/F_{p-1}B(\Gamma), Z_2)$$

under the dual of the projection

$$p\colon B(\Gamma) \to B(\Gamma)/F_{p-1}B(\Gamma) .$$

Then it is seen that (C, δ, F) is a cochain complex with a decreasing filtration. Let us call it Adams filtered complex associated with (Γ, Λ).

Theorem 2. *Let (C, δ, F) be Adams filtered complex associated with a pair of Hopf algebras over Z_2. Then there exist a Z_2-linear map $\underset{i}{\cup}: C \otimes C \to C$ such that $\mathfrak{S}(\Gamma, \Lambda) = \{C, \delta, F, \underset{i}{\cup}\}$ is a graded differential algebra with a decreasing filtration F and with cup-i-products in the sense of Definition* 1.

Proof. Let $h^i\colon B(\Gamma) \to B(\Gamma) \otimes B(\Gamma)$ be the Δ-homomorphism in Theorem 1 and define $\underset{i}{\cup}: C \otimes C \to C$ by $h^{i\#}\chi$ as was considered in Definition 2. Since $\underset{i}{\cup}$

is the cup-i-product in $C=\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$, it is easy to see that $\underset{i}{\cup}$ satisfies all the necessary conditions except the filtration condition. Consequently, it is sufficient to show that if $f\in F^pC^{m,s}$ and $g\in F^qC^{n,t}$, then $f\underset{i}{\cup}g\in F^\alpha C^{m+n-i,s+t}$ for $\alpha=\mathrm{Max}\{p+q-i, p, q\}$. Consider first the case when

$$\alpha=\mathrm{Max}\{p+q-i, p, q\}=p+q-i,$$

then

$$\mathrm{Min}\{(\alpha-1)+i, 2(\alpha-1)\}=(\alpha-1)+i=p+q-1$$

except the case when $p=q=i$. By Theorem 1

$$h^i(F_{\alpha-1}B(\Gamma))\subset \overset{\times}{F}_{p+q-1}(B(\Gamma)\otimes B(\Gamma)).$$

If

$$h^i(x)=\sum x'\otimes x''$$

for $x\in F_{\alpha-1}B(\Gamma)_{m+n-i,s+t}$, then $x'\in F_\xi B(\Gamma)_{\rho,\theta}$ and $x''\in F_\eta B(\Gamma)_{\sigma,\nu}$ with the property that $\xi+\eta=p+q-1$, $\rho+\sigma=m+n$, and $\theta+\nu=s+t$. Then

$$(f\underset{i}{\cup}g)(x)=\sum f(x')\cdot g(x'')=0,$$

because $\xi<p$ or $\eta<q$. Therefore, $f\underset{i}{\cup}g\in F^{p+q-i}C^{m+n-i,s+t}$. If $\alpha=p$, then $p\geq q$ and $i\geq q$. In this case also,

$$(f\underset{i}{\cup}g)(F_{p-1}B(\Gamma))=0$$

can be shown because

$$h^i(F_{p-1}B(\Gamma))\subset \overset{\times}{F}_{p-1+i}(B(\Gamma)\otimes B(\Gamma))\cap \overset{\times}{F}_{2p-2}(B(\Gamma)\otimes B(\Gamma)).$$

Hence, the proof is completed.

From Theorem 2 and Proposition 1 we obtain

Theorem 3. *Let* (Γ, Λ) *be a pair of connected locally finite cocommutative Hopf algebras over* Z_2 *such that* Λ *is central in* Γ, *and let* $\{E_\gamma, d_\gamma\}$ *be Adams spectral sequence associated with the system* $\mathfrak{S}(\Gamma, \Lambda)$. *Then there exist algebraic Steenrod operations* ${}_BSt_i$: $E_\gamma^{p,q}\rightarrow E_{2\gamma-2}^{2p-i,2q}$ *for* $\infty\geq\gamma\geq 2$ *and* ${}_FSt_i$: $E_\gamma^{p,2q+p-i}$ *for* $\infty\geq\gamma\geq 1$.

4. Some properties of algebraic Steenrod operations

Theorem 4. ${}_BSt_i$ *and* ${}_FSt_i$ *defined in Adams spectral sequence satisfy Propositions* 2, 3, *and* 4.

Theorem 5. *Let* (Γ, Λ) *and* (Γ', Λ') *be pairs of Hopf algebras over* Z_2 *both of which satisfy the conditions stated before, and let* E_γ *and* E'_γ *be Adams spectral sequences associated with* $\mathfrak{S}(\Gamma, \Lambda)$ *and* $\mathfrak{S}(\Gamma', \Lambda')$ *respectively. If* $f: (\Gamma, \Lambda) \to (\Gamma', \Lambda')$ *be a morphism of pairs of Hopf algebras, then* f *induces a sequence of homomorphisms* $\phi_\gamma: E'_\gamma \to E_\gamma$ *for* $\gamma \geq 1$ *such that*

$$\phi_\gamma\ {}_FSt_i = {}_FSt_i\ \phi_\gamma \quad \text{and} \quad \phi_{2\gamma-2}\ {}_BSt_i = {}_BSt_i\ \phi_\gamma$$

for $\gamma \geq 2$.

Proof. It is obvious that f induces a chain map $B(f): B(\Gamma) \to B(\Gamma')$ preserving filtrations and gradings. If h^i and h'^i are Δ-homomorphisms in Theorem 1, then we have $h'^i B(f) = (B(f) \otimes B(f)) h^i$. Consequently, $B(f)$ induces a morphism $\mathfrak{S}(f): \mathfrak{S}(\Gamma', \Lambda') \to \mathfrak{S}(\Gamma, \Lambda)$. By a morphism $\mathfrak{S}(f)$ of the system $\mathfrak{S}$ we mean that $\mathfrak{S}(f)$ is a chain map compatible with gradings, filtrations, and cup-i-products. Therefore, it is straightforward to verify the theorem.

Theorem 6. *Let* $\Lambda \xrightarrow{i} \Gamma \xrightarrow{\pi} \Omega$ *be a sequence of Hopf algebras as stated before, and let* $\{E_\gamma\}$ *be Adams spectral sequence associated with* (Γ, Λ). *Then the natural maps* $B(\pi): B(\Gamma) \to B(\Omega)$ *and* $B(i): B(\Lambda) \to B(\Gamma)$ *induce isomorphisms* $B(\pi)^*: H^p(\Omega) \to E_2^{p,0}$ *and* $B(i)^*: E_2^{0,q} \to H^q(\Lambda)$ *respectively. If* $E_2^{p,0}$ *and* $E_2^{0,q}$ *are identified with* $H^p(\Omega)$ *and* $H^q(\Lambda)$ *respectively, then* ${}_BSt_i: E_2^{p,0} \to E_2^{2p-i,0}$ *coincides with* ${}_\Omega Sq_i: H^p(\Omega) \to H^{2p-i}(\Omega)$, *and* ${}_FSt_i: E_2^{0,q} \to E_2^{0,2q-i}$ *coincides with* ${}_\Lambda Sq_i: H^q(\Lambda) \to H^{2q-i}(\Lambda)$. *Moreover,* ${}_BSt_i: E_\infty^{p,q} \to E_\infty^{2p-i,2q}$ *for* $i \leq p$ *and* ${}_FSt_i: E_\infty^{p,q} \to E_\infty^{p,2q+p-i}$ *for* $i \geq p$ *are induced by* ${}_\Gamma Sq_i: H^{p+q}(\Gamma) \to H^{2p+2q-i}(\Gamma)$.

Proof. Adams has shown in [1] that $B(\pi)^*$ and $B(i)^*$ are isomorphisms. Hence, a morphism of pairs of Hopf algebras $\pi: (\Gamma, \Lambda) \to (\Omega, Z_2)$ induces the isomorphism $\phi_2: E_2'^{p,0} \to E_2^{p,0}$ for each p, because

$$E_2'^{p,0} = E_\infty'^{p,0} = H^p(\Omega)\,.$$

Since ${}_BSt_i: E_2'^{p,0} \to E_2'^{2p-i,0}$ is exactly ${}_\Omega Sq_i: H^p(\Omega) \to H^{2p-i}(\Omega)$, we obtain ${}_BSt_i\ \phi_2 = \phi_2\ {}_\Omega Sq_i$. Similarly, $\phi_2\ {}_FSt_i = {}_\Lambda Sq_i\ \phi_2$. From the facts that $H^{p+q}(\Gamma)$ is filtered by $F^pH^{p+q}(\Gamma) = Z_\infty^{p,q}/B_\infty^{p,q}$ with the property that $E_\infty^{p,q} = F^pH^{p+q}(\Omega)/F^{p+1}H^{p+q}(\Omega)$ and that ${}_\Gamma Sq_i$ maps $F^pH^{p+q}(\Gamma)$ into $F^{2p-i}H^{2p+2q-i}(\Gamma) \subset H^{2p+2q-i}(\Gamma)$, it is immediate to see that ${}_BSt_i: E_\infty^{p,q} \to E_\infty^{2p-i,2q}$ is induced by ${}_\Gamma Sq_i$. The rest of the proof is obvious. Hence, the proof is complete.

In a subsequent paper the author wishes to discuss higher cohomology operations involved in the Cartan formula and Massey-Uehara products.

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