

BELTRAMI DIFFERENTIAL EQUATION AND QUASICONFORMAL MAPPING

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1. Introduction

Let $\varphi(z)$ be a holomorphic function in the disk $D = \{z \mid |z| < 1\}$ and $k (< 1)$ a positive constant. Put $\mu(z) = k\bar{\varphi}'(z)/\varphi'(z)$ in D and $\mu(z) = 0$ outside D . Then the Beltrami differential equation $w_{\bar{z}} = \mu(z)w_z$ is known to have a homeomorphic solution $w = f(z)$ in $|z| \leq \infty$: $f(z)$ is a Teichmüller mapping in D and is meromorphic outside D : further the solution $f(z)$ is unique if normalized by the condition $f(1) = 1, \lim_{z \rightarrow \infty} f(z)/z = 1$. (see [1], p. 91). In this paper we restrict ourselves to the case in which $\varphi(z)$ is rational and investigate the solutions of those Beltrami equations. First we introduce a function $\Phi(z)$ which is defined by means of $\varphi(z)$ and satisfies the relation $g \circ f(z) = \Phi(z)$ for some rational function $g(z)$. Next we find the conditions for $\varphi(z)$ under which $f(z)$ maps D onto itself. These are equivalent to the condition for $f(z)$ to fix the boundary of D pointwise. From this we shall obtain short proofs of Theorem 6 in [2] and Theorem 2.3 in [3]. Finally we have an example which fixes the boundary of D pointwise for some k but not for k' other than k .

2. $\Phi(z)$ is a branched covering

Let $\varphi(z)$ be a non-constant rational function holomorphic in D . Put with some $k, 0 < k < 1$,

$$\Phi(z) = \begin{cases} \varphi(z) + k\bar{\varphi}(z) & \text{for } z \text{ in } D, \\ \varphi(z) + k\bar{\varphi}(1/\bar{z}) & \text{for } z \text{ outside } D. \end{cases}$$

Then we have

Lemma. $\Phi(z)$ is a branched covering and has the same number of sheets as $\varphi(z)$.

Proof. By definition $\Phi(z)$ is a branched covering in D and outside \bar{D} . On the boundary of D , $\varphi(z) + k\bar{\varphi}(z)$ and $\varphi(z) + k\bar{\varphi}(1/\bar{z})$ have the same values and the same orientation. Therefore $\Phi(z)$ has the same multiplicities as $\varphi(z)$ on the

boundary of D , so that it is an unlimited branched covering. Next we count the number of sheets. Writing

$$\varphi(z) = \gamma \frac{(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)}$$

we have

$$\Phi(z) = \gamma \frac{(z-\alpha_1)\cdots(z-\alpha_m)}{(z-\beta_1)\cdots(z-\beta_n)} + k\bar{\gamma} \frac{z^{n-m}(1-\bar{\alpha}_1 z)\cdots(1-\bar{\alpha}_m z)}{(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_n z)}$$

on the complement of D . Since $\beta_i, i=1, \dots, n$, lie outside D , the number of z with the multiplicities at which $\Phi(z)=\infty$ is $n+\max(m-n, 0)=\max(m, n)$ which is equal to the number of sheets of $\varphi(z)$. q.e.d.

Let $f(z)$ be the normalized solution of $f_{\bar{z}}(z)=\mu(z)f_z(z)$ with $\mu(z)=k\bar{\varphi}'(z)/\varphi'(z)$ in D and $=0$ outside D . Then $\Phi \circ f^{-1}$ is a branched covering with the same number of sheets as $\varphi(z)$. $f(z)$ is meromorphic outside \bar{D} with a simple pole at ∞ so that $\Phi \circ f^{-1}$ is meromorphic outside $f(\bar{D})$. It will be shown as follows that $\Phi \circ f^{-1}$ is holomorphic in $f(D)$. The differentiation of $f \circ f^{-1}(w)=w$ with respect to \bar{w} gives

$$(f_z \circ f^{-1}(w))f^{-1}(w)_{\bar{w}} + (f_{\bar{z}} \circ f^{-1}(w))\bar{f}^{-1}(w)_{\bar{w}} = 0 \quad a.e.$$

or

$$f^{-1}(w)_{\bar{w}} = -\frac{f_{\bar{z}}(z)}{f_z(z)}\bar{f}^{-1}(w)_{\bar{w}} = -k\frac{\bar{\varphi}'(z)}{\varphi'(z)}\bar{f}^{-1}(w)_{\bar{w}} \quad a.e..$$

So, we have

$$\begin{aligned} (\Phi \circ f^{-1}(w))_{\bar{w}} &= (\Phi_z \circ f^{-1}(w))f^{-1}(w)_{\bar{w}} + (\Phi_{\bar{z}} \circ f^{-1}(w))\bar{f}^{-1}(w)_{\bar{w}} \\ &= (\varphi' \circ f^{-1}(w))(-k)\frac{\bar{\varphi}'(z)}{\varphi'(z)}\bar{f}^{-1}(w)_{\bar{w}} + k(\bar{\varphi}' \circ f^{-1}(w))\bar{f}^{-1}(w)_{\bar{w}} = 0 \quad a.e.. \end{aligned}$$

This shows the holomorphy of $\Phi \circ f^{-1}$ in $f(D)$. Except for a finite number of points which are f -images of the critical points of $\varphi(z)$, $\Phi \circ f^{-1}(w)$ is holomorphic on the boundary of $f(D)$ because it is locally a composition of the quasiconformal mappings and 1-quasiconformal. By the finite multivalency of $\Phi \circ f^{-1}$ it is meromorphic at the excepted points so that it is a rational function. We formulate this as

Theorem 1. $f(z)$ and $\Phi(z)$ are related with a rational function $g(z)$ such that $g \circ f(z)=\Phi(z)$.

Application. We consider the expansion of $f(z)$ outside \bar{D} . Under the normalization $f(0)=0$, instead of $f(1)=1$, $f(z)$ has an expansion

$$f(z) = z + P(\mu(h+1)) = z + P\mu + P\mu T\mu + P\mu T\mu T\mu + \dots,$$

where

$$Ph(\zeta) = -\frac{1}{\pi} \iint h(z) \left(\frac{1}{z-\zeta} - \frac{1}{z} \right) dx dy \text{ and } Th(\zeta) = \lim_{\varepsilon \rightarrow 0} -\frac{1}{\pi} \iint_{|z-\zeta|>\varepsilon} \frac{h(z)}{(z-\zeta)^2} dx dy.$$

(see [1]). If $g(z)$ is determined explicitly we shall be able to see $P\mu, P\mu T\mu, P\mu T\mu T\mu, \dots$ successively. For example consider the case of

$$\varphi(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z$$

where a_i are real and $\{z | \varphi'(z)=0\}$ lies on the segment $[-1, 1]$. Then we find $g(w)=(1+k)\varphi((1+k)^{-1/n}w)$ and $\Phi(z)=\varphi(z)+k\varphi(1/z)$ outside D . Substituting these into $g \circ f(z)=\Phi(z)$ we have

$$\begin{aligned} \sum_{i=1}^n a_i(1+k)^{1-i/n}(z+kP\mu'+k^2P\mu'T\mu'+k^3P\mu'T\mu'T\mu'+\dots)^i \\ = \varphi(z)+k\varphi(1/z), a_n = 1. \end{aligned}$$

Here we put $\mu'=\mu/k=\bar{\varphi}'(z)/\varphi'(z)$. Comparing the coefficients of j -th power of $k, j=0, 1, 2, \dots$, of both sides we have

$$\begin{aligned} P\mu' &= (\varphi(1/z)-\varphi(z))/\varphi'(z)+z/n, \\ P\mu'T\mu' &= -\frac{1}{2\varphi'(z)}(\varphi''(z)(P\mu')^2+2P\mu'\sum_{i=1}^n i(1-i/n)a_i z^{i-1} \\ &\quad + \sum_{i=1}^n (1-i/n)(-i/n)a_i z^i), \\ &\dots \end{aligned}$$

3. The case in which $f(z)$ keeps every boundary point of D fixed

Let $F(z)$ be a quasiconformal mapping of D onto D which satisfies $F_{\bar{z}}(z)=\mu(z)F_z(z)$. If $F(z)$ fixes the boundary of D pointwisely, then we have a normalized solution of $f_{\bar{z}}(z)=\mu(z)f_z(z)$ by setting $f(z)=F(z)$ in D and $f(z)=z$ outside D . This implies that $f(z)$ maps D onto D . Conversely if a normalized solution $f(z)$ maps D onto D then $f(z)=z$ outside D and therefore it fixes the boundary of D pointwisely. The restriction of $f(z)$ to D is a solution of $F_{\bar{z}}(z)=\mu(z)F_z(z)$ which fixes the boundary of D pointwisely. Therefore we can say that $F(z)$ fixes the boundary of D pointwisely if and only if the normalized solution $f(z)$ maps D onto D . If $f(z)$ maps D onto D , then we have $g(z)=\varphi(z)+k\bar{\varphi}(1/\bar{z})$ outside D and therefore everywhere. In this case all poles of $\varphi(z)$ lie on the boundary of D . More precisely, $m \leq n$ and $|\beta_i|=1, i=1, 2, \dots, n$. Proof is as follows;

First we observe that the number of sheets of $g(z)$ is equal to $\max(m, n)$. This follows readily from Lemma and Theorem 1. $g(z)$ has poles at $\beta_i, \bar{\beta}_i^{-1}, i=$

1, 2, ..., n , and at 0, ∞ if $m > n$. If $m > n$ then the number of z with the multiplicities at which $g(z) = \infty$ is not less than $2(m-n) + n = m + (m-n)$, which is a contradiction. Therefore $m \leq n$. If there is a β_i , $|\beta_i| \neq 1$, then the number of z with the multiplicities at which $g(z) = \infty$ is greater than n , a contradiction. The assertion follows.

The identity $g(z) = \varphi(z) + k\bar{\varphi}(1/\bar{z})$ implies that $\varphi(z) + k\bar{\varphi}(1/\bar{z})$ has the branch points at w_i , $i=1, 2, \dots, l$, and only there in D , where w_i is the f -image of the branch point z_i , $i=1, 2, \dots, l$, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i , and that $\varphi(w_i) + k\bar{\varphi}(1/\bar{w}_i) = \varphi(z_i) + k\bar{\varphi}(z_i)$, $i=1, 2, \dots, l$. Conversely if $g(z) = \varphi(z) + k\bar{\varphi}(1/\bar{z})$, this is true when the above conditions on $\varphi(z)$ are satisfied, then $f(z)$ maps D onto D . We summarize those as

Theorem 2. *The followings are all equivalent.*

- a) $F(z)$ fixes the boundary of D pointwisely,
- b) $f(z)$ maps D onto D ,
- c) $g(z) = \varphi(z) + k\bar{\varphi}(1/\bar{z})$,
- d) $\varphi(z)$ has poles only on the boundary of D , $\varphi(z) + k\bar{\varphi}(1/\bar{z})$ has the branch points at w_i , $i=1, 2, \dots, l$, and only there in D , where w_i is the f -image of the branch point z_i , $i=1, 2, \dots, l$, of $\varphi(z)$ in D with the same order as $\varphi(z)$ has at z_i , and $\varphi(w_i) + k\bar{\varphi}(1/\bar{w}_i) = \varphi(z_i) + k\bar{\varphi}(z_i)$, $i=1, 2, \dots, l$.

4. Short proofs

If $\varphi(z)$ has no branch point in D , then d) implies that $\varphi(z)$ has poles only on the boundary of D . This is Theorem 6 in [2]. On the other hand if $\varphi(z)$ has the branch points in D and if a) is true for all k , $0 < k < 1$, then we can show $w_i = z_i$, $i=1, 2, \dots, l$. In this case d) implies that $\varphi(z)$ has poles only on the boundary of D and $\bar{\varphi}(1/\bar{z})$ has the branch points at z_i , $i=1, 2, \dots, l$, and only there in D with the same order as $\varphi(z)$ has at z_i and that $\varphi(1/\bar{z}_i) = \varphi(z_i)$, $i=1, 2, \dots, l$. Conversely if the above conditions on $\varphi(z)$ are satisfied then d) is satisfied with $w_i = z_i$, $i=1, 2, \dots, l$, and hence a) is true for all k , $0 < k < 1$. This is Theorem 2.3 in [3].

Proof of $w_i = z_i$, $i=1, 2, \dots, l$. By the well known fact that $|w_i - z_i| < 2k$ for all k , $0 < k < 1$, d) implies that for all k

$$k(\bar{\varphi}(z_i) - \bar{\varphi}(1/\bar{w}_i)) = \varphi(w_i) - \varphi(z_i) = \varphi'(z_i)(w_i - z_i) + O((w_i - z_i)^2).$$

Dividing both sides by k and letting $k \rightarrow 0$, we have $\varphi(1/\bar{z}_i) = \varphi(z_i)$. Therefore $\varphi(z) + k\bar{\varphi}(1/\bar{z}) = \varphi(z_i) + k\bar{\varphi}(z_i) = \varphi(z_i) + k\bar{\varphi}(1/\bar{z}_i)$ is satisfied by z_i and w_i . We set $E_i = \{z \mid \varphi(z) = \varphi(z_i)\} \cap D$ and $E_i' = \{z \in E_i \mid z \neq z_i\}$. Then for sufficiently small k , w_i lies near z_i , and $f(E_i')$ and w_i have a positive distance which tends to the distance between E_i' and z_i as $k \rightarrow 0$, hence we have $w_i = z_i$ for small k . By the continuity of $f(z)$ in k we have $w_i = z_i$ for all k , $0 < k < 1$, because all w_i , $i=$

1, 2, ..., l , are fixed for small k and they do not change with each other without a jump.

5. Special solution

In general c) in Theorem 2 does not imply that $w_i = z_i$, $i = 1, 2, \dots, l$, for there are $\varphi(z)$ and k , that is $\mu(z) = k\bar{\varphi}'(z)/\varphi'(z)$, such that $f(z)$ maps D onto D and z_1 to $w_1 \neq z_1$. This gives an example of $\mu(z)$ for which $F(z)$ fixes the boundary of D pointwisely but not for $k' \neq k$, $0 < k' < 1$. The existence of such $\mu(z)$ is known in [2] where $\mu(z)$ is not restricted to be the Teichmüller type.

Let $\varphi(z) = z^2(z - (5 + \sqrt{13})/2)/(z - 1)^3$, $k = (3 + \sqrt{13})/8$. Then we have $\varphi'(z) = 0$ at $z = 0$ and $\varphi(0) + k\bar{\varphi}(0) = 0$. On the other hand $\varphi(z) + k\bar{\varphi}(1/\bar{z}) = (z - 1/2)^2(z - (3 + \sqrt{13})/2)/(z - 1)^3$, hence $(\varphi(z) + k\bar{\varphi}(1/\bar{z}))' = 0$ at $z = 1/2$ and $\varphi(1/2) + k\bar{\varphi}(2) = 0$. Therefore we have $g(z) = \varphi(z) + k\bar{\varphi}(1/\bar{z})$ with $k = (3 + \sqrt{13})/8$ and $z_1 = 0$, $w_1 = 1/2$.

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References

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