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ACYCLIC FAKE SURFACES WHICH ARE SPINES OF 3-MANIFOLDS

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1. Introduction

In [1], we defined fake surfaces to study 3–manifolds with boundary from their spines. Let $\mathcal{F}$(s, t) denote the set of all the acyclic closed fake surfaces P with $\#\mathfrak{S}_2(P)=s$ and $\#\mathfrak{S}_3(P)=t$ ($\#$ means the number of the connected components). In this paper, we consider about the subset $\mathcal{E}(s, t)$ of $\mathcal{F}(s, t)$ each of whose elements can be embedded in some 3–manifold.

A connected closecd fake surface P is called a *normal spine*, if P can be embedded in a 3–manifold. That is, taking the regular neighborhood, we can regard P as a spine of a 3–manifold, when P is a normal spine. Of course, every element of $\mathcal{E}(s, t)$ is a normal spine.

We use the following notations. For a polyhedron P, $\dot{P}$ means the boundary of P, that is, $\dot{P}$ is the union of the free faces of P, and $\mathring{P}$ means the interior of P defined by $\mathring{P}=P-\dot{P}$. $\bar{P}$ means the closure of P, and I is the closed unit interval [0, 1]. For the other unexplained notations, see [1].

In §2, we prepare some lemmas for acylic normal spines by defining the *connected sum* of closed fake surfaces and the *r-th complement*. In §3, we obtain the sufficient condition that $\mathcal{E}(s, t)$ is empty, that is, Theorem 1 states that $\mathcal{E}(s, t)$ is empty if $s\geqq 2t$, (and, in the last section, we show that this is also the necessary condition). In §4, two types of *elementary deformation* of normal spines in the respective 3–manifolds are introduced and two invariants $\alpha(P)$ and $\beta(P)$ are defined for a closed fake surface P. And, in Theorem 2, we prove $\alpha(P)=r=\beta(P)$ when P is a r-th complement. In §5, all the elements of the set $\mathcal{E}(s, 2)$ are characterized geometrically using the concept of the *union* of closed fake surfaces, from which the Zeeman's conjecture is shown to be true for any element of $\mathcal{E}(s, 2)$, easily.

Zeeman's conjecture [2] : *If P is a contractible 2–polyhedron, then $P\times I$ is collapsible where $I=[0, 1]$ is the closed unit interval.*

In the last section, we obtain the geometrical characterizations of the elements of $\mathcal{E}(2t-1, t)$ and $\mathcal{E}(2t-2, t)$ for all integers $t\geqq 1$ and $t\geqq 2$, respectively. And, as the consequences, the Zeeman's conjecture for them follows.

Furthermore, in Theorem 6, we show that $\mathcal{E}(s, t)$ contains a spine of a 3-ball for any pair (s, t) with $1 \leqq s \leqq 2t-1$. Combining this with Theorem 1, we obtain the following.

Theorem. *$\mathcal{E}(s, t)$ is empty if and only if $s \geqq 2t$.*

On the other hand, it is easily seen that $\mathcal{F}(s, t)$ is empty if and only if $t=0$. The sufficiency follows from Theorem 1 [1]. To show the necessity, replace a 2-ball B in $\mathring{M}(P)$ of an element P of $\mathcal{E}(2t-1, t)$ by the element $\mathcal{N}_{s-2t+1}$ so that $\dot{B}=\mathcal{N}_{s-2t+1}$ (for the definition of $\mathcal{N}_{s-2t+1}$, see Definition 6, §6, [1]).

Note that $\mathcal{E}(1, 1)$ consists of a unique element $F^1_{1,1}$ by Theorem 4 [1] which is named "Abalone" by *H.* Noguchi and the realization of an abalone in the Euclidean 3-space R^3 is written in Figure 0 which is shown by Y. Tsukui.

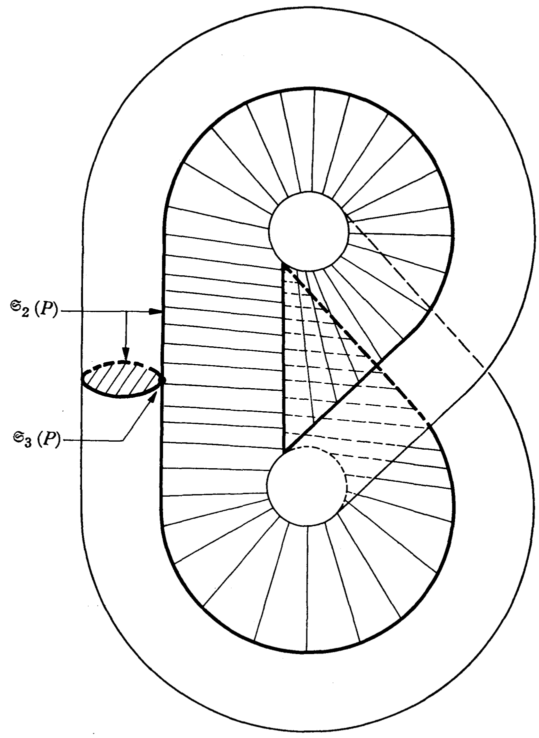

Fig. 0

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2. Lemmas

DEFINITION 1. Let P_i be a closed fake surface with a 2–ball B_i in $\mathring{M}(P)$, $i=1, 2$, and f a homeomorphism from $\dot{B}_1$ to $\dot{B}_2$. We define the *connected sum* $P_1 \circ P_2$ of P_1 and P_2 with respect to B_1, B_2 and f by $P_1 \circ P_2 = ((P_1 - \mathring{B}_1) \cup (P_2 - \mathring{B}_2))/f$.

DEFINITION 2. First, define the 0-th *complement* to be an acyclic normal spine. A connected closed fake surface X is said to be a r-*th complement* if there exists an acyclic fake surface P such that $X \circ P$ is a $(r-1)$-th complement.

DEFINITION 3. Let P be a fake surface. We say that a connected component U of $U(P)$ is *isolated* if $\mathfrak{S}_3(U)$ is empty. And let $\nu(P)$ denote the number of the isolated components of $U(P)$.

Lemma 1. *Let P be a closed fake surface. If U(P) is embeddable in an orientable 3–manifold, P is a normal spine.*

Proof. Let W be an orientable 3–manifold in which $U(P)$ is embedded, and let M be an element of $M(P)$ with boundary $\dot{M} = b_1 \cup \cdots \cup b_j$. Let us consider $M \times I$ and $A_i = b_i \times I$ where I denote the closed unit interval [0, 1] and $M = M \times 1/2$, and the 2–nd derived neighborhood N_i of b_i in the boundary of the regular neighborhood N of $U(P)$ in W mod $\dot{U}(P)$, $i=1, \cdots, j$. Since $\dot{N}$ is a disjoint union of orientable closed 2–manifolds, there is a homeomorphism f_i from A_i onto N_i which is the identity on b_i. Then, we obtain a homeomorphism h_M from $\bigcup_i A_i = \dot{M} \times I$ onto $\bigcup_i N_i$ defined by f_i on each A_i. Define the 3–manifold

$$V = \bigcup_M ((N \cup (M \times I))/h_M),$$

that is, V is the 3–manifold obtained from N and $M(P) \times I$ by identifying A_i and N_i by f_i for all $i=1, \cdots$, j and for all elements M of $M(P)$. Obviously, P is embedded in the 3–manifold V, completing the proof.

Lemma 2. *Let P be a closed fake surface with $H_1(P)=0$. Then, P is a normal spine if and only if U(P) can be embedded in R^3, the Euclidean 3–space.*

Proof. Sufficiency follows immediately from Lemma 1. So, we prove Necessity. Let W be a 3–manifold in which P is embedded. Since W is orientable and $U(P)$ collapses to the 1–polyhedron $\mathfrak{S}_2(P)$, the regular neighborhood N of $U(P)$ in W is a disjoint union of solid tori with certain genus. Then, N is embeddable in R^3, and hence, so is the subpolyhedron $U(P)$.

Lemma 3. (i) *Let X be a r-th complement. Then, we have $H_1(X)=0$ and $H_2(X)=Z+\cdots+Z$ of rank r.*

(ii) *A r-th complement X is a normal spine.*

(iii) *Let* $X=X_1\circ X_2$ *be a r-th complement. Then,* X_i *is a* r_i*-th complement for* $i=1, 2$, *and* $r_1+r_2=r+1$.

Proof. The proof goes by induction on r. When $r=0$, there is nothing to prove (i) and (ii). So, we prove (iii). By Lemma 14 [1], we may assume that X_1 is acyclic. Then, X_2 is a 1–st complement from the definition. Since X is a normal spine, X_1 is also a normal spine, by Lemma 2, because $U(X_1)$ is contained in $U(X)$ and is embeddable in R^3. Thus, X_1 is a 0-th complement. Now, we consider the case $r\geqq 1$. That is, there is an acyclic closed fake surface P such that $X\circ P$ is a $(r-1)$-th complement, where the connected sum is taken with respect to the 2–balls B_X and B_P contained in $M(X)$ and $M(P)$ and a homeomorphism f from $\dot{B}_X$ to $\dot{B}_P$. Define $Q=(X\circ P)\cup(\dot{B}_P*v)$ where v is an ideal coing point over $\dot{B}_P$, that is, $(\dot{B}_P*v)$ is the cone from v over $\dot{B}_P$ and $(X\circ P)\cap(\dot{B}_P*v)=\dot{B}_P$. Using the inductive hypothesis $H_1(X\circ P)=0$ and $H_2(X\circ P)=Z+\cdots+Z$ of rank $r-1$, we obtain $H_1(Q)=0$ and $H_2(Q)=Z+\cdots+Z$ of rank r by the Mayer-Vietoris exact sequence. Since $H_q(Q)=H_q(X)+H_q(P)$ and P is acyclic, we see $H_1(X)=0$ and $H_2(X)=Z+\cdots+Z$ of rank r. This proves (i). By the inductive hypothesis, $U(X\circ P)=U(X)\cup U(P)$ can be embedded in R^3. Then $U(X)$ is, of course, embeddable in R^3, and hence, by Lemma 2, X is a normal spine. This proves (ii). Now, we may assume that the 2–ball B_X is contained in X_1, because B_X can be moved away from X_2 when $B_X\cap(X_1\cap X_2)$ is non-empty by an isotopy of X. Then, we can write $X\circ P=(X_1\circ P)\circ X_2$. Then, by the inductive hypothesis, $(X_1\circ P)$ is a r'-th complement and X_2 a r_2-th one and $r'+r_2=r$. Then, X_1 is a $(r'+1)$-th complement, because P is acyclic. Thus, we have $r_1=r'+1$, and hence $r_1+r_2=r+1$. This completes the proof of Lemma 3.

Lemma 4. *Let P be a normal spine with* $H_1(P)=0$ *and* $H_2(P)=Z$. *Then,* $\mathfrak{S}_3(P)$ *is empty if and only if P is a 2–sphere.*

Proof. Sufficiency is trivial. We prove Necessity. It is clear that a 2–sphere satisfies the required conditions and the other 2–manifolds do not. Hence Lemma 4 is true if P is a 2–manifold. So, we assume that $\mathfrak{S}_2(P)$ is non-empty and try to prove that such P does not exist. Let $U(P)=U_1\cup\cdots\cup U_n$ where U_i means a connected component of $U(P)$ for $i=1,\cdots, n$. Then, each U_i must be isolated, because $\mathfrak{S}_3(P)$ is empty. And since P is a normal spine with $H_1(P)=0$, U_i is neither $S\times\tau T$ nor $S\times\sigma T$, by Lemma 24 [1], Lemma 2 and Corollary to Theorem 1[1]. That is, $U_i=S\times T$ for any $i=1,\cdots, n$. The proof goes by induction on n. When $n=1$, $M(P)$ consists of three 2–balls by Lemma 12 [1] and Proposition 4 [1], and P is obtained from $M(P)$ by identifying their boundaries as indicated in Figure 1.

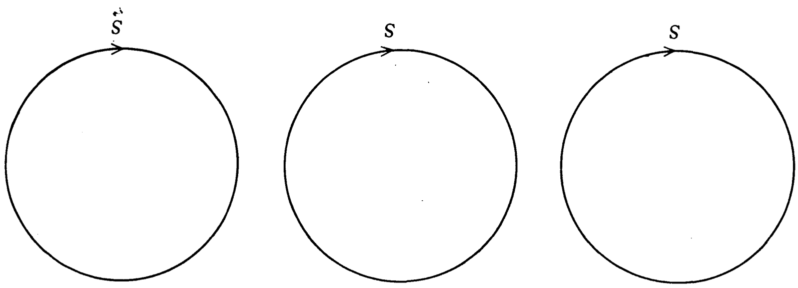


Fig. 1

Then, we have $H_2(P)=Z+Z$ which contradicts to our hypothesis $H_2(P)=Z$. Now, we deal with the case $n\geqq 2$. Then, there is an element M with $\# \dot{M}\geqq 2$ in $M(P)$ by Lemma 14 [1], and a boundary component b of M disconnects P into two fake surfaces P_1 and P_2 such that $\mathfrak{S}_2(P_i)$ is non-empty for both $i=1, 2$, by Lemma 14 of [1]. Let $\tilde{P}=P\cup(b^*v)$ and $\tilde{P}_i=P_i\cup(b^*v)$, $i=1, 2$, where v is an ideal coning point over b. Then, by the Mayer Vietoris exact sequence, we obtain $H_1(\tilde{P})=0$ and $H_2(\tilde{P})=Z+Z$, and hence $H_1(\tilde{P}_i)=0$ for both $i=1, 2$, and $H_2(\tilde{P}_1)+H_2(\tilde{P}_2)=Z+Z$. Suppose $H_2(\tilde{P}_1)=0$. Then, $\tilde{P}_1$ is an acyclic closed fake surface without 3–rd singularity, which is a contradiction to Theorem 1 [1]. Thus, we see $H_2(\tilde{P}_i)=Z$ for both $i=1, 2$. Since P is a normal spine, $\tilde{P}_i$ is also a normal spine by Lemma 2. And, clearly, $1\leqq \# U(\tilde{P}_i)\leqq n-1$ holds true, because $\mathfrak{S}_2(\tilde{P}_i)$ is non-empty. This contradicts to our inductive hypothesis, competing the proof.

REMARK. It is easy to see that a 2–sphere S^2 is a 1–st complement, because $S^2\circ F^1_{1,1}$ is homeomorphic to $F^1_{1,1}$.

Lemma 5. *Let $P=P_1\circ P_2$ be an element of $\mathcal{E}(s, t)$. Suppose that P_1 is not acyclic. Then, P_1 is either a 2–sphere or a 1–st complement with $1\leqq \#\mathfrak{S}_2(P_1)\leqq t-1$.*

Proof. By Lemma 14 [1], P_2 is acyclic, and hence $\#\mathfrak{S}_3(P_2)\geqq 1$, by Theorem 1 [1]. Then, P_1 is a 1–st complement. Suppose $\#\mathfrak{S}_3(P_1)=0$. Then, by Lemma 4, P_1 is a 2–sphere. And when $1\leqq \#\mathfrak{S}_3(P_1)$, we see $\#\mathfrak{S}_3(P_1)\leqq t-1$, because $\#\mathfrak{S}_3(P_2)\geqq 1$ and $\#\mathfrak{S}_3(P_1)+\#\mathfrak{S}_3(P_2)=t$.

Lemma 6. *Let P be an element of $\mathcal{F}(s, t)$ with an isolated component $U=S\times T$. Then, just one of the connected components of $\overline{P-U}$ is acyclic.*

Proof. By Lemma 13 [1], $\overline{P-U}$ is the disjoint union of three connected fake surfaces P_1, P_2 and P_3. First, we show that at least one of P_1, P_2 and P_3 is acyclic. Suppose that P_3 is not acyclic. Then, by Lemma14 [1], we obtain an acyclic fake surface $P_0=P_1\cup U\cup P_2$. Since $U=S\times T$, we obtain an acyclic closed fake surface Q from P_0 by collapsing P_0 from its boundary $\dot{P}_0$ by the natural way. And the 1–sphere $\mathfrak{S}_2(U)$ disconnects Q into two fake surfaces Q_1 and Q_2 so that P_i is contained in Q_i, for $i=1, 2$. Note that P_i is homeomorphic to Q_i, $i=1, 2$. Then, by the Mayer-Vietoris exact sequence, we obtain $H_2(Q_i)=0$ for both $i=1, 2$, and $H_1(Q_1)+H_1(Q_2)=Z$. Hence, either Q_1 or Q_2 is acyclic, that is, either P_1 or P_2 is acyclic. Suppose that there are two acyclic components P_1 and P_2. Define $P_0=P_1 \cup U \cup P_2$. Then, we easily have $H_1(P_0)=0$ and $H_2(P_0)=Z$ which implies $H_2(P)\neq 0$. This proves Lemma 6.

Lemma 7. *Let P be an element of $\mathcal{E}(s, t)$ with $\nu(P)\geqq 1$. Then, there is an isolated component U in U(P) such that there exists a connected component Q of $\overline{P-U}$ with $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q)\neq 0$.*

Proof. Let U_i be an isolated component of $U(P)$. Then, $U_i=S\times T$ by the same reason as in the proof of Lemma 4. And hence $\overline{P-U_i}$ has three connected components P_{i1}, P_{i2} and P_{i3}. By Lemma 6, we assume that P_{i3} is acyclic. Then, of course, P_{ij} is not acyclic, for $j=1, 2$. If we consider $\tilde{P}_{ij}=P_{ij}\cup(\dot{P}_{ij}*v_j)$, we see that $\tilde{P}_{ij}$ is acyclic, for $j=1, 2$, by Lemma 14 [1]. And $\#\mathfrak{S}_3(P_{ij})=\#\mathfrak{S}_3(\tilde{P}_{ij})\neq 0$, by Theorem 1 [1], for $J=1, 2$. Now, it is sufficient to prove the following statement (*) by induction on $\nu=\nu(P_{i1})$.

(*) *Either* (1) *U_i is a required isolated component U in U(P), or* (2) *we can find U in P_{i1}, holds true.*

Proof of (*). When $\nu=0$, there is nothing to prove by taking $U=U_i$ and $Q=P_{i1}$. So, we assume that (*) is true for $\nu(P_{i1})\leqq\nu-1$, and we deal with the case $\nu\geqq 1$. Let U_k be an isolated component of $U(P)$ contained in P_{i1}. Then, either P_{k1} or P_{k2} is contained in P_{i1}, say P_{k1}. Then, (*) is true for P_{k1}, by the inductive hypothesis, because

$$\nu(P_{k1})\leqq\nu(P_{i1})-1=\nu-1.$$

Then, clearly, U is contained in P_{i1}, completing the proof.

3. The sufficient condition that $\mathcal{E}$(s, t) be empty

Proposition 1. *Let P be an element of $\mathcal{E}(s, t)$. Then, we obtain $s\geqq 2\nu(P)+1$.*

Proof. The proof goes by induction on s. We see $s\geqq 1$ by Theorem 1 [1], and when $s=1$, there is nothing to prove, because $\nu(P)=0$ by Theorem 1 [1] again. We deal with the case $s\geqq 2$. If $U(P)$ contains no isolated component, that is, $\nu(P)=0$, Proposition 1 is trivially ture for P. Thus, we may assume that there exist an isolted component U and a connected component Q of $\overline{P-U}$ with $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q)\neq 0$ obtained in Lemma 7. Let us consider $X=\overline{P-Q}$, $Y=X\cup(\dot{X}*v)$ and $W=Q\cup(\dot{Q}*v)$ where v is an ideal coning point over the

1–sphere $\dot{X}=\dot{Q}$. Then, we can write $P=W\circ Y$, by identifying the 2–balls $(\dot{X}*v)$ and $(\dot{Q}*v)$. And, by Lemma 3, there are following two cases.

Case 1. *W is a 0–th complement and Y is a 1–st one.*

By Lemma 14 [1], X must be acyclic, and hence we can collapse X to an acyclic closed fake surface X' from $\dot{X}$ by the natural way, because $U=S\times T$. Then, X' is also a normal spine by Lemma 2, and we easily have $1\leqq\#\mathfrak{S}_2(X')=s'\leqq s-1$, because X' is acyclic and does not contain U. Hence, we have $s'\geqq 2\nu(X')+1$, by the inductive hypothesis. Put $s''=\#\mathfrak{S}_2(W)$. Then, we see $s=s'+s''+1$ and $\nu(P)=\nu(X')+1$. Hence,

$$\begin{aligned} s-2\nu(P) &= (s'-s''+1)-2(\nu(X')+1.) \\ &= (s'-2\nu(X'))+(s''-1) \\ &\geqq 1, \end{aligned}$$

because $\#\mathfrak{S}_3(W)=\#\mathfrak{S}_3(Q)\neq 0$ means $s''\neq 0$. Therefore, we obtain $s\geqq 2\,\nu(P)+1$.

Case 2. *W is a 1–st complement and Y is a 0–th one.*

In this case, we see $1\leqq\#\mathfrak{S}_2(Y)=s_1\leqq s-1$, by the condition $s''\neq 0$. Then, by the inductive hypothesis, we obtain $s_1\geqq 2\nu(Y)+1$, because Y is an acyclic normal spine by Lemma 2. And, in this case, we see $s=s_1+s''$ and $\nu(P)=\nu(Y)$ from which $s\geqq 2\nu(P)+1$ follows by a similar calculation to Case 1. Thus, Proposition 1 is established.

Theorem 1. *$\mathcal{E}(s, t)$ is empty if $s\geqq 2t$.*

Proof. Suppose that $\mathcal{E}(s, t)$ is non-empty. And let P be an element of $\mathcal{E}(s, t)$. Then, we have

$$s\geqq 2\,\nu(P)+1\geqq 2(s-t)+1$$

from Proposition 1. Hence $s\leqq 2t-1$. This proves Theorem 1.

4. Elementary deformations of normal spines in the 3-manifolds

Let P be a normal spine in a 3–manifold V with nonempty 2–nd singularity, i. e. $\mathfrak{S}_2(P)\neq\phi$. Suppose that there is a 1–ball A in P satisfying the following conditions (1) and (2).

(1) $A\cap\mathfrak{S}_2(P)=\dot{A}=a_1\cup a_2$.

(2) a_1 *and a_2 are vertices of $\mathfrak{S}_2(P)-\mathfrak{S}_3(P)$.*

Taking the 2–nd derived neighborhood N of A in V, $\dot{N}-(\dot{N}\cap P)$ consists of four open 2–balls each of whose closures is a 2–ball B_i, $i=1,\cdots,4$. Let B_1 be the 2–ball contained in st (a_1, V). Note that such a 2–ball is uniquely determined (see Figure 2). Then, we may regard the 3–ball $N=B_1\times I$ and hence we can collapse N to $\dot{N}-\mathring{B}_1$ from the free face $B_1=B_1\times 0$.

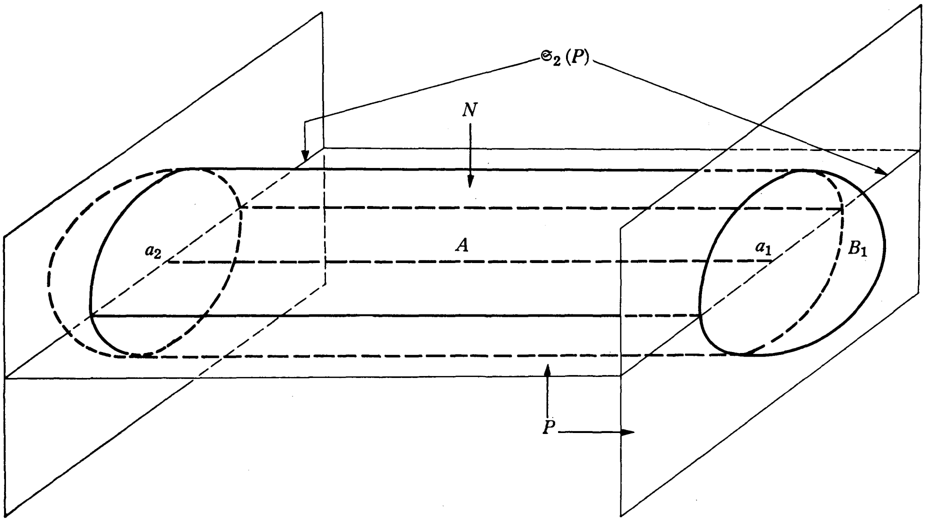


Fig. 2

DEFINITION 4. Define the normal spine $P(1)$ by

$$P(1) = (P-(P\cap N))\cup(\dot{N}-\mathring{B}_1),$$

and we say that $P(1)$ *is obtained from P by an elementary deformation in V* (*with respect to A*). Inductively, we can define $P(r)$ as a normal spine obtained from $P(r-1)$ by an elementary deformation in V, and we say that $P(r)$ is obtained from P by r times of elementary deformation in V.

DEFINITION 5. An elementary deformation is said to be of type I, if the boundary $\mathring{A}$ is contained in a connected component of $\mathfrak{S}_2(P)$, and of type II otherwise.

DEFINITION 6. Let P be a closed fake surface. We define the invariants $\alpha(P)$ and $\beta(P)$ by

$$\alpha(P) = \sharp M(P)-\sharp\mathfrak{S}_2(P)-\sharp\mathfrak{S}_3(P), \text{ and}$$
$$\beta(P) = \sharp\dot{M}(P)-2\sharp\mathfrak{S}_2(P)-\sharp\mathfrak{S}_3(P)+1.$$

Lemma 8. *Let P be a normal spine of a 3–manifold V and P(r) a normal spine obtained from P by r times of elementary deformation in V. Then, P(r) is also a spine of V.*

Proof. From the definition of $P(r)$, it is sufficient to prove that P and $P(1)$ are simple homotopy equivalent in V. Let N be the 2–nd derived neigh neighborhood of A in V in the above definition. Then, P expands to $P\cup N$ and $P\cup N$ collapses to $P(1)$ in V, and hence P and $P(1)$ are simple homotopy equivalent in V.

The following two lemmas are immediate from Figure 2.

Lemma 9. *Let P be a normal spine in a 3–manifold V and $P(r)$ a normal spine obtained from P by r times of elementary deformation of type I in V. Then, we have;*

(1) $\#\mathfrak{S}_2(P(r)) = \#\mathfrak{S}_2(P)$, *and*

(2) $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3(P)+2r$.

Lemma 10. *Let P be a normal spine in a 3–manifold V and $P(r)$ a normal spine obtained from P by r times of elementary deformation of type II in V. Then, we have;*

(1) $\#\mathfrak{S}_2(P(r)) = \#\mathfrak{S}_2(P)-r$,

(2) $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3P)+2r$,

(3) $\#M(P(r)) = \#M(P)+r$, *and*

(4) $\#\dot{M}(P(r)) = \#\dot{M}(P)$.

Proposition 2. *Let P be an element of $\mathcal{E}(s, t)$. Then, we obtain $\alpha(P)=0=\beta(P)$.*

Proof. The proof is done by induction on s. When $s=1$, Proposition 4 and Proposition 5 [1] give the answer. Suppose $s\geqq 2$. Since P is connected, we can apply an elementary deformation of type *II* to P in some 3–manifold, and we obtain $P(1)$ which belongs to $\mathcal{E}(s-1, t+2)$ by Lemma 10. Then, by the inductive hypothesis and Lemma 10, we have

$$\begin{aligned}\alpha(P) &= \#M(P)-\#\mathfrak{S}_2(P)-\#\mathfrak{S}_3(P)\\ &= (\#M(P(1))-1)-s-t\\ &= ((s-1)+(t+2)-1)-s-t\\ &= 0.\end{aligned}$$

And, by the same way, we can prove $\beta(P)=0$.

Theorem 2. *Let X be an r-th complement. Then, we obtain $\alpha(X)=r=\beta(X)$..*

Proof. The proof is done by induction on r. When $r=0$, Proposition 2 gives the answer. We assume $r\geqq 1$. Let P be an acyclic fake surface (closed) such that $X\circ P$ becomes an $(r-1)$-th complement. Note that P is necessarily a 0-th complement. Clearly, the followings hold true.

$$\begin{aligned}\#\mathfrak{S}_2(X\circ P) &= \#\mathfrak{S}_2(X)+\#\mathfrak{S}_2(P),\\ \#\mathfrak{S}_3(X\circ P) &= \#\mathfrak{S}_3(X)+\#\mathfrak{S}_3(P),\\ \#M(X\circ P) &= \#M(X)+\#M(P)-1,\\ \#\dot{M}(X\circ P) &= \#\dot{M}(X)+\#\dot{M}(P).\end{aligned}$$

Then, we have $\alpha(X\circ P)=\alpha(X)+\alpha(P)-1$ and $\beta(X\circ P)=\beta(X)+\beta(P)-1$. Thus, by the inductive hypothesis and Proposition 1 which means $\alpha(P)=0=\beta(P)$, we easily obtain $\alpha(X)=r=\beta(X)$.

5. $\mathcal{E}(s, 2)$.

DEFINITION 7. Let P_i be a closed fake surface with a 2–ball B_i in $\mathring{M}(P_i)$, $i=1, 2$, and let f be a homeomorphism from B_1 onto B_2. We define the *union* $P_1\oplus P_2$ of P_1 and P_2 with respect to B_1, B_2 and f by $P_1\oplus P_2=(P_1\cup P_2)/f$.

Proposition 3. *Let P be an element of* $\mathcal{E}(3, 2)$. *Then, we obtain* $P=F^1_{1,1}\oplus F^1_{1,1}$.

Proof. First, we obtain $\nu(P)=1$, because

$$\nu(P)\geqq\#\mathfrak{S}_2(P)-\#\mathfrak{S}_3(P) = 1,$$
$$\nu(P)\leqq(\#\mathfrak{S}_2(P)-1)/2 = 1.$$

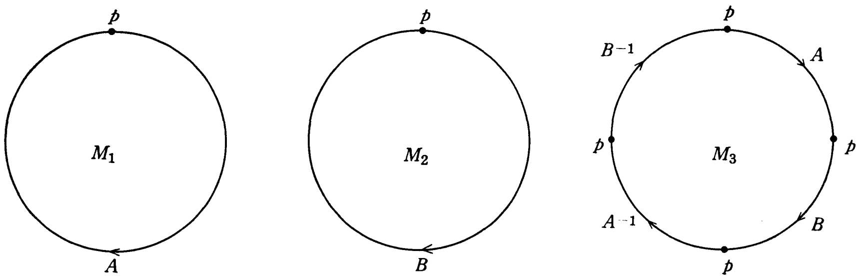
The 2–nd inequality follows from Proposition 1. Let U denote the isolated component of $U(P)$ and P_i the connected component of $\overline{P-U}$, $i=1, 2, 3$. Since $\#\mathfrak{S}_3(P)=2$, we may assume that P_2 contains no point of $\mathfrak{S}_3(P)$. We show that P_2 is acyclic. Suppose not. Then, $\tilde{P}_2=P_2\cup(\dot{P}_2*v)$ is a acyclic closed fake surface without 3–rd singularity. This contradicts to Theorem 1 [1]. Putting $Q=\overline{P-P_2}$, we define $\tilde{Q}=Q\cup(\dot{Q}*v)$. Then, clearly, we can write $P=\tilde{P}_2\circ\tilde{Q}$ using the 2–balls $(\dot{P}_2*v)$ and $(\dot{Q}*v)$. Since P_2 is acyclic, $\tilde{P}_2$ is not acyclic, by Lemma 14 [1]. Then, by Lemma 5, $\tilde{P}_2$ is a 2–sphere, because $\#\mathfrak{S}_3(\tilde{P}_2)=\#\mathfrak{S}_3(P_2)=0$. Hence P_2 is a 2–ball. Define $\tilde{P}_i=P_i\cup(\dot{P}_i*v_i)$, for $i=1, 3$. Then, $\tilde{P}_i$ is an acyclic normal spine by Lemma 14 [1] and Lemma 2, because P_i is not acyclic by Lemma 6 for $i=1, 3$. Since $\tilde{P}_i$ is acyclic, we see $\#\mathfrak{S}_3(\tilde{P}_i)\geqq 1$, and hence $\#\mathfrak{S}_3(\tilde{P}_i)=1$ by $\#\mathfrak{S}_3(\tilde{P}_1)+\#\mathfrak{S}_3(\tilde{P}_3)=\#\mathfrak{S}_3(P)=2$. Similarly, we have $\#\mathfrak{S}_2(\tilde{P}_i)=1$ for $i=1, 3$. Thus, $\tilde{P}_i$ is an element of $\mathcal{E}(1, 1)$, that is, $\tilde{P}_i=F^1_{1,1}$, for $i=1, 3$. It is clear that P is obtained from $\tilde{P}_1$ and $\tilde{P}_2$ by identifying the 2–balls $(\dot{P}_1*v_1)$ and $(\dot{P}_2*v_2)$ to the 2–ball P_2, that is, $P=\tilde{P}_1\oplus\tilde{P}_3=F^1_{1,1}\oplus F^1_{1,1}$.

REMARK. The number of the elements of $\mathcal{E}(3, 2)$ is, clearly, at most 6.

Lemma 11. *Let G be a 1-st complement. Suppose that* $\#\mathfrak{S}_2(G)=1=\#\mathfrak{S}_3(G)$. *Then, G is uniquely determined as described in Fig.* 3.

Proof. We obtain the Homology groups $H_1(G)=0$ and $H_2(G)=Z$ by Lemma 3. By Theorem 2, we see $\alpha(G)=1=\beta(G)$ which impies $\#M(G)=3$. Then, by Lemma 12 [1] and Proposition 4 [1], it is known that $M(G)$ consists of three 2–balls M_1, M_2 and M_3. Then, we check all the possible cases as explained in the last half part of the proof of Theorem 2[1]. And we obtain the identification of M_1, M_2 and M_3 as shown in Fig. 3, uniquely.

1-st complement G.



1-st complement G.

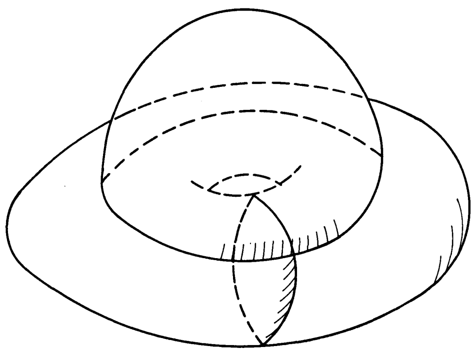


Fig. 3

REMARK. From now on, let G denote the unique 1–st complement obtained in Lemma 11.

REMARK. Let B_G be a 2–ball in $\dot{M}(G)$ and P an acyclic closed fake surface with a 2–ball B_P in $\dot{M}(P)$. Let $G\circ P$ be the connected sum with respect to B_G and B_P. Then, it is easy to see that $G\circ P$ is acyclic if and only if B_G is contained in M_3 (for M_3, see Fig. 3). And, from now on, B_G denotes the 2–ball contained in M_3.

Proposition 4. *Let P be an element of $\mathcal{E}(2, 2)$. Then, we obtain $P=G\circ F^1_{1,1}$.*

Proof. There exists an element M in $M(P)$ with $\sharp\dot{M}=2$, because $\sharp M(P)=4$ and $\sharp\dot{M}(P)=5$ by Theorem 2. By cutting P along a boundary component of M and attaching a 2–ball to the boundary of each connected components, we can write $P=P_1\circ P_2$ and we have $\sharp\mathfrak{S}_3(P_i)\neq 0$ for $i=1, 2$, because $\sharp\mathfrak{S}_2(P_i)\neq 0$ is clear and $\nu(P)=0$ implies $\nu(P_i)=0$ for both $i=1, 2$. Note that $\nu(P)=0$ follows from Proposition 1. Then, by Lemma 3, We may assume that P_1 is a

1–st complement and P_2 is a 0-th one. Since $\#\mathfrak{S}_2(P_i)=1=\#\mathfrak{S}_3(P_i)$ for both $i=1, 2$, we have $P_1=G$ and $P_2=\mathrm{F}^1_{1,1}$, completing the proof.

REMARK. The number of the elements of $\mathcal{E}(2, 2)$ is at most 4.

Proposition 5. *$\mathcal{E}(1, 2)$ consists of three elements $F^1_{1,2}$, $F^2_{1,2}$, and $F^3_{1,2}$ which are described in Fig.* 4.

Proof. By the same way as expained in the last half part of the proof of Theorem 2 [1], we obtain the elements as shown in Fig. 4.

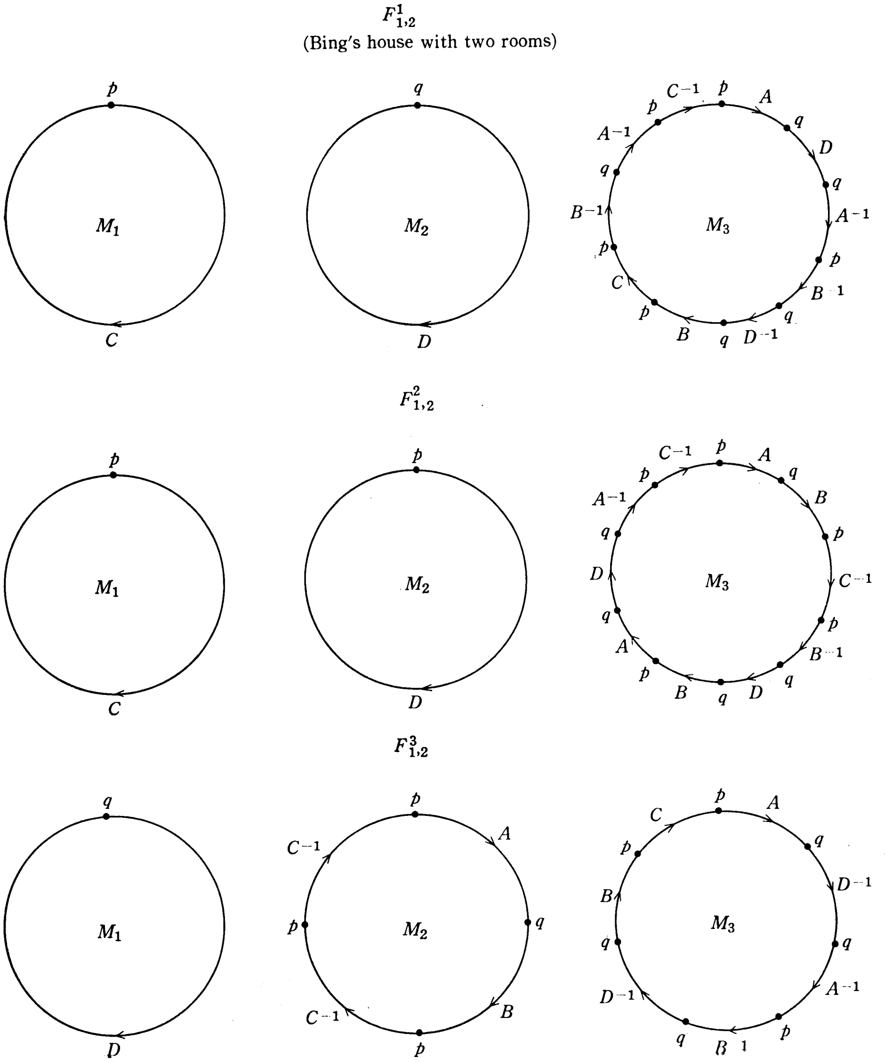


Fig. 4

REMARK. The element $F^1_{1,2}$ of $\mathcal{E}(1, 2)$ is well-known as "Bing's House with two rooms".

Theorem 3. *Zeeman's conjecture holds true for any element P of $\mathcal{E}(s, 2)$, that is, $P\times I$ is collapsible.*

Proof. Case 1. When $s=3$, we see $P=F^1_{1,1}\oplus F^1_{1,1}$ by Proposition 3, and hence, $P\times I$ is collapsible by Proposition 8 of [1].

Case 2. When $s=2$, we obtain $P=G\circ P$, from Proposition 4. Then, by the same way as Case 2 in the proof of Theorem 3 [1], $P\times I$ is collapsible, because $G-\mathring{B}_G$ is collapsible.

Case 3. When $s=1$, $P\times I$ is collapsible by the same way as Case 1 in the proof of Theorem 3 [1], by attaching a 3-ball to M_1 (for M_1, see Fig. 4).

6. $\mathcal{E}(s, t)$ with $1\leqq s\leqq 2t-1$.

In this section, we characterize, geometrically, the elements of the sets $\mathcal{E}(2t-1, t)$ and $(2t-2, t)$ and prove the converse of Theorem 1.

Theorem 4. *Let P be an element of $\mathcal{E}(s, t)$ with $s=2t-1$ and $t\geqq 2$. Then, we can write $P=P_1\oplus P_2$ where P_i belongs to $\mathcal{E}(s_i, t_i)$ with $s_i=2t_i-1$, $t_1+t_2=t$ and $t_i\geqq 1$, $i=1, 2$.*

Proof. The proof goes by induction on t. When $t=2$, Proposition 3 gives the answer. So, we assume $t\geqq 3$. Since $s=2t-1$, we obtain $\nu(P)=t-1$, because

$$t-1=s-t\leqq \nu(P)\leqq (s-1)/2=t-1.$$

by Proposition 1. Hence $\nu(P)\geqq 1$. Let U and Q be the isolated component of $U(P)$ and the connected component of $\overline{P-U}$ obtained in Lemma 7. Now, we show that Q is not acyclic. Suppose not. Then, $\hat{A}=A\cup(\mathring{A}*v)$ must be acyclic by Lemma 14 [1], where $A=\overline{P-Q}$. And we have $\nu(\hat{A})=\nu(P)$ and $\#\mathfrak{S}_2(\hat{A})\leqq s-1$, because, by Lemma 7, $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q)\neq 0$ implies $\#\mathfrak{S}_2(Q)\neq 0$. Then, we obtain

$$\#\mathfrak{S}_2(\hat{A})\leqq s-1 = 2t-2 = 2\nu(A)$$

which contradicts to Proposition 1, because $\hat{A}$ is a normal spine by Lemma 2. Thus, Q is not acyclic and hence A is acyclic. Then, A collapses naturally to an acyclic normal spine A_1 from $\mathring{A}$. Note that $U=S\times T$. And $\nu(A_1)=\nu(P)-1$ is trivial. Then, we have $\#\mathfrak{S}_2(A_1)=s-2$, because

$$\begin{aligned} s-2 \geqq \#\mathfrak{S}_2(A_1) &\geqq 2\nu(A_1)+1 \\ &= 2\nu(P)-1 \\ &= 2t-3 \\ &= s-2. \end{aligned}$$

And we see $\#\mathfrak{S}_3(A_1)\geqq t-1$, because

$$t-2 = \nu(P)-1 = \nu(A_1)\geqq s-2-\#\mathfrak{S}_3(A_1).$$

Since $\#\mathfrak{S}_3(Q)\neq 0$ by Lemma 7, we obtain $\#\mathfrak{S}_3(A_1)=t-1$. Therefore, A_1 is an element of $\mathcal{E}(s_1', t_1')$ with

$$s_1' = s-2 = 2t-3 = 2(t-1)-1 = 2t_1'-1.$$

And consequently, we see $\#\mathfrak{S}_2(Q)=1=\#\mathfrak{S}_3(Q)$. Let S denote the base space of the T-bundle $U=S\times T$.

Case 1. Suppose that S bounds a 2–ball in $M(A_1)$. Let $\tilde{Q}=Q\cup(\dot{Q}*v)$. Then, $\tilde{Q}$ belongs to $\mathcal{E}(1, 1)$. And it is easy to write $P=A_1\oplus Q$ by identifying the 2–balls B and $(\dot{Q}*v)$. Putting $P_1=A_1$ and $P_2=Q$, the required conditions in Theorem 4 are satisfied.

Case 2. Suppose that S does not bound a 2–ball in $M(A_1)$. By the inductive hypothesis, we can write $A_1=A_2\oplus A_3$ with respect to the 2–balls B_2 and B_3 contained in $M(A_2)$ and $M(A_3)$, respectively, where A_i belongs to $\mathcal{E}(s_i', t_i')$ with $s_i^1=2t_i'-1$, $t_2'+t_3'=t_1'$ and $t_i\geqq 1$, $i=1, 2$. Since S does not bounds a 2–ball in $M(A_1)$, S is contained in either A_2-B_2 or A_3-B_3, say A_2-B_2. Let us define $P_1=A_2\cup U\cup Q$ and $P_2=A_3$. Then, using the 2–balls B_2 and B_3, we can write $P=P_1\oplus P_2$. And it is clear that P_1 belongs to $\mathcal{E}(s_2'+2, t_2'+1)$. And hence, $s_2'+2=(2t_2'-1)+2=2(t_2'+1)-1$. Thus, the reaquired conditions in Theorem 4 are satisfied. And Theorem 4 is now established.

Corollary to Theorem 4. *For any element P of $\mathcal{E}(2t-1, t)$ with $t\geqq 1$, the Zeeman's conjecture holds true, that is, $P\times I$ is collapsible.*

Proof. By Theorem 4, $\mathcal{E}(2t-1, t)$ is contained in C_t defined in §9 [1], for any integer $t\geqq 1$. Then, $P\times I$ is collapsible by Proposition 8 [1].

In order to characterize the elements of $\mathcal{E}(s, t)$ in the case $s=2t-2$, we extend the definition of the union of closed fake surfaces as follows.

DEFINITION 8. Let P_i be a closed fake surface with an acyclic fake surface A_i such that the boundary $\dot{A}_i$ is a 1–sphere contained in $\mathring{M}(P_i)$ and A_i is a connected component of P disconnected by $\dot{A}_i$, $i=1, 2$. Suppose that there is a homeomorphism f from A_1 onto A_2. Define the *union* $P_1\underset{A}{\oplus}P_2$ of P_1 and P_2 with respect to $A=A_1=A_2$ and f by $P_1\underset{A}{\oplus}P_2=(P_1\cup P_2)/f$.

Then, in general, we obtain the following.

Proposition 6. (1) *Let P be an element of $\mathcal{E}(s, t)$ with $\nu(P)\geqq 1$. Then, there exists an acyclic fake surface A in P such that we can write $P=P_1\underset{A}{\oplus}P_2$.*

(2) *If we can write* $P=P_1\underset{A}{\oplus}P_2$ *for an element* P *of* $\mathcal{E}(s, t)$, *we obtain the following conditions.*

(i) P_i *belongs to* $\mathcal{E}(s_i, t_i)$, $i = 1, 2$.

(ii) $s_i \geqq \#\mathfrak{S}_2(A)+1, i = 1, 2$.

(iii) $t_i \geqq \#\mathfrak{S}_3(A)+1, i = 1, 2$.

(iv) $s_1+s_2-\#\mathfrak{S}_2(A) = s-1$.

(v) $t_1+t_2-\#\mathfrak{S}_3(A)=t$.

Proof. Since $\nu(P)\geqq 1$, there exists an isolated component U in $U(P)$. And we see $U=S\times T$, because P belongs to $\mathcal{E}(s, t)$. Then, by Lemma 6, there exists an acyclic component A in $\overline{P-U}$, uniquely, and the other components than A of $\overline{P-U}$ are denoted by Q_1 and Q_2. Note that $\#\mathfrak{S}_3(Q_i)\neq 0$ for $i=1, 2$, because $\tilde{Q}_i=Q_i\cup(\dot{Q}_i*v_i)$ is an acyclic normal spine and hence $\#\mathfrak{S}_3(Q_i)=\#\mathfrak{S}_3(\tilde{Q}_i)\neq 0$, by Theorem 1 [1]. Now, unpasting P at A, we obtain two closed fake surfaces P_1 and P_2, and it is clear that P can be written $P=P_1\underset{A}{\oplus}P_2$. This proves (1). And it is also clear that P_i is an acyclic normal spine for $i=1, 2$, that is, P_i belongs to $\mathcal{E}(s_i, t_i)$, because both P and A are acyclic. We may assume $P_i\supset Q_i$, for $i=1, 2$. Then, the conditions (ii) and (iii) are proved by $\mathfrak{S}_j(Q_i)\cup\mathfrak{S}_j(A)=\mathfrak{S}_j(P_i)$, for $i=1, 2$, and $j=2, 3$. The condition (iv) follows from the facts $\mathfrak{S}_2(P_1)\cup U\cup\mathfrak{S}_2(P_2)=\mathfrak{S}_2(P)$ and $\mathfrak{S}_2(A)\subset\mathfrak{S}_2(P_i)$, for both i—1, 2. The last condition (v) is also satisfied by $\mathfrak{S}_3(P_1)\cup\mathfrak{S}_3(P_2)=\mathfrak{S}_3(Q_1)\cup\mathfrak{S}_3(Q_2)\cup\mathfrak{S}_3(A)=\mathfrak{S}_3(P)$ and $\mathfrak{S}_3(A)\subset\mathfrak{S}_3(P_i)$ for both $i=1, 2$.

REMARK. Let G be the 1–st complement obtained in Lemma 11 and B_G the 2–ball in M_3 of G (see Remark to Lemma 11). From now on, $G-\mathring{B}_G$ is denoted by G_0.

Theorem 5. *Let* P *be an element of* $\mathcal{E}(s, t)$ *with* $s=2t-2$ *and* $t\geqq 3$. *Then, we can write* $P=P_1\underset{A}{\oplus}P_2$ *so that* A *is either a 2–ball or* G_0 *and* P_i *belongs to* $\mathcal{E}(s_i, t_i)$, $i=1, 2$. *And if* A *is a 2–ball, we obtain* $s_1=2t_1-1$ *and* $s_2=2t_2-2$. *If* A *is* G_0, *we obtain* $s_i=2t_i-2$, *for both* $i=1, 2$.

Proof. This theorem is also proved by induction on t by the similar argument to the proof of Theorem 4. However, the preparation is more complicated. In this case, we obtain $\nu(P)=t-2$, because

$$t-2 = s-t\leqq\nu(P)\leqq(s-1)/2<t-1.$$

We can find an isolated component U in $U(P)$ and a connected component Q in $\overline{P-U}$ with $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q)\neq 0$, by Lemma 7.

Step 1. In this step, we study about Q.

Case 1. Suppose that Q is acyclic.

In this case, we show $\#\mathfrak{S}_2(Q)=1=\#\mathfrak{S}_3(Q)$ which implies $Q=G_0$, because $\tilde{Q}=\dot{Q}\cup(Q^*v)$ is a 1-st complement.

Since Q is acylic and $\nu(Q)=0$, we obtain $\tilde{F}=F\cup(\dot{F}^*v)$ is a acyclic normal spine with $\nu(\tilde{F})=\nu(P)$, where $F=\overline{P-Q}$. Then, we see $\#\mathfrak{S}_2(\tilde{F})=s-1$ and $\#\mathfrak{S}_3(\tilde{F})=t-1$. because

$$\begin{aligned} s-1 \geqq \#\mathfrak{S}_2(\tilde{F}) &\geqq 2\nu(\tilde{F})+1 \\ &= 2t-3 \\ &= s-1, \end{aligned}$$

and

$$\begin{aligned} t-1 \geqq \#\mathfrak{S}_3(\tilde{F}) &\geqq (\#\mathfrak{S}_2(\tilde{F})+1)/2 \\ &= t-1, \end{aligned}$$

by Proposition 1 and Theorem 1. Hence, we obtain the required condition $\#\mathfrak{S}_2(Q)=1=\#\mathfrak{S}_3(Q)$, because

$$\#\mathfrak{S}_j(\tilde{F})+\#\mathfrak{S}_j(Q) = \#\mathfrak{S}_j(P)$$

is true for $j=2, 3$.

Case 2. Suppose that Q is not acyclic.

In this case, F is acyclic and hence we obtain an acyclic normal spine F_1 from F by a natural collapsing. And we have $\nu(F_1)=\nu(P)-1=t-3$. Then, by the similar arargument to the proof of Theorem 4 and Case 1 in this step, we can prove that the pair $(\#\mathfrak{S}_2(F_1), \#\mathfrak{S}_3(F_1))$ is either $(s-2, t-1)$ or $(s-3, t-2)$. Thus, we obtain the following statement (*).

(*) $(\#\mathfrak{S}_2(Q), \#\mathfrak{S}_3(Q))=(k, k)$ if and only if $(\#\mathfrak{S}_2(F_1), \#\mathfrak{S}_3(F_1))=(s-1-k, t-k)$, for $k=1, 2$.

Step 2. Suppose $t=3$ (the 1-st step of induction).

Then, $\nu(P)=t-2=1$. Then, by Proposition 6, we can write $P=P_1\underset{A}{\oplus}P_2$. Since $\nu(P)=1$ implies $\nu(P_i)=0$ for both $i=1, 2$, we obtain the two possibility. That is, if $\#\mathfrak{S}_3(A)=0$, then A is a 2-ball by Lemma 4 or Lemma 5. And hence P_i belongs to $\mathcal{E}(i, i)$ for $i=1, 2$. And if $\#\mathfrak{S}_3(A)\neq 0$, we see $A=G_0$ by Step 1 (Case 1), because $\nu(A)=0$. Hence, we can write $P=P_1\underset{G_0}{\oplus}P_2$, and P_i belongs to $\mathcal{E}(2, 2)$ for both $i=1, 2$, by Proposition 6.

Step 3. We deal with the case $t\geqq 4$.

Case 1. Suppose that Q is acyclic.

In this case, take $A=Q$. Then, $Q=G_0$ by Case 1 of Step 1, and hence, $P=P_1\underset{G_0}{\oplus}P_2$ and P_i belongs to $\mathcal{E}(s_i, t_i)$, $i=1, 2$. By Proposition 6, we obtain $s_1+s_2=s$ and $s_i\geqq 2$ and $t_1+t_2=t+1$ and $t_i\geqq 2$. Put $s_i=2t_i-u_i$, $i=1, 2$. Then, we obtain $u_1+u_2=4$, because

$$
\begin{aligned}
2t-(u_1+u_2-2) &= (2t_1-u_1)+(2t_2-u_2)\\
&= s_1+s_2\\
&= s\\
&= 2t-2.
\end{aligned}
$$

Since $u_i \geqq 1$ by Theorem 1, for both $i=1, 2$, we see that the pair (u_1, u_2) is either (1, 3) or (2, 2). Suppose $u_1=1$. Then, P_1 must be an element of $\mathcal{E}(2t_1-1, t_1)$. But, for any integer $t_1 \geqq 1$, it is clear, from Theorem 4, that no element of $\mathcal{E}(2t_1-1, t_1)$ contains G_0 as a subpolyhedron. Thus, (u_1, u_2) must be (2, 2), and hence $s_i=2t_i-2$ for both $i=1, 2$. This completes the proof of this case.

Case 2. Suppose that Q is not acyclic.

In this case, the construction of P_1 and P_2 highly resembles to the last Case 2 in the proof of Theorem 4. We use the statement (*) in Case 2 in Step 1. When $k=1$, we can write $F_1=F_2 \underset{A}{\oplus} F_3$ by the inductive hypothesis. And if $k=2$, we can write $F_1=F_2 \oplus F_3$ by Theorem 4. And we obtain P_1 and P_2 as required in Theorem 5.

Thus, Theorem 5 is established.

When we define the set $\mathcal{C}$ of acyclic normal spines obtained from $\mathcal{E}(1, 1)$ and $\mathcal{E}(2, 2)$ using $P_1 \underset{G_0}{\oplus} P_2$ and $P_1 \oplus P_2$ as the set $\mathcal{C}_t$ defined in §9 in [1], we have the following proposition by the similar reason to that of Proposition 8 [1].

Proposition 7. *Let P be an element of $\mathcal{C}$. Then, $P \times I$ is collapsible.*

And we have the following as a corollary to Theorem 5, because $\mathcal{E}(2t-2, t)$ is contained in $\mathcal{C}$ by Theorem 5.

Corollary to Theorem 5. *For any element P of $\mathcal{E}(2t-2, t)$ with $t \geqq 2$, the Zeeman's conjecture is true, that is, $P \times I$ is collapsible.*

We prepare the following lemmas to prove Theorem 6.

Lemma 12. *$\mathcal{E}(1, t)$ contains a spine of a 3-ball, for any integer $t \geqq 1$.*

Proof. Suppose that t is odd, that is, $t=2r+1$. When $r=0$, there is nothing to prove, because the unique element $F^1_{1,1}$ (abalone) of $\mathcal{E}(1, 1)$ is a spine of a 3-ball by Theorems 3 and 4 [1]. We construct a normal spine of a 3-ball in $\mathcal{E}(1, t)$ inductively. Let P be an element of $\mathcal{E}(1, 2(r-1)+1)$ which is a spine of a 3-ball V. Then, we can apply an elementary deformation of type I to P in V, and we obtain a normal spine $P(1)$ of V, by Lemma 8. Then, by Lemma 9, it is clear that $P(1)$ belongs to $\mathcal{E}(1, t)$. When t is even, we obtain a spine of a 3-ball in $\mathcal{E}(1, t)$ by the same way as above from an element of $\mathcal{E}(1, 2)$ which is non-empty by Proposition 5 and it is known, by Theorem 3, that any element of $\mathcal{E}(1, 2)$ is a spine of a 3–ball.

Lemma 13. *Suppose that G_0 is embedded in a 3–ball V properly, that is, $G_0 \cap \dot{V} = \dot{G}_0$. Then, V collapses to G_0.*

Proof. Let N be the regular neighborhood of G_0 in V meeting the boundary regularly, that is, $N \cap \dot{V}$ is a regular neighborhood of $\dot{G}_0$ in $\dot{V}$. Since G_0 is collapsible and $\dot{G}_0$ *is a* 1–sphere, N is a 3–ball and $N \cap \dot{V}$ is an annulus. Then, $\overline{V-N}$ is the disjoint union of two 3–balls V_1 and V_2. And, clearly, $N \cap V_i = \dot{N} \cap \dot{V}_i = F_i$ is a 2–ball for $i=1, 2$. Then, V collapses to N by collapsing each V_i to F_i and N collapses to G_0. Thus, V collapses to G_0.

Lemma 14. *Let P be a normal spine of a 3–manifold W, that is, W collapses to P. Then, $G \circ P$ is also a spine of W, where the connected sum is taken with respect to B_G.*

Proof. Let B_P be the 2–ball of P used in the connected sum $G \circ P$, and let N be the 2–nd derived neighborhood of B_P in W mod $\dot{B}_P$. Note that we can expand P to $P \cup N$ in W. It is possible to replace B_P by G_0 in N to satisfy $G_0 \cap \dot{N} = \dot{G}_0 = \dot{B}_P$, because N is a 3–ball and $\dot{G}_0$ and $\dot{B}_P$ are 1–spheres. Then, by Lemma 13, N collapses to G_0, and hence $P \cup N$ collapses to $(P - B_P) \cup G_0$ which is clearly $G \circ P$. Thus, $G \circ P$ is a spine of W.

Theorem 6. *$\mathcal{E}(s, t)$ contains a spine of a 3–ball for any pair (s, t) with $1 \leqq s \leqq 2t-1$.*

Proof. By Lemma 12 and Corollary to Theorem 4, each of $\mathcal{E}(1, t)$ and $\mathcal{E}(2t-1, t)$ contains a spine of a 3–ball for any integer $t \geqq 1$. So, assuming $2 \leqq s \leqq 2t-2$, we construct a spine Q of a 3–ball in $\mathcal{E}(s, t)$ inductively. Suppose that P is a spine of a 3–ball in $\mathcal{E}(s-1, t-1)$. Define $Q = G \circ P$. Then, by Lemma 14, Q is also a spine of a 3–ball and clearly Q belongs to $\mathcal{E}(s, t)$.

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