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# EQUIVALENT SIZES OF LIPSCHITZ MANIFOLDS AND THE SMOOTHING PROBLEM

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## 1. Introduction

A criterion for smoothing a closed topological manifold $M$ was given by Y. Shikata in [3]. His criterion was based on the notion of the *size* of $M$, which will be denoted by $|M|_S$.

**Shikata's Theorem.** *A closed topological manifold M is smoothable if and only if* $|M|_S=0$.

In this paper we introduce another "size" of $M$, denoted $||M||$, which is conceptually somewhat simpler and which is definable for all open or closed Lipschitz manifolds. We then can prove the following result.

**Theorem.** *A closed Lipschitz manifold M is smoothable if and only if* $||M||=0$.

This theorem follows directly from Shikata's theorem and the following proposition, which shows that the two "sizes" are equivalent.

**Proposition.** *For each positive integer n there are positive numbers* $\alpha(n)$ *and* $\beta(n)$ *such that if M is a closed n-dimensional Lipschitz manifold then*

$$\alpha(n)||M|| \leq |M|_S \leq \beta(n)||M|| .$$

REMARK. Neither of these sizes is trivial, for L. Siebenmann has given examples in [4, pp. 135–137] of Lipschitz manifolds $X^n$, $n\geq 6$, having no piecewise linear manifold structure. In particular, $X^n$ is not smoothable, and so $0<||X^n||<\infty$.

## 2. Lipschitz manifolds and their sizes

We recall from [2] that if $(X_1, d_1)$ and $(X_2, d_2)$ are metric spaces and $f:X_1\to X_2$ is a map then the *Lipschitz size of f* (relative to $d_1$ and $d_2$) is

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$$l(f)=\begin{cases}\text{Inf } \{k\geq 1 \mid k^{-1}d_1(x, y)\leq d_2(fx, fy)\leq kd_1(x, y) \text{ for all } x, y\in X_1\} \\ \infty, \text{ if the above set is empty.}\end{cases}$$

If $l(f)<\infty$, $f$ is a *regular Lipschitz map*. It is clearly seen that relative to the obvious distances $l(f)=l(f^{-1})$ and $l(gf)\leq l(f)\cdot l(g)$.

We further recall from [6, p. 165] that a topological $n$-manifold $M$ is a *Lipschitz manifold* if there is a coordinate cover $C=\{(U_i, h_i)\}_{i\in I}$ such that $U_i$ is an open subset of $M$, $h_i$ is a homeomorphism of an open subset of Euclidean $n$-space onto $U_i$, and
$h_i^{-1}h_j$: $h_j^{-1}(U_i\cap U_j)\rightarrow h_i^{-1}(U_i\cap U_j)$ is a regular Lipschitz homeomorphism for all $i, j$. We may always assume $C$ is finite when $M$ is compact.

For any metric $d$ on $M$ denote

$$l_d(C)=\text{Sup}_{i\in I}\, l(h_i) \text{ relative to Euclidean metric and } d,$$
$$l(C)=\text{Sup}_{i,j\in I}\, l(h_i^{-1}h_j) \text{ relative to Euclidean metric.}$$

Then we may define two "sizes" for $M$:

$$|M| = \text{Inf}\log l_d(C) \quad \text{and}$$
$$||M|| = \text{Inf}\log l(C)\,,$$

where both infima are taken over all coordinate covers $C$ and all metrics $d$ on $M$.

**Lemma 1.** *For all closed Lipschitz manifolds $M$,*

$$|M|\leq ||M||\leq 2\cdot |M|\ .$$

Proof. For the first inequality let $\varepsilon^*>0$ be arbitrary and pick $C$ as above so that $\log l(C)\leq ||M||+\varepsilon^*$. Denote $L=l(C)$. Let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$, $V_i=$Carrier $(f_i)$ and $W=\cup_i V_i\times V_i$. For $(x, y)\in W$, the functions $\varphi_i(x, y)=f_i(x)f_i(y)[\sum_j f_j(x)f_j(y)]^{-1}$ define a partition of unity on $W$ so that $\varphi_i(x, y)=\varphi_i(y, x)$. For $(x, y)\in W$ define $\rho(x, y)=\sum_i\varphi_i(x, y)||h_i^{-1}x-h_i^{-1}y||$.

We can choose sequences $S=\{x=x_0, x_1,\cdots, x_t=y\}$ so that for all $i$, $(x_{i-1}, x_i)\in W$ and define $[S]=\sum_i\rho(x_{i-1}, x_i)$ and $d(x, y)=\text{Inf}_S[S]$. It is easily verified that $d$ is a pseudo-metric on $M$ giving the original topology on $M$. By [1, Theorem 5.26, p. 154] there is a Lebesgue number $\eta$ for the cover $\{V_i\}$ with respect to $d$.

Assertion. $d$ is a metric, i.e., $d(x, y)=0$ implies $x=y$.

Choose $\varepsilon>0$ with $\varepsilon<\eta$. Then pick $[S]<\varepsilon$. Since $d(x_i, x_j)\leq[\{x_i, x_{i+1},\cdots, x_j\}]\leq[S]<\varepsilon$, diameter $(S)<\eta$ and $S\subset V_j$ for some $j$. Now

$$L^{-1}||h_j^{-1}x_{i-1}-h_j^{-1}x_i||\leq||h_k^{-1}h_jh_j^{-1}x_{i-1}-h_k^{-1}h_jh_j^{-1}x_i|| \quad \text{and}$$
$$\begin{aligned}L^{-1}||h_j^{-1}x_{i-1}-h_j^{-1}x_i|| &= L^{-1}\textstyle\sum_k\varphi_k(x_{i-1}, x_i)\cdot||h_j^{-1}x_{i-1}-h_j^{-1}x_i||\\ &\leq\textstyle\sum_k\varphi_k(x_{i-1}, x_i)\cdot||h_k^{-1}x_{i-1}-h_k^{-1}x_i||\\ &= \rho(x_{i-1}, x_i)\end{aligned}$$

and

$$L^{-1}\cdot\|h_j^{-1}x-h_j^{-1}y\|\leq L^{-1}\sum_i\|h_j^{-1}x_{i-1}-h_j^{-1}x_i\|$$
$$\leq\sum_i\rho(x_{i-1},x_i)=[S]<\varepsilon\,.$$

Thus, $x=y$ since $\varepsilon$ is arbitrary.

Now let $\{W_i\}$ be a cover of $M$ so that diameter $(W_i)<\eta$. Then $W_i\subset V_j$ for some $j$. Let $k_i^{-1}=h_j^{-1}\mid W_i$ and $C^*=\{(W_i,k_i)\}$.

Assertion. $k_i$ is regular Lipschitz relative to Euclidean metric and $d$.
If $h_jx, h_jy\in W_i\subset V_j$ and $d(h_jx,h_jy)<\eta$, then $d(h_jx,h_jy)\leq\rho(h_jx,h_jy)=\sum_k\varphi_k(h_jx,h_jy)\cdot\|h_k^{-1}h_jx-h_k^{-1}h_jy\|\leq\sum_k\varphi_k(h_jx,h_jy)\cdot\mathrm{L}\cdot\|x-y\|=L\cdot\|x-y\|$.
Also we may pick $S=\{h_jx=x_0,x_1,\cdots,x_p=h_jy\}$ with $d(h_jx,h_jy)\leq[S]<\eta$. Then for all $t$, $d(h_jx,x_t)\leq\sum_{k=1}^t\rho(x_{k-1},x_k)\leq[S]<\eta$ and $x_t\in V_j$. Hence

$$\begin{aligned}[S] &= \textstyle\sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_k^{-1}h_jh_j^{-1}x_{l-1}-h_k^{-1}h_jh_j^{-1}x_l\|\\ &\geq L^{-1}\cdot\textstyle\sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\ &= L^{-1}\cdot\textstyle\sum_l\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\ &\geq L^{-1}\cdot\|x-y\|\,.\end{aligned}$$

Thus, $L^{-1}\cdot\|x-y\|\leq d(k_ix,k_iy)\leq L\cdot\|x-y\|$ and $l(k_i)\leq L=l(C)$. Hence, $l_d(C^*)\leq l(C)$ and $|M|\leq\|M\|+\varepsilon^*$. Since $\varepsilon^*$ is arbitrary, $|M|\leq\|M\|$.

For the second inequality let $\varepsilon>0$ and choose $C$ and $d$ so that $2\log l_d(C)\leq 2\cdot|M|+\varepsilon$. Then $l(h_i^{-1}h_j)\leq l(h_i)\cdot l(h_j)\leq l_d(C)^2$ and $l(C)\leq l_d(C^*)^2$. Thus $\|M\|\leq 2\cdot|M|+\varepsilon$ and $\|M\|\leq 2\cdot|M|$.

The above proof also shows an equivalent way of defining Lipschitz manifolds. We state this as a corollary.

**Corollary.** *A closed manifold $M$ is Lipschitz manifold if and only if there is a coordinate cover $C=\{(U_i,h_i)\}_{i\in I}$ such that each homeomorphism $h_i$ is regular Lipschitz with respect to the usual metric on Euclidean space and some metric $d$ on $M$.*

## 3. A reformulation of Shikata's criterion

Shikata [3] defined the "size" of a compact topological $n$-manifold $M$ to be

$$|M|_S=\operatorname{Inf}_{C,d}(8\gamma)^{m(C)}\cdot\log l_d(C)\,,$$

where $\gamma>0$ depends only on $n$ and $m(C)$ is the maximum number of $U_j$ that any $U_i$ can intersect. It is clear that any $C$ can be replaced by a coordinate cover $C'$ refining it, provided we use the restrictions of the appropriate homeomorphisms $h_i$ from $C$ in computing $l_d(C')$. In [5] we show that there is a positive integer $\mu(n)$ depending only on $n$ such that any $C$ has a refinement $C'$ as above with $m(C')\leq\mu(n)$. Let $\alpha'(n)=(8\gamma)^{\mu(n)}$ and $\beta'(n)=1$ if $8\gamma\leq 1$, and vice versa if $8\gamma \geqq 1$. Then noting the definitions of $|M|$ and $|M|_S$, we have proved the following lemma.

**Lemma 2.** *There are positive numbers $\alpha'(n)$ and $\beta'(n)$ depending only on $n$ such that for all closed manifolds $M$*

$$\alpha'(n)|M| \leq |M|_S \leq \beta'(n)|M| .$$

Taken together, Lemmas 1 and 2 clearly yield the proposition. Hence, the theorem is established.

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