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ON THE GENERALIZED SOLITON SOLUTIONS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

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In this paper, we discuss the asymptotic property of the generalized soliton solutions of the modified Korteweg-de Vries (*K-d*v) equation

$$(1) \qquad v_t+6v^2v_x+v_{xxx}=0\,, \qquad -\infty<x,\ t<\infty$$

where v_t and v_x denote partial derivatives of $v=v(x, t)$ with respect to t and x respectively. This equation gives one of the simplest modifications of the *K-dV* equation

$$u_t-6uu_x+u_{xxx}=0\,, \qquad -\infty<x,\ t<\infty\,.$$

Both of the *K-dV* equation and the modified *K-dV* equation are known to have progressive wave solutions;

$$u(x, t)=-2^{-1}c\ \mathrm{sech}^2\,(2^{-1}c^{1/2}(x-ct-\delta))\,, \qquad c>0$$

for the *K-dV* equation,

$$(2) \qquad v(x, t)=\pm c^{1/2}\ \mathrm{sech}\,(c^{1/2}(x-ct-\delta))\,, \qquad c>0$$

for the modified *K-dV* equation.

Each of such solutions is called a solitary wave solution or a soliton on account of its shape.

On the other hand, Gardner, Greene, Kruskal and Miura [1] have related the solution $u(t)=u(x, t)$ of the *K-dV* equation to the scattering theory of the one dimensional Schrödinger operator with the potential $u(t)$ and found that discrete eigenvalues are invariants and the reflection coefficient and normalization coefficients vary exponentially with respect to t (see also Lax [3]). Here a soliton of the *K-dV* equation is characterized as the solution with one discrete eigenvalue and the zero reflection coefficient. Furthermore, the reflectionless potential with N discrete eigenvalues can be written in closed form in terms of exponentials by the method of Kay and Moses [2] for each t. These potentials are called N-tuple wave solutions. N-tuple wave solution satisfies actually the K-dV equation and behaves like a superposition of N solitons for large t (see Tanaka [7]).

Recently, Tanaka [5], [6] has related the solution of the modified K-dV equation (1) to the scattering theory of the differential operator

$$(3) \qquad L_u = i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D - i\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix} \qquad D = d/dx$$

where u^* denote the complex conjugate of u and, on the basis of this relation, constructed a family of particular solutions which includes solitons (2) as the simplest case (see also Wadati [8]). We call these solutions the generalized soliton solutions.

In this paper, we study the generalized soliton solution with purely imaginary discrete eigenvalues. In this case, the generalized soliton solutions decompose into N solitons for large t.

Our result is similar to that of the N-tuple wave solution of the K-dV equation. But comparing with the N-tuple wave solution, the expression of the generalized soliton solution is somewhat complicated.

In section 1, we summarize the general properties of the scattering data of L_u. In section 2, following [6], we describe the relation between the scattering data of L_u and a solution of the modified K-dV equation, and introducing the generalized soliton solutions, we state our main theorem. In section 3, we rewrite the formula of the generalized soliton solution into the form which is more convenient for our consideration. In section 4, we give the proof of our main theorem.

1. The scattering data of L_u

We summarize the general properties of the scattering data of L_u from [4], [9]. Consider the eigenvalue problem

$$(4) \qquad L_u y = \zeta y \qquad y = {}^t(y_1, y_2)$$

on the real axis $(-\infty, \infty)$. If $u(x)$ is integrable, then, for $\zeta = \xi + i\eta$, $\eta \geqslant 0$, there are unique solutions $\phi_\pm$ of (4) (called Jost solutions) which behave as

$$\phi_+(x, \zeta) = {}^t(0, 1)\exp(i\zeta x) + o(1) \qquad x \to \infty\,.$$
$$\phi_-(x, \zeta) = {}^t(1, 0)\exp(-i\zeta x) + o(1) \qquad x \to -\infty\,.$$

Then $\phi_\pm(x, \zeta)$ are analytic in ζ, $I_m\zeta > 0$. Note that if y is a solution of (4), then

$$y^\# = {}^t(y_2^*, -y_1^*)$$

is a solution of (4) where ζ is replaced by ζ^*. If $\zeta = \xi$ is a non zero real number, then ϕ_+ and$\phi_+^\#$ are linearly independent solutions of (4). So there are $a(\xi)$ and $b(\xi)$ such that

$$\phi_- = a(\xi)\phi_+^\# + b(\xi)\phi_+ . \tag{5}$$

From (5), $a(\xi)=\det(\phi_-, \phi_+)$, so $a(\xi)$ can be extended to the analytic function $a(\zeta)$, $I_m \zeta>0$.

If $u(x)$ satisfies an additional condition

$$\int_{-\infty}^{\infty}(1+|x|)|u(x)|dx<\infty$$

then there is a function $f\in L^1(0, \infty)$ such that

$$a(\zeta) = 1+\int_0^\infty f(t)\, exp\, (i\zeta t)\, dt \qquad I_m\zeta\geqslant 0 .$$

Moreover, if we assume that $a(\xi)$ is not equal to zero for $\xi\in \boldsymbol{R}$, then $a(\zeta)$ has finite number of zeros $\zeta_1, \zeta_2, \cdots, \zeta_N, I_m\zeta_k>0$. For $\zeta=\zeta_j$, $\phi_\pm(x, \zeta_j)$ are linearly dependent, so there is a non zero constant d_j such that

$$\phi_-(x, \zeta_j) = d_j\phi_+(x, \zeta_j) \qquad (1\leqslant j\leqslant N) .$$

By the asymptotic properties of $\phi_\pm$, they are square integrable *i.e* ζ_j are discrete eigenvalues of L_u. We have

$$a'(\zeta_j) = -2id_j\int_{-\infty}^{\infty}\phi_{+1}(x, \zeta_j)\phi_{+2}(x,\zeta_j)\, dx .$$

Now suppose that ζ_j are simple zeros. Put

$$c_j=d_j/a'(\zeta_j)$$

(called normalization coefficients of eigenfunctions) and

$$r(\xi)=b(\xi)/a(\xi)$$

under the assumption, $a(\xi)\neq 0$ for any $\xi\in \boldsymbol{R}$ (called the reflection coefficient). The triplet $S=\{r(\xi), \zeta_j, c_j; 1\leqslant j\leqslant N\}$ is called the scattering data of L_u. Suppose that $u=iv$ is purely imaginary, then eigenvalues ζ_j are distributed symmetrically in $I_m\zeta>0$ with respect to the imaginary axis, *i.e* let M be a non negative integer $2M\leqslant N$, and π be a permutation among natural number 1 and N defined $\pi(j)=j+1, j$ odd $\leqslant 2M;\pi(j)=j-1, j$ even $\leqslant 2M$; $\pi(j)=j, j>2M$, then we have

$$\zeta_{\pi(j)} = -\zeta_j^* \qquad c_{\pi(j)} = c_j^* \tag{6}$$

(see Tanaka [5] [6]).

2. The generalized soliton solutions

For a smooth real valued function $v=v(x)$, put

$$B_v = -4D^3+3\begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix}D+3D\begin{bmatrix} -v^2 & iv_x \\ iv_x & -v^2 \end{bmatrix}$$

then the operator $[B_v, L_{iv}]=B_vL_{iv}-L_{iv}B_v$ is the multiplication by the matrix

$$\begin{bmatrix} 0 & -6v^2v_x-v_{xxx} \\ 6v^2v_x+v_{xxx} & 0 \end{bmatrix}.$$

Therefore, for a smooth real valued function $v(t)=v(x, t)$, the operator evolution equation

$$\frac{d}{dt}L_{iv(t)} = [B_{v(t)}, L_{iv(t)}] \tag{7}$$

is equivalent to the modified *K-dV* equation (1). Now suppose that a smooth real valued function $v(t)=v(x, t)$ is a solution of the modified *K-dV* equation (1) which satisfies the conditions for the existence of the scattering data for each t. Consider the operator $L_{iv(t)}$, then we have the scattering data $S_t=\{r(\xi, t), \zeta_j(t), c_j(t)\}$ depending on t as a parameter. Then we have

$$\begin{aligned} \zeta_j(t) &= \zeta_j(0) \\ c_j(t) &= c_j(0)\exp(8i\zeta_j^3t) \\ r(\xi, t) &= r(\xi, 0)\exp(8i\xi^3t)\,. \end{aligned} \tag{8}$$

These remarkable facts have been derived from the relation (7) (see Tanaka [5] and Wadati [8]). Conversely to the above, Tanaka [6] has shown the following.

Theorem (Tanaka [6]). *Let π be a permutation defined above, and ζ_j ($\zeta_i\neq\zeta_j$, $i\neq j$), c_j ($1\leqslant j\leqslant N$) be complex numbers which satisfy the condition* (6). *Put*

$$\begin{aligned} c_j(t) &= c_j\exp(8i\zeta_j^3t) \\ \lambda_j = \lambda_j(x, t) &= c_j(t)^{1/2}\exp(i\zeta_jx)\,. \end{aligned}$$

Let $\psi_{ij}=\psi_{ij}(x, t)$, $i=1, 2$, $j=1, 2, \cdots, N$ be the solution of

$$\begin{aligned} \psi_{1j}+\textstyle\sum_{k=1}^N \lambda_j\lambda_k^*(\zeta_j-\zeta_k^*)^{-1}\psi_{2k}^* &= 0 \\ \textstyle\sum_{k=1}^N\lambda_k\lambda_k^*(\zeta_k-\zeta_j^*)^{-1}\psi_{1k}+\psi_2^* &= \lambda_j^* \qquad (1<j<N) \end{aligned}$$

with non degenerate coefficient matrix, then the function

$$v(x, t) = -2\textstyle\sum_{j=1}^N\lambda_j^*(x, t)\psi_{2j}^*(x, t) \tag{10}$$

is a real valued function and satisfies the modified K-dV equation (1).

If $N=1$, then $\zeta_1=i\eta$, $\eta>0$ and $c_1=c$ is non zero real number. Therefore we have solutions

$$v(x, t) = (\text{sign } c)\, s\,(x-4\eta^2 t-\delta, \eta)$$

where

$$s(x, \eta) = -2\,\eta \text{ sech } 2\eta x$$

and

$$\delta = \delta(c, \eta) = (2\eta)^{-1} \log(|c|/2\eta).$$

These solutions coincide with solutions (2). So we call the solutions $v(x, t)$ defined by (10) the generalized soliton solutions.

Now let $M=0$, then $\zeta_j=i\eta_j$ $(\eta_j>0)$ and $c_j=c_j(0)$ are real. We can assume $0<\eta_1<\eta_2<\cdots<\eta_N$ without loss of generality. We can now state our main result.

Theorem. *Let $M=0$, then the generalized soliton solutions decompose into N solitons as $t\to\pm\infty$;*

$$v(x, t)-\sum_{j=1}^{N} (\text{sign } c_j) s(x-4\eta_j^2 t-\delta_j^{\pm}, \eta_j)\to 0$$

uniformly in x, where

$$\delta_j^{+} = \delta(c_j, \eta_j)+\eta_j^{-1} \log \prod_{i=j+1}^{N} (\eta_i-\eta_j)(\eta_i+\eta_j)^{-1}$$
$$\delta_j^{-} = \delta(c_j, \eta_j)+\eta_j^{-1} \log \prod_{i=1}^{j-1} (\eta_j-\eta_i)(\eta_j+\eta_i)^{-1}.$$

3. Rearrangement of the formula (10)

In this section we rewrite the generalized soliton solution $v(x, t)$ (10) into the form which is more convenient for our purpose.

Put $\phi_{ij}=\phi_{ij}(x, t)=\lambda_j(x, t)\,\psi_{ij}(x, t)$ $(i=1, 2,\ 1\leqslant j\leqslant N)$. Then we have

$$v(x, t) = -2\sum_{j=1}^{N}\phi_{2j}(x, t).$$

Eliminating ϕ_{1j} from (9), we have a system of N linear algebraic equations for ϕ_{2j}

$$\sum_{l=1}^{N}\alpha_{jl}\phi_{2l} = 1 \quad (1\leqslant j\leqslant N) \tag{11}$$

where

$$\alpha_{jl} = \alpha_{jl}(x, t) = \sum_{i=1}^{N}\lambda_i^2(\eta_j+\eta_i)^{-1}(\eta_i+\eta_l)^{-1}+\lambda_j^{-2}\delta_{jl} \tag{12}$$

(δ_{jl} being Kronecker's symbol).

Let $V=V(x, t)$ be the determinant of the coefficient matrix $(\alpha_{kl}(x, t))$ of (11) and $V_j = V_j(x, t)$ be the determinant obtained by replacing the j-th column of V by ${}^t(1, 1, \cdots, 1)$. Then by the Cramer's formula we have

$$\phi_{2j}(x, t) = V_j(x, t)\ V(x, t)^{-1}.$$

Put

$$n_j = {}^t((\eta_1+\eta_j)^{-1}, (\eta_2+\eta_j)^{-1}, \cdots, (\eta_N+\eta_j)^{-1}).$$

$$e_j = {}^t(0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0)$$

$$e = {}^t(1, 1, \cdots, 1)$$

$$p_{ij} = p_{ij}(x, t) = c_j(\eta_i+\eta_j)^{-1} \exp(-2\eta_j z_j) n_j$$

$$q_j = q_j(x, t) = c_j^{-1} \exp(2\eta_j z_j) e_j$$

where $z_j = x - 4\eta_j^2 t$. Put $\Lambda_j = \{1, 2, \cdots, j\}$, $\Lambda^j = \{j, j+1, \cdots, N\}$ (we denote Λ_N by Λ.). We use the notations as follows.

For $J \subset \Lambda$, $\sigma \in I(J)$ means that σ is an injective mapping from J into Λ, for $j, k \in \Lambda$, $k \leqslant j, \sigma \in B(k, j)$ means that σ is a bijective mapping from $\Lambda^k - \{j\}$ onto Λ^{k+1}, and $\sigma \in S^j$ means that σ is a permutation among natural numbers j and N. For $J, J' \subset \Lambda$, put

$$e(J, J'; x, t) = \exp(2\textstyle\sum_{i\in J}\eta_i z_i - 2\sum_{i\in J'}\eta_i z_i). \tag{13}$$

Now, for $J \subset \Lambda$ and $\sigma \in I(J^c)$, let $V(J, \sigma) = V(J, \sigma; x, t)$ be the determinant of N-th order whose i-th column coincides with $q_i(x, t)$ if $i \in J$, and $P_{i\sigma(i)}(x, t)$ if $i \in J$. Note that

$${}^t(\alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{Nj}) = \textstyle\sum_{i=1}^N p_{ji}(x, t) + q_j(x, t)$$

and $p_{1j}, p_{2j}, \cdots, p_{Nj}$ are linearly dependent for each x, t. Hence, by the N linearlity, we have

$$V(x, t) = \textstyle\sum_J \sum_\sigma V(J, \sigma; x, t)$$

By the way, we can express $V(J, \sigma)$ as

$$V(J, \sigma; x, t) = K_{J\sigma} e(J, \sigma(J^c); x, t)$$

($K_{J\sigma}$ being a constant which can be determined easily). So $V(x, t)$ is some linear combination of functions (13).

Similarly to the above, for $j \in \Lambda$, $J \subset \Lambda - \{j\}$, and $\sigma \in I((J \cup \{j\})^c)$, let $V(j, J, \sigma) = V(j, J, \sigma; x, t)$ be the determinant of N-th order whose i-th column coincides with $q_i(x, t)$ if $i \in J$, $p_{i\sigma(i)}(x, t)$ if $i \in (J \cup \{j\})^c$ and e if $i = j$, then we have

$$V_j(x, t) = \textstyle\sum_J \sum_\sigma V(j, J, \sigma; x, t).$$

Now, Theorem (for $t\to\infty$) reduces to

Theorem′. *Put, for $j\in\Lambda$*

$$v_j = v_j(x, t) = -2\sum_{i=j}^{N}\sum_{\sigma\in B(j,i)}V(i, \Lambda_{j-1}, \sigma; x, t)\ V(x, t)^{-1}$$

and

$$R = R(x, t) = v(x, t)-\sum_{j=1}^{N}v_j(x, t)$$

then as $t\to\infty$

$$v_j(x, t)-(\text{sign } c_j)s(x-4\eta_j^2t-\delta_j^+, \eta_j)\to 0$$

and

$$R(x, t)\to 0$$

uniformly in x.

4. Proof of Theorem ′

It is necessary for our purpose to determine the coefficients of $e(\Lambda_j, \Lambda^{j+1}; x, t)$, $j=0, 1, 2, \cdots, N$, in $V(x, t)$ (we denote them by K_j) and $e(\Lambda_{j-1}, \Lambda^{j+1}; x, t)$, $j=1, 2, \cdots, N$, in the numerator of the generalized soliton solution (expressed by the Cramer's formula)

$$-2\sum_{j=1}^{N}V_j(x, t)$$

(we denote them by M_j). Then we have

Lemma 1.

$$K_j = \prod_{i=1}^{j}c_i^{-1}\prod_{i=j+1}^{N}c_i \det(e_1, \cdots, e_j, n_{j+1}, \cdots, n_N)^2$$

and

$$M_j = \prod_{i=1}^{j-1}c_i^{-1}\prod_{i=j+1}^{N}c_i \det(e_1, \cdots, e_{j-1}, e, n_{j+1}, \cdots, n_N)^2 .$$

Proof. One can see immediately that

$$\sum_{\sigma\in s^{j+1}}V(\Lambda_j, \sigma; x, t) = K_je(\Lambda_j, \Lambda^{j+1}; x, t) .$$

Therefore, we have

$$K_j = \prod_{i=1}^{j}c_i^{-1}\prod_{i=j+1}^{N}c_i\sum_{\sigma\in s^{j+1}}\prod_{i=j+1}^{N}(\eta_i+\eta_{\sigma(i)})^{-1} \\ det(e_1, \cdots, e_j, n_{\sigma(j+1)}, \cdots, n_{\sigma(N)}\mathbf{n}) .$$

By the way, the relations

$$\det(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_j\ \boldsymbol{n}_{\sigma(j+1)}, \cdots, \boldsymbol{n}_{\sigma(N)}) = (\mathrm{sign}\sigma) \det (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_j, \boldsymbol{n}_{j+1} \cdots, \boldsymbol{n}_N)$$

and

$$\sum_{\sigma\in S^{j+1}} (\mathrm{sing}\sigma) \prod_{i=j+1}^{N} (\eta_i+\eta_{\sigma(i)})^{-1} = \det(\boldsymbol{e}_1, \cdots, \boldsymbol{e}_j, \boldsymbol{n}_{j+1}\cdots, \boldsymbol{n}_N)$$

hold.

Next, note that

$$(14) \qquad -2\sum_{i=j}^{N} \sum_{\sigma\in B(j,i)} V(i, \Lambda_{j-1}\ \sigma; x, t) = M_j e(\Lambda_{j-1}, \Lambda^{j+1}; x, t).$$

The calculation for M_j is completely parallel to the above. Q.E.D.

Next we have

Lemma 2. *As* $t\to\infty$

$$v_j(x, t)-(\mathrm{sign}\ c_j)s(x-4\eta_j^2 t-\delta_j^+, \eta_j)\to 0$$

uniformly in the infinite sector

$$(15) \qquad -\varepsilon t\leqslant z_j = x-4\eta_j^2 t\leqslant\varepsilon t,\ t>0$$

where ε *is sufficiently small positive constant e.g.* $0<\varepsilon< 2\,\mathrm{Min}_{1\leqslant i\leqslant N-1}(\eta_{i+1}^2-\eta_i^2)$.

Proof. Recall that $V(x, t)$ is a linear combination of functions (13). Therefore, it is easy to see that

$$(16) \qquad e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)V(x, t) = K_{j-1} \exp(-2\eta_j z_j)(1+B_{j,1}(x, t)) + K_j \exp(2\eta_j z_j)(1+B_{j,2}(x, t))+B_{j,3}(x, t)$$

where $B_{jk}(x, t)$ $(k=1, 2, 3)$ are polynomials of $\exp(\pm 2\eta_i z_i), \pm i\geqslant \pm j+1$ whose constant terms are equal to zero. If (x, t) is in the sector (15), then we have

$$(17) \qquad \exp(\pm 2\eta_i z_i)\leqslant\exp(\pm 4\eta_i(\eta_j^2-\eta_j^2)t), \qquad (\pm i\geqslant \pm j+1).$$

Therefore, for $t\to\infty$, $B_{jk}(x, t)$ converge to zero uniformly in the sector (15). Hence, for large t

$$(e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)V(x, t))^{-1}$$

behaves like

$$(K_{j-1} \exp(-2\eta_j z_j)+K_j \exp(2\eta_j z_j))^{-1}$$

in the sector (15). Recall that (see (14))

$$v_j(x, t) = M_j e(\Lambda_{j-1}, \Lambda^{j+1}; x, t)V(x, t)^{-1}$$

and put

$$u_j = u_j(x, t) = M_j \cdot (K_{j-1} \exp(-2\eta_j z_j) + K_j \exp(2\eta_j z_j))^{-1}.$$

Then, for large t, $v_j(x, t)$ behaves like $u_j(x, t)$ in the sector (15). One can see that

$$u_j(x, t) = (\text{sign } c_j) s(x - 4\eta_j^2 t - \delta_j^+, \eta_j)$$

by the relations (see Lemma 1)

$$K_{j-1}/M_j = c_j(2\eta_j)^{-2} \Pi_{i=j+1}^{N} (\eta_i + \eta_j)^2 (\eta_i + \eta_j)^{-2}$$

and

$$K_j/M_j = c_i^{-1} \Pi_{i=j+1}^{N} (\eta_i + \eta_j)^2 (\eta_i - \eta_j)^{-2}. \qquad \text{Q.E.D.}$$

Next we have

Lemma 3.

i) *$(e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)\ V(x, t))^{-1}$ and* $\exp(\pm 2\eta_j z_j)$ $(e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)\ V(x, t)^{-1}$ *are bounded in the sector* (15).

ii) *$(e(\Lambda^{j+1}, \Lambda_j; x, t)\ V(x, t))^{-1}$ is bounded in the sector*

(18) $$(4\eta_j^2 + \varepsilon)t \leqslant x \leqslant (4\eta_{j+1}^2 - \varepsilon)t,\ t > 0.$$

(If $j=0$, N, then (18) *are the half spaces $x \leqslant (4\eta_1^2 - \varepsilon)t$ and $(4\eta_N^2 + \varepsilon)t \leqslant x$ respectively.)*

Proof. i) is a direct consequences of the fact that a soliton is bounded. ii) Similarly to Lemma 2, one can show that for large t

$$(e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)\ V(x, t))^{-1} \qquad j = 0, 1, \cdots, N$$

behaves like $1/K_j$ in the sector (18). And K_j is not equal to zero. Q.E.D.

For $k \in \Lambda$, $J \subset \Lambda - \{k\}$, and $\sigma \in I((J \cap \{k\})^c)$ put

$$(k, J, \sigma)_+ = J - J \cap \sigma(J \cup \{k\})^c$$

and

$$(k, J, \sigma)_- = \sigma(J \cup \{k\})^c - J \cap \sigma(J \cup \{k\})^c$$

then we have

(19) $$V(k, J, \sigma; x, t)\ V(x, t)^{-1} = Ke((k, J, \sigma)_+, (k, J, \sigma)_-; x, t)\ V(x, t)^{-1}.$$

Next we have

Lemma 4. *If*

(20) $$\{(k, J, \sigma)_-^c \cap \Lambda^{j+1}\} \cup \{(k, J, \sigma)_+^c \cap \Lambda_{j-1}\}$$

is not empty, then $V(k, J, \sigma) V^{-1}$ *converges to zero uniformly in the infinite sector*

$$-\varepsilon t \leqslant x - 4\eta_j^2 t \leqslant \varepsilon t .$$

Proof. By (19) and Lemma 3, there is $K>0$ such that $|V(k, J, \sigma; x, t)\ V(x, t)^{-1}| \leqslant Ke((k, J, \sigma)'_+, (k, J, \sigma)'_+; x, t)e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)$ where $(k, J, \sigma)'_\pm = (k, J, \sigma)_\pm - \{j\}$ if $j \in (k, J, \sigma)_\pm$. If (20) is not empty, then $e((k, J, \sigma)'_+, (k, J, \sigma)'_-; x, t)\ e(\Lambda^{j+1}, \Lambda_{j-1}, x, t)$ is a finite product of exp $(\pm 2\eta_i z_i)$, $\pm i \geqslant \pm j+1$, and is not constant. Q.E.D.

If (20) is empty, then

$$(k, J, \sigma)_+ \supset \Lambda_{j-1},\quad (k, J, \sigma)_- \supset \Lambda^{j+1} .$$

Note that

$$(k, J, \sigma)_+ \cap (k, J, \sigma)_- = \phi$$

(21) $$|(k, J, \sigma)_+| + |(k, J, \sigma)_-| \leqslant N-1 .$$

Therefore we have

$$(k, J, \sigma)_+ = \Lambda_{j-1} \quad (k, J, \sigma) = \Lambda^{j+1} .$$

This implies that

(22) $$k \geqslant j,\ J = \Lambda_{j-1}, \text{ and } \sigma \in B(j, k) .$$

Hence, (20) is empty, if and only if (22) holds.

Therefore, by Lemma 4, as $t \to \infty$, $v_i(x, t)$, $i \neq j$, and $R(x, t)$ converge to zero uniformly in the sector $-\varepsilon t \leqslant x - 4\eta_j^2 t \leqslant \varepsilon t$, $t>0$.

Next we have

Lemma 5. *As* $t \to \infty$, $V(k, J, \sigma; x, t)\ V(x, t)^{-1}$ *converges to zero uniformly in the infinite sector*

$$(4\eta_j^2 + \varepsilon)t \leqslant x \leqslant (4\eta_{j+1}^2 - \varepsilon)t,\ t>0$$

for any k, j, J and σ.

Proof. By Lemma 3, there is $K>0$ such that

$$|V(k, J, \sigma; x, t)\ V(x, t)^{-1}| \leqslant Ke((k, J, \sigma)_+, (k, J, \sigma)_-; x, t)\ e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)$$

$e((k, J, \sigma)_+, (k, J, \sigma)_-; x, t)\ e(\Lambda^{j+1}, \Lambda_{j-1}; x, t)$ is a finite product of exp $(-2\eta_i z_i)$, $i \leqslant j$ and exp $(2\eta_i z_i)$, $i \geqslant j+1$, and is not constant, because of the relation (21). Similary to (17), it is easy to see that $-2\eta_i z_i$, $i \leqslant j$ and $2\eta_i z_i$, $j \geqslant 1$ can be estimated by $B_i t, B_i < 0$. Q.E.D.

This complete the proof of Theorem ′.

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References

[1] O. S. Gardner, J.M. Greene, M.D. Kruskal and R. M. Miura: *A method for solving the Korteweg-de Vries equations*, Phys. Rev. Lett. **19** (1967), 1095–1097.

[2] I. Kay and H. E. Moses: *Reflectionless transmission through dielectrics and scattering potentials*, J. Appl. phys. **27** (1956), 1503–1508.

[3] P. D. Lax: *Integrals of non linear equations of evolution and solitary wave*, Comm. Pure Appl. Math. **21** (1968), 467–490.

[4] A. B. Shabat: *One dimensional scattering theory*, Differential Equations, **8** (1972), 164–178.

[5] S. Tanaka: *Modified Korteweg-de Vries equation and scattering theory*, Proc. Japan Acad. **48** (1972), 466–469.

[6] S. Tanaka: *Some remarks on the modified Korteweg-de Vries equations*, Publ. Res. Inst. Math. Sci. **8** (1973), 429–437.

[7] S. Tanaka: *On the N-tuple wave solutions of the Korteweg-de Vries equation*, Publ. Res. Inst. Math. Sci. **8** (1973), 419–427.

[8] M. Wadati: *The exact solutions of the modified Korteweg-de Vries equation*, J. Phys. Soc. Japan **32** (1972), 1681.

[9] V. E. Zakharov and A. B. Shabat: *Exact theory of two dimensional self focusing and one dimensional self modulation of waves in non linear media*, J. Exp. Theor. Phys. **61** (1971), 118–134.