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ORIENTED AND WEAKLY COMPLEX BORDISM OF FREE METACYCLIC ACTIONS

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Abstract. Oriented and weakly complex bordism modules of free metacyclic actions are determined up to the Kasparov formula which describes the bordism classes of generalized lens spaces in terms of a linear combination of those of the standard lens spaces. In the oriented case for $p=2$ (the dihedral case), the module structure is particularly simple because the corresponding Kasparov formula reduces to the multiplication by ± 1. We also compute the abelian group structure of these bordisms in case $p\geqslant 2$ a prime and $q\geqslant 3$ an odd prime. Of independent interest is the canonical projections defined on these bordism modules which select a direct summand with one generator in each $2pj-1$dimension ($j=1, 2, \cdots$).

1. Introduction.

Let $Z_{q,p}$ be the metacyclic group

$$Z_{q,p}=\{x, y\,|\, x^q=y^p=1,\ yxy^{-1}=x^r\}$$

where $p\geqslant 2$ is a prime integer, $q\geqslant 3$ is an odd integer and r is a primitive p-th root of 1 mod q such that $(r-1, q)=1$. (So $r\equiv -1$ mod q when $p=2$.) By virtue of Fermat's theorem, these conditions imply $(p, q)=1$.

Obviously there is an exact sequence

$$1\longrightarrow Z_q \xrightarrow{\ i\ } Z_{q,p} \underset{s}{\overset{\pi}{\rightleftarrows}} Z_p \longrightarrow 1$$

with s a corss-section defined by $s(\bar{y})=y$.

Kamata—Minami [3] determined the additive structure of the weakly complex reduced bordism group of the free dihedral group actions $\tilde{\Omega}^U_m(Z_{q,2})$ in case q is an odd prime. Here we generalize their results to the cases for the oriented and weakly complex bordism modules $\tilde{\Omega}^{SO}_*(Z_{q,p})$ and $\tilde{\Omega}^U_*(Z_{q,p})$ of the free metacyclic actions.

For the basic notations and prerequisites, we refer the reader to the introductory part and §1 of Kamata—Minami [3].

Thanks are due to Professor Minoru Nakaoka for suggesting me the subject.

2. The module structure of $\tilde{\Omega}^L_*(Z_{q,p})$; $L=SO$, U

First we recall the basic fact about $\tilde{\Omega}(Z_{q,p})$ from Lazarov [5].

Lemma 2.1. (Lazarov [5]).

(1) $i_*\colon \tilde{\Omega}^L_*(Z_q)\to\tilde{\Omega}^L_*(Z_{q,p})$ *is surjective onto the q-torsion.*

(2) $s_*\colon \tilde{\Omega}^L_*(Z_p)\to\tilde{\Omega}(Z_{q,p})$ *is injective onto the p-torsion (which is a direct summand as an* $\tilde{\Omega}^L_*$*-module because* $\pi_*\circ s=\mathrm{id}$).

The proof is done by calculating the integral homology $H_*(Z_{q,p};Z)$. Thanks to our assumption on p, q and r stated in the introduction, Lazarov's proof still works here in a slightly generalized situation.

Therefore it suffices to know the kernel of i_* for the determination of the module structure $\tilde{\Omega}^L_*(Z_{q,p})$ because we already know the structure of $\tilde{\Omega}^L_*(Z_m)$ (Conner-Floyd [1], Kamata [2], Shibata [6]).

Let

$$T_{(q,j)}\colon Z_q\times S^{2n-1}\to S^{2n-1}$$

denote the Z_q-action on the $(2n-1)$-dimensional sphere defined by $T_{(q,j)}(x^h,z)=\rho^{hj}z$, where $\rho=\exp(2\pi\sqrt{-1}/q)$. This is a free action if j is a unit in Z_q.

Let us consider the images of the $[T_{(q,rj)}, S^{2n-1}]$ by the canonical homomorphism

$$i_*\colon \tilde{\Omega}^L_*(Z_q)\to\tilde{\Omega}^L_*(Z_{q,p}).$$

Lemma 2.2.

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\hat{T}_{(q,rj)}, Z_p\times S^{2n-1}],$$

where

$$\hat{T}_{(q,rj)}(x, (\bar{y}^h, z)) = (\bar{y}^h, \rho^{rj-h}z)$$

and

$$\hat{T}_{(q,rj)}(y, (\bar{y}^h, z)) = (\bar{y}^{h+1}, z).$$

Proof. The map i_* is the extension (see Conner-Floyd [1] page 53), and so

$$i_*[T_{(q,rj)}, S^{2n-1}] = [\tilde{T}_{(q,rj)}, Z_{q,p}\times_{Z_q} S^{2n-1}]$$

where $\tilde{T}_{(q,rj)}$ is the natural operation of $Z_{q,p}$ on $Z_{q,p}\times_{Z_q} S^{2n-1}$ from the left. There is an L-structure preserving ($L=$SO or U), $Z_{q,p}$-equivariant diffeomorphism

$$\phi_j\colon (\tilde{T}_{(q,rj)}, Z_{q,p}\times_{Z_q} S^{2n-1})\to(\hat{T}_{(q,rj)}, Z_p\times S^{2n-1})$$

defined by $\phi_j([x^a y^b, z])=(\bar{y}^b, \rho^{arj-b}z)$.

Hence the lemma follows.

Corollary 2.3.

$$i_*[T_{(q,1)}, S^{2n-1}] = i_*[T_{(q,r^j)}, S^{2n-1}]$$

for $j=0, 1, 2, \cdots, p-1$.

Proof. From the preceding lemma, it suffices to find an L-structure preserving, $Z_{q,p}$-equivariant diffeomorphism

$$\psi_j: (\hat{T}_{(q,1)}, Z_p\times S^{2n-1})\to(\hat{T}_{(q,r^j)}, Z_p\times S^{2n-1}).$$

In fact the formula $\psi_j(\bar{y}^b, z)=(\bar{y}^{b+j}, z)$ defines a desired one.

Let t: $\tilde{\Omega}^L_*(Z_{q,p})\to\tilde{\Omega}^L_*(Z_q)$ be the transfer homomorphism, *i.e.* the homomorphism induced by the restriction of the action on the subgroup (Conner-Floyd [1] page 52).

Lemma 2.4.

$$t\circ i_*[T_{(q,1)}, S^{2n-1}] = \sum_{j=0}^{p-1} [T_{(q,r^j)}, S^{2n-1}].$$

Proof. The lemma is obvious from Lemma 2.2 and the definition of t.

DEFINITION 2.5. We define the elements β_{2n-1} ($n=1, 2, \cdots$) of $\tilde{\Omega}^L_{2n-1}(Z_q)$ as follows.

(1) In case $(n, p) = 1$, $\beta_{2n-1} = \sum_{0\leq j\leq p-1} ([T_{(q,1)}, S^{2n-1}]-[T_{(q,r^j)}, S^{2n-1}])$.

(2) $\beta_{2pm-1} = t\circ i_*[T_{(q,1)}, S^{2pm-1}] = \sum_{0\leq j\leq p-1} [T_{(q,r^j)}, S^{2pm-1}]$.

Lemma 2.6. (1) $i_*\beta_{2n-1} = 0$ *in case* $(n, p) = 1$.
(2) $t\circ i_*\beta_{2pm-1} = p\beta_{2pm-1}$.

Proof. (1) is obvious from definition 2.5 and corollary 2.3. Also definition 2.5, corollary 2.3 and lemma 2.4 imply (2).

At this stage, we need the formula of Kasparov [4], which describes the unitary bordism classes of the generalized lens spaces as a linear combination of those of the standard lens spaces. We restate his formula only in the special case which we concern.

Theorem 2.7 (Kasparov [4]). *In* $\tilde{\Omega}^L_*(Z_q)$, *the class* $[T_{(q,r^j)}, S^{2n-1}]$ *is the coefficient of* X^n *in*

$$(\sum_{1\leq k} [T_{(q,1)}, S^{2k-1}]X^k)\,(X/g^{-1}(r^jg(X)))^n$$

(or its image in $\tilde{\Omega}^{SO}_*(Z_q)$ *by the natural homomorphism* $\tilde{\Omega}^U_*(Z_q)\to\tilde{\Omega}^{SO}_*(Z_q)$*), where* $g(X)=\sum_{1\leq h}([CP_{h-1}]/h)X^h$ *is the logarithm of the cobordism formal group law, i.e.*

"the Miscenko series".

Corollary 2.8.

$$\beta_{2n-1}-p[T_{(q,1)}, S^{2n-1}]\in\Omega_*^L\{[T_{(q,1)}, S^{2k-1}];\ 1\leqslant k\leqslant n-1]\},$$

where $\Omega_*^L\{\cdots\}$ *denotes the* Ω_*^L *-submodule of* $\tilde{\Omega}_*^L(Z_q)$ *generated by the elements* $\{\cdots\}$.

Proof. By the Kasparov formula, we see that

$$[T_{(p,r^j)}, S^{2n-1}]-\left(\frac{1}{r^j}\right)^n[T_{(q,1)}, S^{2n-1}]\in\Omega_*^L\{[T_{(q,1)}, S^{2j-1}];\ 1\leqslant i\leqslant n-1\}\ .$$

Notice that here we can treat everything as reduced mod q since $q[T_{(q,1)}, S^{2j-1}]\in\Omega_*^L\{[T_{(q,1)}, S^{2h-1}];\ 1\leqslant h<j\}$ (Shibata [6]). In case $n=kp$, $\sum_{0\leqslant j\leqslant p-1}\left(\frac{1}{r^j}\right)^n=\sum_{0\leqslant j\leqslant p-1}(1/(r^p)^{kj})\equiv p$. Thus the lemma is true. Otherwise put $n=kp+t(1\leqslant t\leqslant p-1)$. Then $\sum_{0\leqslant j\leqslant p-1}(1/r^j)^n=\sum_{0\leqslant j\leqslant p-1}(r^{-t})^j\equiv 0$ mod q since r^{-t} is a root of the equation $x^p-1=(x-1)\ (x^{p-1}+\cdots+x+1)\equiv 0$ and $r^{-t}-1$ is a unit in Z_q by virtue of the condition $(r-1, q)=1$. Therefore the lemma holds also in case $(n, p)=1$.

Corollary 2.9. $\Omega_*^L\{[T_{(q,1)}, S^{2j-1}];\ 1\leqslant j\leqslant k\}=\Omega_*^L\{\beta_{2j-1};\ 1\leqslant j\leqslant k\}$. *In particular,* $\tilde{\Omega}_*^L(Z_q)=\Omega_*^L\{\beta_{2j-1};\ 1\leqslant j\}$.

Proof. Since we are assuming $(p, q)=1$, this corollary is easily proved by induction on k by virtue of 2.8.

Now we can state the main theorem of this section as follows.

Theorem 2.10. *There are the following exact sequences of* Ω_*^L*-module homomorphisms*

$$(1)\quad 0\to\Omega_*^L\{\beta_{2m-1};\ 1\leqslant m,\ (m, p)=1\}\xrightarrow{\iota\oplus 0}$$
$$\to\tilde{\Omega}_*^L(Z_q)\oplus\tilde{\Omega}(Z_p)\xrightarrow{i_*+s_*}\tilde{\Omega}_*^L(Z_{q,p})\to 0,\quad \textit{and}$$
$$(2)\quad 0\to\tilde{\Omega}_*^L\{\beta_{2pk-1};\ 1\leqslant k\}\oplus\tilde{\Omega}_*^L(Z_p)\xrightarrow{i_*+s_*}\tilde{\Omega}_*^L(Z_{q,p})\to 0,$$

where ι *is the canonical inclusion as a submodule,*

$$\beta_{2m-1}=\begin{cases} p[T_{(q,1)}, S^{2m-1}]-\sum_{0\leqslant j\leqslant p-1}[T_{(q,r^j)}, S^{2m-1}] \\ \qquad \textit{if } (m, p)=1, \textit{ and} \\ \sum_{0\leqslant j\leqslant p-1}[T_{(q,r^j)}, S^{2m-1}] \\ \qquad \textit{if } p\mid m, \end{cases}$$

and the $[T_{(q,r^j)}, S^{2m-1}]$ *can be written down as a linear combination over* Ω_*^L *of the* $[T_{(q,1)}, S^{2n-1}]$ $(1\leqslant n\leqslant m)$ *by the Kasparov formula (Theorem 2.7).*

Proof. The proof is now obvious from 2.1, 2.6 and 2.9. We only indicate the proof of the fact that Ker $i_* \subset \Omega^L_*\{\beta_{2m-1};\ 1\leqslant m,\ (m,p)=1\}$. Suppose x belongs to Ker i_* and is homogeneous of dimension $2t-1$. By 2.9,

$$x=\sum_{m=1}^{t}\alpha_{2(t-m)}\beta_{2m-1}$$

for some $\alpha_{2(t-m)}\in\Omega^L_{2(t-m)}$. Then $0=t\circ i_*(x)=\sum_{m=1}^{[t/p]}p\alpha_{2t-2pm}\beta_{2pm-1}$. So $\sum_{m=1}^{[t/p]}\alpha_{2t-2pm}$ $\beta_{2pm-1}=0$ and this implies $x=x-\sum_{m=1}^{[t/p]}\alpha_{2t-2pm}\beta_{2pm-1}=\sum_{\substack{1\leqslant m\leqslant t\\(m,p)=1}}\alpha_{2(t-m)}\beta_{2m-1}$ as desired.

3. The oriented case for p=2.

There is a special simplicity for the oriented bordism of the free dihedral actions.

Lemma 3.1. *Let s be a unit in Z_q. It holds in $\tilde{\Omega}^{SO}_*(Z_q)$ that*

$$[T_{(q,-s)},S^{2n-1}]=(-1)^n[T_{(q,s)},S^{2n-1}].$$

Proof. Consider the Z_q-equivariant diffeomorphism

$$c\colon (T_{(q,-s)},S^{2n-1})\to(T_{(q,s)},S^{2n-1})$$

defined by $c(z_0,\cdots,z_{n-1})=(\bar{z}_0,\cdots,\bar{z}_{n-1})$, *i.e.* the complex conjugation. Then c preserves the orientation when n is even and reverses when n is odd. *Q.E.D.*

It follows that, in $\tilde{\Omega}^{SO}_*(Z_q)$,

$$\begin{aligned}\beta_{4i+1}&=[T_{(q,1)},S^{4i+1}]-[T_{(q,-1)},S^{4i+1}]\\&=2[T_{(q,1)},S^{4i+1}],\text{ and}\\\beta_{4i-1}&=[T_{(q,1)},S^{4i-1}]+[T_{(q,-1)},S^{4i-1}]\\&=2[T_{(q,1)},S^{4i-1}].\end{aligned}$$

Therefore theorem 2.10 of the last section reduces to the following.

Theorem 3.2. *There are the following exact sequences of Ω^{SO}_*-module homomorphisms*

$$(1)\quad 0\to\Omega^{SO}_*\{[T_{(q,1)},S^{4m+1}];\ 0\leqslant m\}\xrightarrow{\iota\oplus 0}$$
$$\to\tilde{\Omega}^{SO}_*(Z_q)\oplus\tilde{\Omega}^{SO}_*(Z_2)\xrightarrow{i_*+s_*}\tilde{\Omega}^{SO}_*(Z_{q,2})\to 0,$$

and

$$(2)\quad 0\to\Omega^{SO}_*\{[T_{(q,1)},S^{4m-1}];1\leqslant m\}\oplus\tilde{\Omega}^{SO}_*(Z_2)\to$$
$$\xrightarrow{i_*+s_*}\tilde{\Omega}^{SO}_*(Z_{q,2})\to 0.$$

REMARK 3.3. The module structures of $\tilde{\Omega}_*^{SO}(Z_q)$ (q odd) and $\tilde{\Omega}_*^{SO}(Z_2)$ are determined in Shibata [6]. According to 6.1 and 6.3 of Shibata [6], together with the fact that the natural homomorphism $\Omega_*^U \to \Omega_*^{SO}/\mathrm{Tor}$ kills the elements of dimension $4j+2 (j=0, 1, 2, \cdots)$, we see that the restriction of the Smith homomorphism

$$\Delta \colon \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-1}];\ 1 \leqslant m\} \to \Omega_*^{SO}\{[T_{(q,1)}, S^{4m-3}];\ 1 \leqslant m\}$$

is an isomorphism, and thus $\tilde{\Omega}_*^{SO}(Z_q)$ is a direct sum of two isomorphic copies (with dimension shift) of Ω_*^{SO}-submodules.

REMARK 3.4. We can not expect such a simple phenomenon in the unitary bordism of the dihedral actions. For example,

$$[T_{(q,-1)}, S^3] = [T_{(q,1)}, S^3] - 2[CP_1]\,[T_{(q,1)}, S^1],$$
$$[T_{(q,-1)}, S^5] = -[T_{(q,1)}, S^5] + 3[CP_1]\,[T_{(q,1)}, S^3] - 3[CP_1]^2\,[T_{(q,1)}, S^1],$$

and so

$$t \circ i_*[T_{(q,1)}, S^3] = 2[T_{(q,1)}, S^3] - 2[CP_1]\,[T_{(q,1)}, S^1], \neq 2[T_{(q,1)}, S^3],$$

and in case $q > 3$,

$$t \circ i_*[T_{(q,1)}, S^5] = 3[CP_1]\,[T_{(q,1)}, S^3] - 3[CP_1]^2[T_{(q,1)}, S^1] \neq 0$$

in $\tilde{\Omega}_*^U(Z_q)$.
Also when $q=3$,

$$t \circ i_*[T_{(3,1)}, S^9] = 5[CP_1]\,[T_{(3,1)}, S^7] - [CP_1]^2[T_{(3,1)}, S^5] + \cdots \neq 0.$$

REMARK 3.5. Even in the oriented case, if we take the case for $p \geqslant 3$, the Kasparov formula becomes complicated. The lowest dimensional example is the case for $p=3$, $q=7$, $r=2$. The computation shows that

$$t \circ i_*[T_{(7,1)}, S^5] = 3[T_{(7,1)}, S^5] + 4[CP_2]\,[T_{(7,1)}, S^1] \neq 3[T_{(7,1)}, S^5],$$
$$t \circ i_*[T_{(7,1)}, S^9] = 5[CP_2]\,[T_{(7,1)}, S^5] + 2[CP_2]^2[T_{(7,1)}, S^1].$$

Therefore $\qquad t \circ i_*[T_{(7,1)}, S^9] \neq 0$ in $\tilde{\Omega}_*^{SO}(Z_7)$.

4. Computation of abelian group sturcture of $\tilde{\Omega}_*^L(Z_{q,p})$ for q an odd prime

In this section we present a generalization of the main theorem of Kamata-Minami [3] to the case for $\tilde{\Omega}_*^L(Z_{q,p})$ with $p \geqslant 2$ a prime and $q \geqslant 3$ an odd prime. So in this section, we assume q an odd prime.

As in Kamata [2], let $\Gamma_*(q)$ be the polynomial subring of $\Omega_*^U = Z[x_1, x_2, \cdots]$ which is generated by x_i $(i \neq q-1)$ (unitary case) or its image in Ω_*^{SO} by the canonical homomorphism $\Omega_*^U \to \Omega_*^{SO}$ (oriented case).

Analogously to Kamata-Minami [3], proposition 3.1, we obtain;

Proposition 4.1. *The following two conditions for the elements* $[M^{2(l-k)}] \in \Gamma_{2(l-k)}(q)$ *are equivalent;*

(1) $\sum_{k=1}^{n} [M^{2(l-k)}]\beta_{2k-1} = 0$ *in* $\tilde{\Omega}_*^L(Z_q)$, *and*

(2) $[M^{2(l-k)}] \in q^{[k-1/q-1]+1}\Gamma_{2(l-k)}(q)$,

where the β_{2k-1} *are the module generators of* $\tilde{\Omega}_*^L(Z_q)$ *defined in section* 2.

Now $\tilde{\Omega}_*^L(Z_q)$ can be considered as a $\Gamma_*(q)$-module and we denote by $\Gamma_*(q)\{\cdots\}$ the $\Gamma_*(q)$-submodule of $\tilde{\Omega}_*^L(Z_q)$ generated by the elements $\{\cdots\}$.

Lemma 4.2. *Ther is a* $\Gamma_*(q)$*-isomorphism*

$$\nu: \Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}]; 1 \leqslant n\} \to \Gamma_*(q)\{\beta_{2n-1}; 1 \leqslant n\}$$

defined by $\nu[T_{(q,1)}, S^{2n-1}] = \beta_{2n-1}$.

Proof. According to proposition 4.1 and Kamata [2], proposition 2.5, the $[T_{(q,1)}, S^{2n-1}]$ and the β_{2n-1} satisfy the same $\Gamma_*(q)$-module relations. Q.E.D.

Corollary 4.3. $\tilde{\Omega}_*^L(Z_q) = \Gamma_*(q)\{[T_{(q,1)}, S^{2n-1}]; 1 \leqslant n\} = \Gamma_*(q)\{\beta_{2n-1}; 1 \leqslant n\}$.

Proof. The first equality is a consequence of Kamata [2], proposition 2.6. So the map ν of 4.2 defines an injective endomorphism of $\tilde{\Omega}_*^L(Z_q)$ which is dimension preserving. But $\tilde{\Omega}_*^L(Z_q)$ contains only a finite number of elements in each dimension, and thus the injectivity of ν implies the surjectivity. This means $\Gamma_*(q)\{\beta_{2n-1}; 1 \leqslant n\} = \text{Image } \nu = \tilde{\Omega}_*^L(Z_q)$.

Corollary 4.4. $\Omega_*^L\{\beta_{2pm-1}; 1 \leqslant m\} = \Gamma_*(q)\{\beta_{2pm-1}; 1 \leqslant m\}$

Proof. It is obvious that $\Omega_*^L\{\beta_{2pm-1}; 1 \leqslant m\} \supset \Gamma_*(q)\{\beta_{2pm-1}; 1 \leqslant m\}$. Conversely let

(*) $x = \sum_{m=0}^{n} \alpha_{2t+2(n-m)p}\beta_{2pm-1}; \alpha_{2t+2(n-m)p} \in \Omega_*^L$. From the preceding corollary, $\Omega_*^L\{\beta_{2j-1}; 1 \leqslant j\} \subset \Gamma_*(q)\{\beta_{2n-1}; 1 \leqslant n\}$. So

(**) $x = \sum_{j=0}^{t+np} \gamma_{2t-2pn-2j}\beta_{2j-1}; \gamma_{2t+2pn-2j} \in \Gamma_*(q)$.

By (*), we have $t \circ i_*(x) = px$. On the other hand, (**) implies

$$t \circ i_*(x) = \sum_{m=1}^{n+[t/p]} p\gamma_{2t+2p(n-m)}\beta_{2pm-1}.$$

Therefore $x = \sum_{m=1}^{n+[t/p]} \gamma_{2t+2p(n-m)}\beta_{2pm-1}$. Q.E.D.

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *The additive structure of* $\tilde{\Omega}_*^L(Z_{q,p})$ *with* q *an odd prime is determined by the following exact sequence of* $\Gamma_*(q)$*-homomorphisms*

$$0 \to \Gamma_*(q)\{\{q^{[pj-1/q-1]+1}\beta_{2pj-1};\ 1 \leqslant j\}\} \xrightarrow{\iota \oplus 0}$$
$$\to \Gamma_*(q)\{\{\beta_{2pj-1};\ 1 \leqslant j\}\} \oplus \tilde{\Omega}_*^L(Z_p) \xrightarrow{i_*+s_*} \tilde{\Omega}(Z_{q,p}) \to 0\,,$$

where $\Gamma_*(q)\{\{\cdots\}\}$ denotes the free $\Gamma_*(q)$-module generated by $\{\cdots\}$.

Proof. According to 2.10 and 4.4, i_*+s_* is epimorphic. And 4.1 implies that the kernel of i_*+s_* is as stated in the theorem. Q.E.D.

REMARK 4.6. Except for the case $\tilde{\Omega}_*^{SO}(Z_2)$, it holds that additively

$$\tilde{\Omega}_*^L(Z_p) \cong \Gamma_*(p)\{[T_{(p,1)}, S^{2n-1}];\ 1 \leqslant n\}/$$
$$\Gamma_*(p)\{p^{[n-1/p-1]+1}[T_{(p,1)}, S^{2n-1}];\ 1 \leqslant n\}$$

(Kamata [2], proposition 2.6)

And, also additively,

$$\tilde{\Omega}_*^{SO}(Z_2) = \bigoplus_{j=0}^{\infty} E^{2j+1}\mathcal{W}_*\,,$$

where $\mathcal{W}_*$ is Wall's polynomial subalgebra $Z_2[X_{2k-1}, X_{2k};\ k \neq 2^j, (X_{2^j})^2]$ in $\mathfrak{N}_*$ and E^{2j+1} is the isomorphism of raising the dimension of each element by $2j+1$. (Shibata [6], corollary 3.3, lemma 4.1)

5. Canonical splitting for $\tilde{\Omega}_*^L(Z_q)$

According to the results of section 2, we have the following proposition.

Proposition 5.1. *Let* p, q, r *be as stated in the introduction.* (1) *There is the projection homomorphism*

$$\rho_{(p,r)}\colon \tilde{\Omega}_*^L(Z_q) \to \tilde{\Omega}_*^L(Z_q)$$

defined by $\rho_{(p,r)} = \iota \circ (i_* \mid \Omega_*\{\beta_{2pm-1};\ 1 \leqslant m\})^{-1} \circ i_*$.

(2) *The corresponding direct sum decomposition as* Ω_*^L*-modules* Image $\rho_{(p,r)} \oplus$ Ker $\rho_{(p,r)}$ *is*

$$\tilde{\Omega}_*^L(Z_q) = \Omega_*^L\{\beta_{2pm-1};\ 1 \leqslant m\} \oplus \Omega_*^L\{\beta_{2n-1};\ 1 \leqslant n,\ (n, p) = 1\}\,.$$

When $p=2$, r is necessarily equal to -1, or equivalently, $q-1$.

Corollary 5.2. *Let* q *be an odd integer.*
(1) *The formulas*

$$\rho_2(\beta_{4n+1}) = 0,\ \rho_2(\beta_{4n+3}) = \beta_{4n+3}$$

define an Ω_*^L-homomorphism

$$\rho_2\colon \tilde{\Omega}_*^L(Z_q) \to \tilde{\Omega}_*^L(Z_q)$$

which is a projection operator.

(2) *The corresponding direct sum splitting is;*

$$\tilde{\Omega}_*^U(Z_q) = \Omega_*^U\{\beta_{4n-1};\ 1\leqslant n\} \oplus \Omega_*^U\{\beta_{4n-3};\ 1\leqslant n\}\ ,$$
$$\tilde{\Omega}_*^{SO}(Z_q) = \Omega_*^{SO}\{[T_{(q,1)}, S^{4n-1}];\ 1\leqslant n\} \oplus \Omega_*^{SO}\{[T_{(q,1)}, S^{4n-3};\ 1\leqslant n\}\ .$$

I doubt if there is an analogous direct sum splitting for $\tilde{\Omega}_*^U(Z_{2^a})$; $a\geqslant 1$.

In the rest of this section, we assume q an odd prime and p a prime such such that $p\,|\,q-1$.

By elementary number theory arguments we obtain the following fact.

Lemma 5.3. *The equation $x^p-1\equiv 0$ mod q has exactly p distinct roots in Z_q. If $r\neq 1$ is one of them, then $r, r^2, \cdots, r^{p-1}$ are the primitive p-th roots mod q and $x^p-1\equiv\prod_{j=0}^{p-1}(x-r^j)$ mod q.*

Theorem 5.4. *For $q\geqslant 3$ an odd prime and p a prime such that $p\,|\,q-1$, there is the canonical projection*

$$\rho_p\colon \tilde{\Omega}_*^L(Z_q) \to \tilde{\Omega}_*^L(Z_q)$$

whcih gives the canonical direct sum decomposition

$$\Omega_*^L(Z_q) = \tilde{\Omega}_*^L\{\beta_{2pm-1};\ 1\leqslant m\} \oplus \Omega_*^L\{\beta_{2n-1};\ 1\leqslant n,\ (n,p)=1\}\ ,$$

and in particular for p an odd prime,

$$\begin{aligned}\tilde{\Omega}_*^{SO}(Z_q) = {} & \Omega_*^{SO}\{\beta_{4pm-1};\ 1\leqslant m\} \oplus \Omega_*^{SO}\{\beta_{4pm-2p-1};\ 1\leqslant m\} \\ & \oplus\ \Omega_*^{SO}\{\beta_{4n-1};\ 1\leqslant n,\ (n,p)=1\} \\ & \oplus\ \Omega_*^{SO}\{\beta_{4n-3};\ 1\leqslant n,\ (2n-1,p)=1\}\ .\end{aligned}$$

Proof. Lemma 5.3 implies that we can find primitive p-th roots in Z_q and that the definition of the β_{2n-1} does not depend on the choice of a p-th root. Hence the theorem follows from 5.1.

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