Shibata, K.
Osaka J. Math.
11 (1974), 331–346

COBORDISM ALGEBRA OF METACYCLIC GROUPS

KATSUYUKI SHIBATA

(Received October 23, 1973)

1. Introduction and the statement of result

Let $Z_{q,p}$ be *the metacyclic group* $\{x, y \mid x^q=y^p=1, yxy^{-1}=x^r\}$ where $p\geqslant 2$ is a prime integer, $q\geqslant 3$ is an odd integer and r is a primitive p-th root of 1 mod q such that $(r-1, q)=1$ (see Shibata [15], 1. Introduction). The object of the present article is to determine the oriented and weakly complex cobordism algebra $\Omega_L^*(Z_{q,p})$ $(L=SO, U)$ of the classifying space of $Z_{q,p}$. Our main result is the following.

Theorem. *Let $p\geqslant 3$ in case $L=SO$ and $p\geqslant 2$ in case $L=U$.*
(1) *$\Omega_L^*(Z_{q,p})$ is a sum of two Ω_L^*-subalgebras whose intersection is the scalars Ω_L^*. These subalgebras are the quotients of power series rings over Ω_L^* generated respectively by the cobordism Euler classes $e(\nu_{q,p})$ and $e(\tilde{\eta}_p)$.*

Here $\tilde{\eta}_p$ is the pull-back of the Hopf line bundle η_p over B_{Z_p} and $\nu_{q,p}$ is a certain complex vector bundle of dimension p whose "restriction" on B_{Z_q} is $\eta_q\oplus\eta_p^r\oplus\cdots\oplus\eta_q^{r^{p-1}}$ with η_q the Hopf line bundle over B_{Z_q}.

(2) $\Omega_L^*(Z_{q,p})\cong\Omega_L^*[[(\prod_{j=0}^{p-1}[r^j]_F(X)),\ Y]]/(XY,\ [q]_F(X), [p]_F(Y),\ (\mathrm{Tor}\ \Omega_L^*)X,$ $(\mathrm{Tor}\ \Omega_L^*)Y)$, *where $[\ ]_F$ indicates the formal multiplication with respect to the formal group law $F(X, Y)$ of complex cobordism theory (or its canonical reduction to oriented cobordism for $L=SO$) (see Quillen* [12]), Tor $\Omega_U^*=0$ *and* Tor Ω_{SO}^* *consists of elements of order* 2. *This isomorphism is realized by the correspondence*; $e(\nu_{q,p})\mapsto \prod_{j=0}^{p-1}[r^j]_F(X)$ *and* $e(\tilde{\eta}_p)\mapsto Y$.

(3) *In case $L=SO$ and $p=2$, the Ω_L^*-subalgebra generated by $e(\tilde{\eta}_p)$ in* (1) *is replaced by $\pi^*\Omega_{SO}^*(Z_2)$, where π^* is the monomorphism induced by the projection π: $B_{Z_{q,2}}\to B_{Z_2}$. And* (2) *is modified as*

$$\Omega_{SO}^*(Z_{q,2})\cong\Omega_{SO}^*[[\prod_{j=0}^{p-1}[r^j]_F(X)]]/([q]_F(X),\ (\mathrm{Tor}\ \Omega_{SO}^*)X)\oplus\tilde{\Omega}_{SO}^*(Z_2)\,.$$

REMARK. $\Omega_{SO}^*[[Y]]/([2]_F(Y))=\Omega_{SO}^*[[Y]]/(2Y)$ is contained in $\Omega_{SO}^*(Z_2)$ as a proper Ω_{SO}^*-subalgebra, Y being the reduction of $e(\eta_2)$ to $\Omega_{SO}^*(Z_2)=\Omega_{SO}^*(B_{Z_2})$. This is easily derived from the results of Shibata [14] via the Atiyah-Poincaré duality (Atiyah [1]).

By virtue of the Conner-Floyd isomorphism

$$\mu_c\colon \Omega_U^*(X) \underset{\Omega_U^*}{\otimes} Z \cong K^*(X)$$

for finite CW complexes [4], we obtain the following corollary, using the same inverse limit arguments as in section 5.

Corollary. (1) $K_U^1(B_{Z_{q,p}})=0.$

(2) $K_U^0(B_{Z_{q,p}}) \cong Z[\prod_{j=0}^{p-1}(1-\eta_q^{r^j}),\ 1-\eta_p]/((1-\eta_q)(1-\eta_p),\ (1-\eta_q^q),\ (1-\eta_p^p))$

the isomorphism being realized by the correspondence;

$$\gamma^p(\nu_{q,p}) = \sum(-1)^i\lambda^i(\nu_{q,p}) \mapsto \prod_{j=0}^{p-1}(1-\eta_q^{r^j}) \quad \textit{and} \quad 1-\tilde{\eta}_p \mapsto 1-\eta_p\,.$$

Kamata [7] determined the group structure of $\Omega_U^*(Z_{q,2})$ by the use of the spaces $D(2k+1, 4n+3)$ whose direct limit become a classifying space for $Z_{q,2}$ (see also Kamata-Minami [8]). We also construct analogous spaces with their direct limit being a classifying space for $Z_{q,p}$. But our construction slightly deviates from that of Kamata-Minami [8] in case $p=2$ (the dihedral case). The difference is essential for our computation, since every homology class of our spaces can be represented by an L-submanifold ($L=U$, SO). Because of this, the Atiyah-Poincaré duality plays an important role in several occasions in the present article.

I am greatly indebted to Professor Masayoshi Kamata for stimulating discussions on his work in [7]. Thanks are also due to Professors Minoru Nakaoka and Fuichi Uchida for helpful conversations.

2. A classifying space for $Z_{q,p}$

Following the line of Kamata-Minami [8], we construct a classifying space for $Z_{q,p}$.

Consider the product space $S^{2pm-1}\times S^{2k-1}$ and define an action ψ of $Z_{q,p}$ on $S^{2pm-1}\times S^{2k-1}$ by the rule

$$\psi(x, ((z_j), z')) = ((\rho_q^{r^j} z_j), z')\,,$$
$$\psi(y, ((z_j), z')) = ((z_{\sigma(j)}), \rho_p z'); \ 0 \leqslant j \leqslant pm-1$$

where $\rho_q = \exp(2\pi\sqrt{-1}/q)$, $\rho_p = \exp(2\pi\sqrt{-1}/p)$ and $\sigma\colon \{0, 1, 2, \cdots\} \to \{0, 1, 2, \cdots\}$ denotes the mapping of period p sending ap$+t$ $(0 \leq t \leq p-2)$ to $ap+t+1$ and $ap+(p-1)$ to ap.

Then we see that this action is free. Denote by $M(m, k)$ the quotient space $S^{2mp-1}\times S^{2k-1}/\psi$. Then the direct limit space of $M(m, m)$ with respect to the natural inclusions $M(m, m)\subset M(m+1, m+1)$ becomes a classifying space of $Z_{q,p}$.

Let $L^{mp-1}(q;(r^j))$ denote the quotient space of S^{2mp-1} by the Z_q-action $\hat{\psi}: (z_0, z_1, \cdots, z_j, \cdots, z_{mp-1})\mapsto(\rho_q z_0, \rho_q^r z_1, \cdots, \rho_q^{r^j} z_j, \cdots, \rho_q^{r^{p-1}} z_{mp-1})$. Consider the product space $L^{mp-1}(q; (r^j))\times S^{2k-1}$ and define a periodic map of period p

$$T: L^{mp-1}(q; (r^j))\times S^{2k-1} \to L^{mp-1}(q; (r^j))\times S^{2k-1}$$

by $T([z_j], z') = ([z_{\sigma(j)}, \rho_p z'])$. Clearly the quotient space $L^{mp-1}(q; (r^j))\times S^{2k-1}/T$ is $M(m, k)$.

The inclusion map i: $L^{mp-1}(q; (r^j))\to M(m, k)$ defined by $i([z])=[[z], (1, 0, \cdots, 0)]$ corresponds to the inclusion i: $Z_q\subset Z_{q,p}$. The projection π: $M(m, k)\to L^{k-1}(p)=L^{k-1}(p; 1, \cdots, 1)$ sending $[[z], z']$ to $[z']$ corresponds to the projection $\pi: Z_{q,p}\to Z_p$. Finally the cross-section s: $L^{k-1}(p)\to M(m, k)$ defined by

$$s[z'] = [[\underbrace{1/\sqrt{p}, \cdots, 1/\sqrt{p}}_{p \text{ coordinates}}, 0, \cdots, 0], z']$$

corresponds to the cross-section s: $Z_p\to Z_{q,p}$ with $s(\bar{y})=y$.

3. $H_*(M(m, k); Z)$ (see Lazarov [9])

Consider the spectral sequence

$$E^2_{i,j} = H_i(Z_p; H_j(L^{mp-1}(q; (r^h))\times S^{2k-1}; Z))$$

associated to the p-fold covering

$$\begin{aligned} L^{mp-1}(q; (r^h))\times S^{2k-1} &\to L^{mp-1}(q; (r^h))\times S^{2k-1}/T \\ &= M(m, k) \end{aligned}$$

(see Cartan [2]). Then $E^2_{i,j}=0$ unless $i=0$, or $j=0$, $2(mp-1)+1$, $2k-1$ or $2(mp+k-1)$. In case $j=0$, $2(mp-1)+1$, $2k-1$ or $2(mp+k-1)$, $E^2_{i,j}=0$ or Z_p depending upon whether positive i is even or odd. To compute $E^2_{0,j}$ we use the resolution

$$Z[Z_p] \xrightarrow{N} Z[Z_p] \xrightarrow{D} Z[Z_p]$$

where $D=1-\bar{y}$, $N=1+\bar{y}+\cdots\bar{y}^{p-1}$, where $\bar{y}$ is the generator of $Z_p=Z_{q,p}/Z_q$. Tensoring on the right with $H_j(L^{mp-1}(q; (r^h))\times S^{2k-1}; Z)$, we find that $E^2_{0,j}=$ cokernel $(D\otimes 1)$. Now $H_*(L^{mp-1}(q; (r^h))\times S^{2k-1}; Z)\cong H_*(L^{mp-1}(q; (r^h)); Z)\otimes H_*(S^{2k-1}; Z)$ and the generator $\bar{y}$ of Z_p acts trivially on $H_*(S^{2k-1}; Z)=H_0(S^{2k-1}; Z)\oplus H_{2k-1}(S^{2k-1}; Z)$, $H_{2s}(L^{mp-1}(q; (r^h)); Z)$ $(s\geqq 0)$ and $H_{2mp-1}(L^{mp-1}(q; (r^h)); Z)$.

Lemma 3.1.

The generator $\bar{y}$ *of* Z_p *acts on* $H_{2s-1}(L^{mp-1}(q; (r^h)); Z) \cong Z_q$ *as the multiplication by* r^s $(1 \leqslant s < mp)$.

Proof. A generator of $H_{2s-1}(L^{mp-1}(q; (r^h)); Z) \cong Z_q$ $(1 \leqslant s < mp)$ can be chosen as the image of the fundamental class $[L^{s-1}(q; (r^h))] \in H_{2s-1}(L^{s-1}(q; (r^h)); Z)$ by the homomorphism induced by the canonical inclusion map

$$\iota^{(s-1)}: L^{s-1}(q; (r^h)) \subset L^{mp-1}(q; (r^h)).$$

Let $\bar{T}: L^{np-1}(q; (r^h)) \to L^{np-1}(q; (r^h))$ be the restriction of T, i.e. $\bar{T}([z_h]) = [z_{\sigma(h)}]$ $(1 \leqslant n \leqslant m)$. Then $(\bar{T})^*$ is the identity homomorphism on $H_{2np-1}(L^{np-1}(q; (r^h)); Z)$ and so $\bar{y}$ acts on $H_{2np-1}(L^{mp-1}(q; (r^h); Z) \cong Z_q$ as the multiplication by $(r)^{np} \equiv 1 \bmod q$ $(1 \leqslant n < m)$. In case $(s, p) = 1$, define a map $(\ )^r: L^{s-1}(q; (r^h)) \to L^{s-1}(q; (r^h))$ by $(\ )^r([z_h]) = [(z_h)^r]$. Then $\iota^{(s-1)} \circ (\ )^r$ is homotopic to $\bar{T} \circ \iota^{(s-1)}$ because $H: L^{s-1}(q; (r^h)) \times I \to L^{mp-1}(q; (r^h))$ defined by

$$H([z_h], t) = [\{1/||(tz_{\sigma(h)} + (1-t) z_h^r)||\}(tz_{\sigma(h)} + (1-t) z_h^r)]$$

gives a desired homotopy. This proves the lemma.

As a consequence

$$\begin{aligned} E^2_{0,*} \cong & \bigoplus_{1 \leqslant n < m} Z_q[L^{np-1}(q; (r^h))] \oplus Z[L^{mp-1}(q; (r^h))] \\ & \bigoplus_{1 \leqslant n < m} Z_q[L^{np-1}(q; (r^h)) \times S^{2k-1}] \oplus Z[L^{mp-1}(q; (r^h)) \times S^{2k-1}] \\ & \oplus Z[pt] \oplus Z[S^{2k-1}]. \end{aligned}$$

From dimensional reasons, it is easy to determine the differentials (c.f. Théorème 4 of Cartan [2]), and we obtain the following result.

Proposition 3.2.

$$\begin{aligned} \tilde{H}_*(M(m, k); Z) \cong & \bigoplus_{1 \leqslant n < m} Z_q[L^{np-1}(q; (r^h))] \oplus Z[L^{mp-1}(q; (r^h))] \\ & \bigoplus_{1 \leqslant n < m} Z_q[M(n, k)] \oplus Z[M(m, k)] \\ & \bigoplus_{1 \leqslant j < k} Z_p[L^{j-1}(p)] \oplus Z[L^{k-1}(p)] \\ & \bigoplus_{1 \leqslant j < k} Z_p[M(m, j)] \end{aligned}$$

Corollary 3.3.

The Thom homomorphism

$$\mu: \Omega_*^L(M(m, k)) \to H_*(M(m, k); Z)$$

is epic. Equivalently, the bordism spectral sequence for $M(m, k)$ *collapses* (c.f.

Conner-Smith [5], Conner-Floyd [3]).

Corollary 3.4. *Except for the case $p=2$, $L=SO$, $\tilde{\Omega}_*^L(M(m, k))$ is generated as an Ω_*^L-module by the bordism classes of the canonical inclusion maps to $M(m, k)$ of the L-submanifolds* ($L=U$, SO);

$$(L^{np-1}(q; (r^h)), \iota);\ 1\leqslant n\leqslant m\,,$$
$$(M(n, k), \iota);\ 1\leqslant n\leqslant m\,,$$
$$(L^{j-1}(p), s);\ 1\leqslant j\leqslant k \quad and \quad (M(m, j), \iota);\ 1\leqslant j<k\,.$$

Proof. Except for the case $p=2$, $L=SO$,

$$H_*(M(m, k);\ \Omega_*^L) \cong H_*(M(m, k);\ Z)\otimes\Omega_*^L$$

by Proposition 3.2. Therefore the arguments of Conner-Floyd [3], Theorem 18.1 work well in this case to prove the corollary.

4. Cobordism algebra $\Omega_L^*(M(m, k))$; L=U, SO

Now we turn from bordism to cobordism via the Atiyah-Poincaré duality (Atiyah [1], Conner-Floyd [3]);

$$\begin{array}{ccc} \Omega_i^L(M(m, k)) & \xrightarrow[\cong]{D} & \Omega_L^{2(mp+k-1)-i}(M(m, k)) \\ \mu\big\downarrow & & \mu\big\downarrow \\ H_i(M(m, k);\ Z) & \xrightarrow[\cong]{D} & H^{2(mp+k-1)-i}(M(m, k);\ Z)\,. \end{array}$$

Let $\tilde{\eta}_{p;m,k}$ be the normal bundle of the embedding $M(m, k)\subset M(m, k+1)$.

Lemma 4.2. *Let* m, $k\geqslant 1$.

(1) $\tilde{\eta}_{p;m,k}\cong\pi^*\eta_{p;k}$, *where $\eta_{p;k}$ is the Hopf line bundle over $L^{k-1}(p)$.*

(2) $D[M(m, j), \iota]=e(\tilde{\eta}_{p;m,k})^{k-j}$ $(1\leqslant j\leqslant k)$.

(3) $e(\tilde{\eta}_{p;m,k})^k=0$.

(4) $D[M(m-1, k), \iota]^j=D[M(m-j, k), \iota]$ $(0\leqslant j\leqslant m-1)$.

(5) $D[M(m-1, k), \iota]^m=0$.

Proof. The projection $\pi\colon (M(m, k+1), M(m, k))\to(L^k(p), L^{k-1}(p))$ induces a bundle map between the normal bundles, and hence follows (1). For $j=k$, (2) is obvious from the definition of the duality D. From (1) we have $e(\tilde{\eta}_{p;m,k})=\pi^*e(\eta_{p;k})$, and we know that $e(\eta_{p;k})=D[L^{k-2}(p)\subset L^{k-1}(p)]$ (Kamata [6]). (Notice that for $k=1$, $L^{k-2}(p)=\phi$ and $e(\eta_{p;k})=0$. This implies (3) for $k=1$.) The projection $\pi\colon M(m, k)\to L^{k-1}(p)$ being t-regular on $L^{k-2}(p)$, we see that $\pi^*e(\eta_{p;k})=D[\pi^{-1}(L^{k-2}(p))=M(m, k-1)\subset M(m, k)]$. This proves (2) for $j=k-1$. For $k\geqslant 2$, let $\iota'\colon M(m, k-1)\to M(m, k)$ be the embedding defined by

$$\iota'([[u_g], (v_h)] = [[u_g], (0, (v_h))]$$

while

$$\iota([[u_g], (v_h)]) = [[u_g], ((v_h), 0)] .$$

Then ι and ι' are homotopic and t-regular (Kamata [7]). Therefore $e(\tilde{\eta}_{p;m,k})^2 = D([M(m, k-1), \iota]\cdot[M(m, k-1), \iota] = D[(\iota')^{-1}(\iota(M(m, k-1)), \iota'] = D[M(m, k-2), \iota'] = D[M(m, k-2), \iota]$. This proves (2) for $j=k-2$. Repeting this procedure, we inductively obtain (2) and then (3). Parts (4) and (5) are proved similarly.

Q.E.D.

Let us denote by $\hat{\Omega}_L^*(L^{k-1}(p))$ the intersection $\tilde{\Omega}_L^*(L^{k-1}(p)) \cap D(\tilde{\Omega}_*^L(L^{k-1}(p))$. So it holds that $\tilde{\Omega}_L^*(L^{k-1}(p)) = \hat{\Omega}^*(L^{k-1}(p)) \oplus \Lambda_{\Omega_L^*}(D[pt])$, where the second term in the right hand side is the exterior algebra over Ω_L^* generated by the dual class of an inclusion map of a point (Shibata [14]).

Lemma 4.3. *Let m, $k \geqslant 1$.*

$$\Omega_L^*(M(m, k)) = \Omega_L^*(D[M(m-1, k), \iota]) \oplus D\circ i_* \Omega_*^L(L^{mp-1}(q; (r^j))) \oplus \pi^* \hat{\Omega}_L^*(L^{k-1}(p)) \oplus D\circ s_* \tilde{\Omega}_*^L(L^{k-1}(p))$$

as Ω_L^*-modules, where by $\Omega_L^*(D[M(m-1, k), \iota])$ we mean the Ω_L^*-subalgebra generated by $D[M(m-1, k), \iota]$.

Proof. By 4.2 (1) and (2), the dual cohomology class of $[M(m, j)]$ is $\pi^* c_1(\eta_{p;k})^{k-j}$. Therefore, applying the Poincaré duality to 3.2, we see that the E_2-term $H^*(M(m, k); \Omega_L^*)$ of the cobordism spectral sequence is additively generated by the submodules $\Omega_L^*(D[M(m-1, k)])$, $D\circ i_* H_*(L^{mp-1}(q; (r^j)); \Omega_*^L)$, $\pi^* \tilde{H}^*(L^{k-1}(p); \Omega_L^*)$ and $D\circ s_* \tilde{H}_*(L^{k-1}(p); \Omega_*^L)$, where $\Omega_L^*(D[M(m-1, k)])$ means the Ω_L^*-subalgebra of $H^*(M(m, k); \Omega_L^*)$ generated by $D[M(m-1, k)]$. By virtue of the Atiyah-Poincaré duality (4.1), corollary 3.3 implies the collapsibility of the cobordism spectral sequence for $M(m, k)$. Together with this, the naturality of the spectral sequences and the arguments of [3], 18.1 imply that $\Omega_L^*(M(m, k))$ is additively generated by the submodules $\Omega_L^*(D[M(m-1, k), \iota])$, $D\circ i_* \Omega_*^L(L^{mp-1}(q; (r^j)))$, $\pi^* \tilde{\Omega}_L^*(L^{k-1}(p))$ and $D\circ s_* \Omega_L^*(L^{k-1}(p))$. Note that we may replace $\pi^* \tilde{\Omega}_L^*(L^{k-1}(p))$ by $\pi^* \hat{\Omega}_L^*(L^{k-1}(p))$ since $\pi^* D[pt] = D\circ i_*[L^{mp-1}(q; (r^j)), \iota]$. Thus we have

$$\Omega_L^*(M(m, k)) = \Omega_L^*(D[M(m-1, k), \iota]) + D\circ i_* \Omega_*^L(L^{mp-1}(q; (r^j))) + \pi^* \hat{\Omega}_L^*(L^{k-1}(p)) + D\circ s_* \tilde{\Omega}_*^L(L^{k-1}(p)) , \tag{4 4}$$

and it remains to prove that the above sums are direct.

Case 1: even dimensional case for $p \geqslant 3$ or $L=U$

Let u be an even dimensional element of $\Omega_L^*(M(m, k))$. Then, by 4.4, we can express u as

$$u = \sum_{j\geqslant 0} [V_j]\cdot D[M(m-1, k), \iota]^j + [W]\cdot D[pt] + \pi^* v$$

with $v \in \hat{\Omega}_L^{\mathrm{ev}}(L^{k-1}(p))$, taking account of the fact that $\Omega_U^{\mathrm{odd}}=0$, that Ω_{SO}^* contains no odd torsion and that Ω_{SO}^j is a 2-torsion group unless $j\equiv 0 \bmod 4$ (Milnor [11]).

Suppose $u=0$. Then $0=s^*u=[V_0]+s^*\pi^*v=[V_0]+v$, and so $[V_0]=0$ and $v=0$ (Notice that s is homotopic to s': $[x]\mapsto[[0, \cdots, 0, 1/\sqrt{p}, \cdots, 1/\sqrt{p}], z]$ with Image $s'\cap M(m-1, k)=\phi$ when $m>1$, while $D[M(m-1, k), \iota]=0$ when $m=1$ by 4.2 (5).) And so $\sum_{j\geqslant 1} [V_j][M(m-j, k), \iota]+[W][pt]=0$. Taking the augmentation homomorphism, we see that $[W]=0$ since $[M(m-j, k)]= [L^{(m-j)p-1}(q; (r^j))\times S^{2k-1}/Z_p]=0$ in Ω_*^U and hence also in Ω_*^{SO}. Therefore the sums in 4.4 are direct in this case.

Case 2. *even dimensional case for $p=2$, $L=SO$.*

Slightly deviating from the preceding case, an even dimensional element u is expressed as

$$u = \sum_{j\geqslant 0} [V_j]\cdot D[M(m-1, k), \iota]^j + [W]\cdot D[pt] + \pi^* v + D\circ s_* w\,,$$

where w belongs to $\tilde{\Omega}_{\mathrm{ev}}^{SO}(RP(2k-1))$.

Since $s=s_0$ is homotopic to s_1: $[z]\mapsto[[-1/\sqrt{p}, \cdots, -1/\sqrt{p}, 0, \cdots, 0], z]$ by the homotopy s_t: $[z]\mapsto[[(\exp \pi t\sqrt{-1})/\sqrt{p}, \ldots, (\exp \pi t\sqrt{-1})/\sqrt{p}, 0, \cdots, 0], z]$ and since Image $s_1\cap$ Image $s=\phi$, it follows that $s^*\circ D\circ s_* w=(s_1)^*\circ D\circ s_* w=0$ and so $u=0$ implies $0=s^*u=[V_0]+v$. Hence $[V_0]=0$ and $v=0$ as in the preceding case. Consequently, $\sum_{j\geqslant 1} [V_j][M(m-j, k), \iota]+[W]\cdot[pt]+s_* w=0$. Again we obtain $[W]=0$ by considering the augmentation. Now

$$\begin{aligned} 0 &= \sum_{j\geqslant 1} [V_j]\pi_*[M(m-j, k), \iota]+\pi_*\circ s_* w \\ &= \sum_{j\geqslant 1} [V_j]\pi_*[M(m-j, k), \iota]+w\,. \end{aligned}$$

So it suffices to prove $\pi_*[M(m-j, k), \iota]=0$ in $\Omega_{\mathrm{ev}}^U(RP(2k-1))$ and hence also in $\Omega_{\mathrm{ev}}^{SO}(RP(2k-1))$. But the augmentation ε: $\Omega_{\mathrm{ev}}^U(RP(2k-1))\to\Omega_{\mathrm{ev}}^U(pt)$ is an isomorphism and the fact that $\varepsilon[M(m-j, k), \iota]=[M(m-j, k)]=0$ in Ω_{ev}^U as shown in the preceding case implies the desired result.

Case 3. *odd dimensional case for $p\geqslant 3$ or $L=U$.*

An odd dimensional u' can be expressed as

$$u' = D\circ i_* v' + D\circ s_* w'$$

with $v' \in \Omega_*^L(L^{mp-1}(q; (r^j)))$ and $w' \in \tilde{\Omega}_*^L(L^{k-1}(p))$. So $u'=0$ implies $w'=\pi_*(i_*v'+s_*w')=\pi_* D^{-1}(u')=0$. Therefore $D\circ s_*w'=D\circ i_*v'=0$.

Case 4. *odd dimensional case for* $p=2$, $L=SO$.

In this case, we have

$$u' = [V_0]+D\circ i_*v'+\pi^*t'+D\circ s_*w' ,$$

with $[V_0]\in\Omega_{SO}^{\text{odd}}$, $v'\in\Omega_{\text{odd}}^{SO}(L^{mp-1}(q: (r^j)))$, $t'\in\hat{\Omega}_{SO}^{\text{odd}}(RP(2k-1))$ and $w'\in\tilde{\Omega}_{\text{odd}}^{SO}(RP(2k-1))$.

Suppose $u'=0$. First consider a map $c: pt\mapsto M(m, k)$ such that $c(pt)\cap$ Image $i=c(pt)\cap$ Image $s=\phi$. Then $0=c^*u'=[V_0]$. Next observe that v' is decomposed as $v'=[V_0'][pt]+v''+[V_1'][L^{mp-1}(q; (r^j)), id]$ with v'' an odd torsion element. Because v'' is the only odd torsion term in the expression of u', it follows that $v''=0$. Therefore $0=u'=[V_0']\cdot D[pt]+[V_1']\cdot D\circ i_*[L^{mp-1}(q; (r^j)),id]+\pi^*t'+D\circ s_*w'$. Then $0=[V_1']s^*\circ D\circ i_*[L^{mp-1}(q: (r^j)), id]+s^*\pi^*t'+(s_1)^*D\circ s_*w'=[V_1']\cdot D[pt]+t'$, and consequently $[V_1']=\mathcal{E}\circ D^{-1}([V_1']\cdot D[pt]+t')=0$. Thus $t'=0$. These facts imply $[V_0'][pt]+s_*w'=0$ and hence $[V_0']=\mathcal{E}([V_0'][pt]+s_*w')=0$ and $s_*w'=0$. Summarizing, we conclude that the sums in 4.4 are also direct in this final case. Q.E.D.

Now let $\xi=\xi_{q;m}$ and respectively $\eta=\eta_{q:m}$ be the complex line bundles associated to the coverings $S^{2mp-1}\to L^{mp-1}(q: (r^j))$ and $S^{2mp-1}\to L^{mp-1}(q)=L^{mp-1}(q; 1, \cdots, 1)$. The maps f: $L^{mp-1}(q)\to L^{mp-1}(q; (r^j))$ and g: $L^{mp-1}(q; (r^j))\to L^{mp-1}(q)$ defined respectively by $f([(z_j)])=[(z_j^{r^j})]$ and $g([(z_j)])=[(z_j^{r^{p-j}})]$ satisfy $f^*\xi=\eta$ and $g^*\eta=\xi$.

Lemma 4.5. *Let* $m, k\geqslant 1$.

(1) $f^*\circ i^*\circ D[M(m-1, k), \iota]=e(\eta\oplus\eta^r\oplus\cdots\oplus\eta^{r^{p-1}})=\prod_{i=0}^{p-1} e(\eta^{r^i})$.

(2) *The restriction of the homomorphism*

$$f^*\circ i^*\colon \Omega_L^*(M(m, k)) \to \Omega_L^*(L^{mp-1}(q))$$

upon the subalgebra $\Omega_L^*(D[M(m-1, k), \iota])$ *is monomorphic.*

(3) $D[M(m-1, k), \iota]=e(\nu_{q,p;m,k})$, *where* $\nu_{q,p;m,k}$ *is the normal bundle of the natural embedding* $M(m, k)\subset M(m+1, k)$.

Proof. First observe that for $m=1$, it holds that $[M(m-1, k), \iota]=0$ and that $\prod_{i=0}^{p-1} e(\eta^{r^i})=0$. And so (1) and (2) are obvious in this case. So we suppose $m\geqslant 2$ in the following proof of (1) and (2).

Now $i^*\circ D[M(m-1, k), \iota]=D[L^{(m-1)p-1}(q; (r^j)), \iota]\in\Omega_L^{2p}(L^{mp-1}(q; (r^j)))$. Define $L_i^{mp-2}(r^j)$ $(i=1, 2, \cdots, p)$ by

$$L_i^{mp-2}(r^j) = \{[(z_j)] \in L^{mp-1}(q; (r^j)); z_{(m-1)p-1+i} = 0\}.$$

Then it is easy to see that $D[L^{(m-1)p-1}(q; (r^j)), \iota] = \prod_{i=1}^{p} D[L_i^{mp-2}(r^j), \iota]$. Define $\psi_i\colon L^{mp-1}(q; (r^j)) \to CP(mp-1)$ by $\psi_i[(z_j)] = [(z_j^{r^{i-j-1+mp}})]$ and $CP_i(mp-2) \subset CP(mp-1)$ by $CP_i(mp-2) = \{[(z_j)] \in CP(mp-1); z_{(m-1)p-1+i} = 0\}$ for $i = 1, 2, \ldots, p$. Then ψ_i is t-regular on $CP_i(mp-2)$ and $D[L_i^{mp-2}(r^j), \iota] = \psi_i^* D[CP_i(mp-2), \iota]$. We can easily construct a homotopy between $CP(mp-2) = CP_p(mp-2) \subset CP(mp-1)$ and

$$CP(mp-2) \xrightarrow{h_i} CP_i(mp-2) \subset CP(mp-1),$$

where $h_i[z_0, \cdots, z_{mp-2}] = [z_0, \cdots, z_{(m-1)p-1+i-1}, 0, z_{(m-1)p-1+i}, \cdots, z_{mp-2}]$. Hence $\psi_i^* D[CP_i(mp-2), \iota] = \psi_i^* D[CP(mp-2), \iota] = \psi_i^* e(\eta_H) = e(\psi_i^* \eta_H)$ with η_H the Hopf line bundle over $CP(mp-1)$. Therefore $f^* \circ i^* \circ D[M(m-1, k), \iota] = \prod_{i=1}^{p} e(f^* \circ \psi_i^* \eta_H) = \prod_{i=1}^{p} e(\eta^{r^{i-1}}) = \prod_{i=0}^{p-1} e(\eta^{r^i})$ since $\psi_i \circ f[(z_j)] = [(z_j)^{r^{i-1+mp}}]$. This proves (1) for $m > 1$.

By (1), we have $f^* \circ i^* \circ D[M(m-1, k)]^j = \{\prod_{i=0}^{p-1} c_1(\eta^{r^i})\}^j = \{(\prod_{i=0}^{p-1} r^i) c_1(\eta)^p\}^j = \{(-1)^{p-1} c_1(\eta)^p\}^j$ in $H^{2pj}(L^{mp-1}(q); Z)$. (Notice that r is a primitive p-th root of 1 mod q, and so $\prod_{i=0}^{p-1} r^i \equiv (-1)^{p-1} \bmod q$.) By 4.2 (5), $D[M(m-1, k)]^m = 0$ and by 3.2, the order of $D[M(m-1, k)]^j = D[M(m-j, k)]$ is q, that is, the order of $\{(-1)^{p-1} c_1(\eta)^p\}^j$ for $1 \leqslant j \leqslant m-1$. Thus the restricition of the homomorphism

$$f^* \circ i^*\colon H^*(M(m, k); Z) \to H^*(L^{mp-1}(q); Z)$$

upon the subring

$$Z(D[M(m-1, k)]) \cong Z_q[D[M(m-1, k)]]/(D[M(m-1, k)]^m)$$

is monomorphic. Then consider the E_2-terms of the cobordism spectral sequences for $M(m, k)$ and $L^{mp-1}(q)$. Since q is odd, Ω_L^* has no q-torsion. And by 4.3, the subalgebra $\Omega_L^*(D[M(m-1, k), \iota])$ is a direct summand as an Ω_L^*-submodule. These facts together with the triviality of the spectral sequence, imply that $\Omega_L^*(D[M(m-1, k), \iota])$ corresponds to $Z(D[M(m--1, k)]) \otimes \Omega_L^*$ in the E_2-term. Therefore the triviality and the naturality of the spectral sequences imply (2) for $m > 1$.

Now by 4.3, $e(\nu_{q,p;m+1,k}) \in \Omega_L^*(M(m+1, k))$ is expressed as $e(\nu_{q,p;m+1,k}) = \sum_{j \geqslant 0} [V_j] \cdot D[M(m, k), \iota]^j + [W] \cdot D[pt] + \pi^* v + D \circ s_* w$ with $v \in \hat{\Omega}_L^{\mathrm{ev}}(L^{k-1}(p))$ and $w \in \tilde{\Omega}^L_{\mathrm{ev}}(L^{k-1}(p))$. Observe that the induced bundle $s^* \nu_{q,p;m+1,k}$ is the complex vector bundle $C^p \times_{Z_p} S^{2k-1} \to S^{2k-1}/Z_p = L^{k-1}(p)$ with Z_p acting on C^p as the regular representation space. This complex bundle is well-known to be isomorphic to

$\eta^0_{p\,;\,k}\oplus\eta^1_{p\,;\,k}\oplus\cdots\oplus\eta^{p-1}_{p\,;\,k}$. Thus $s^*e(\nu_{q,p\,;\,m+1,k})=\prod_{i=0}^{p-1}e(\eta^i_{p\,;\,k})=0\cdot\{\prod_{i=1}^{p-1}e(\eta^i_{p\,;\,k})\}=0$, and this should be equal to $[V_0]+s^*\pi^*v+(s_1)^*\circ D\circ s_*w=[V_0]+v$. Hence $[V_0]=0$ and $v=0$.

Consider the natural inclusion map ι_m: $M(m, k)\subset M(m+1, k)$. One can easily verify that $(\iota_m)^*e(\nu_{q,p\,;\,m+1,k})=e(\nu_{q,p\,;\,m,k})$, $(\iota_m)^*\circ D[M(m, k), \iota]^j=D[M(m-1, k), \iota]^j$, $(\iota_m)^*\circ D[pt]=0$ and $(\iota_m)^*\circ D\circ s_*w=0$. (See the proof of 5.1 in the following section.) Therefore $e(\nu_{q,p\,;\,m,k})=(\iota_m)^*e(\nu_{q,p\,;\,m+1,k})=\sum_{j\geqslant 1}[V_j]\cdot D[M(m-1, k), \iota]^j$. So $e(\nu_{q,p\,;\,m,k})=0$ when $m=1$ and this proves (3) for $m=1$. For $m>1$, $e(\nu_{q,p\,;\,m,k})$ belongs to $\Omega^*_L(D[M(m-1, k), \iota])$.

Consequently, it suffices to prove $f^*\circ i^*e(\nu_{q,p\,;\,m,k})=f^*\circ i^*\circ D[M(m-1,k), \iota]=e(\eta\oplus\eta^r\oplus\cdots\oplus\eta^{r^{p-1}})$ by virtue of (1) and (2). But by definition we have $f^*\circ i^*\nu_{q,p\,;\,m,k}\cong f^*(\xi\oplus\xi^r\oplus\cdots\oplus\xi^{r^{p-1}})=\eta\oplus\eta^r\oplus\cdots\oplus\eta^{r^{p-1}}$. Q.E.D.

We summarize the results in 4.3 and 4.5 in the following form.

Theorem 4.6. *Let* $m,\ k\geqslant 1$.

$$\begin{aligned}\Omega^*_L(M(m, k)) &= \Omega^*_L(e(\nu_{q,p\,;\,m,k}))\oplus D\circ i_*\Omega^L_*(L^{mp-1}(q;\,(r^j)))\\ &\quad\oplus\pi^*\hat{\Omega}^*_L(L^{k-1}(p))\oplus D\circ s_*\Omega^L_*(L^{k-1}(p))\,,\end{aligned}$$

where (1) *$\Omega^*_L(e(\nu_{q,p\,;\,m,k}))$ is the Ω^*_L-subalgebra generated by the cobordism Euler class $e(\nu_{q,p\,;\,m,k})$ of the normal bundle $\nu_{q,p\,;\,m,k}$ of embedding $M(m, k)\subset M(m+1, k)$,*

(2) *$\Omega^*_L(e(\nu_{q,p\,;\,m,k}))$ is mapped isomorphically onto the subalgebra $\Omega^*_L(\prod_{i=0}^{p-1}e(\eta^{r^i}))$ of $\Omega^*_L(L^{mp-1}(q))$ by $f^*\circ i^*$*

(3) *$\hat{\Omega}^*_L(L^{k-1}(p))=\tilde{\Omega}^*_L(L^{k-1}(p))\cap D\tilde{\Omega}^L_*(L^{k-1}(p))$,*

(4) *π^* and $D\circ s_*$ are Ω^*_L-module isomorphisms onto direct summands and*

(5) *$(\sum_{j\geqslant 0}[V_j]\cdot e(\nu_{q,p\,;\,m,k})^j)\cdot\pi^*v=[V_0]\cdot\pi^*v$ since the $e(\nu_{q,p\,;\,m,k})^j$ $(j\geqslant 1)$ are q-torsion while $\Omega^*_L(L^{k-1}(p))$ is a p-torsion group.*

5. Proof of Theorem

Let $\iota_{m,k}$: $M(m, k)\rightarrow M(m+1, k+1)$ be the canonical inclusion. Then, for the induced homomorphism

$$(\iota_{m,k})^*\colon\ \Omega^*_L(M(m+1, k+1))\rightarrow\Omega^*_L(M(m, k))\,,$$

we have the following result.

Lemma 5.1.

(1) $(\iota_{m,k})^*e(\nu_{q,p\,;\,m+1,k+1}) = e(\nu_{q,p\,;\,m,k})\,.$

(2) $(\iota_{m,k})^*e(\tilde{\eta}_{p\,;\,m+1,k+1}) = e(\tilde{\eta}_{p\,;\,m,k})\,.$

(3) $(\iota_{m,k})^* D\circ s_* \Omega_*^L(L^k(p)) = 0$.

(4) $(\iota_{m,k})^* D\circ i_* \Omega_*^L(L^{(m+1)p-1}(q; (r^j))) = 0$.

(5) $(\iota_{m,k})^*\colon \pi^*\bar{\Omega}_L^*(L^k(p)) \to \pi^*\bar{\Omega}_L^*(L^{k-1}(p))$ *is an epimorphism, where* $\bar{\Omega}_L^*(L^k(p)) = \Omega_L^*(L^k(p))/(D[pt])$.

Proof. Part (1) and (2) are obvious from the definition. Let us consider part (3). Let $f\colon M \to L^k(p)$ represent an element $[M, f]$ of $\Omega_*^L(L^k(p))$. In order to obtain $(\iota_{m,k})^* D\circ s_*([M, f])$, we should convert $\iota_{m,k}$ within homotopy to become t-regular to $s\circ f$, and then we take the inverse image of $s\circ f(M)$ (c.f. Quillen [13]). But $\iota_{m,k}$ is homotopic to $\iota_{m',k}$ defined by

$$\iota_{m',k}([(z_h), (w_j)]) = [((\underbrace{0, \cdots, 0}_{p}), (z_h)), (w_j)] .$$

Since

$$s[(w_j)] = [(\underbrace{1/\sqrt{p}, \cdots, 1/\sqrt{p}}_{p}, 0, \cdots 0), (w_j)] ,$$

Image $\iota_{m',k} \cap$ Image $s\circ f = \phi$ and hence they are t-regular. Therefore $(\iota_{m',k})^*(D\circ s_*([M, f])) = 0$. This proves (3). Part (4) is shown similarly. Part (5) follows from the commutativity of the diagram

$$\begin{array}{ccc} \Omega_L^*(M(m+1, k+1)) & \xleftarrow{\pi^*} & \Omega_L^*(L^k(p)) \\ \Big\downarrow (\iota_{m,k})^* & & \Big\downarrow (\iota_k)^* \\ \Omega_L^*(M(m, k)) & \xleftarrow{\pi^*} & \Omega_L^*(L^{k-1}(p)) \end{array} .$$

Q.E.D.

Now Kamata [7] defined a Z_2-action on

$$\bar{\Omega}_U^*(L^{m-1}(q)) \cong \Omega_U^*[[X]]/([q]_F(X), X^m)$$

by the correspondence $X \mapsto [-1]_F(X)$, where $[\ ]_F$ denotes the formal multiplication by the formal group law of the complex cobordism theory. We can generalize this to a Z_p-action ($p \geqslant 2$) on

$$\bar{\Omega}_L^*(L^{m-1}(q)) \cong \Omega_L^*[[X]]/([q]_F(X), X^m, (Tor\ \Omega_L^*)\cdot X)$$

for $1 \leqslant m \leqslant \infty$ as follows. (Notice that Tor $\Omega_U^* = 0$ and that (Tor $\Omega_{SO}^*)\cdot X \subset ([q]_F(X), X^m)$ when $m < \infty$.) For an element g of $\bar{\Omega}_L^*(L^{m-1}(q))$ represented by a formal power series $G(X) = \sum_{i=0}^{\infty} a_i X^j \in \Omega_L^*[[X]]$, we define $[r]_F(g)$ to be the class represented by $G([r]_F(X))$. This is easily seen to be a well-defined endomorphism on $\bar{\Omega}_L^*(L^{m-1}(q))$ for $1 \leqslant m \leqslant \infty$. Further, $([r]_F)^p(X) = [r^p]_F(X) = [nq+1]_F(X)$ for some n by definition. And $[nq+1]_F(X) = F([nq]_F(X), X) = [nq]_F(X) + X + [nq]_F(X)\cdot X\cdot P([nq]_F(X), X)$ for some formal power series $P(X, Y) \in \Omega_L^*[[X, Y]]$. As $[nq]_F(X) = [n]_F([q]_F(X))$, it follows that $[r^p]_F(X) -$

$X\in([q](X))$ Hence $([r]_F)^pG(X)\equiv G(X) \bmod ([q]_F(X))$ for any power series $G(X)$. This proves that $[r]_F$ defines a Z_p-action on $\bar{\Omega}_L^*(L^{m-1}(p))$.

Lemma 5.2. *For* $1\leqslant m\leqslant\infty$, *the invariant set*

$$\{\Omega_L^*[[X]]/([q]_F(X), (\mathrm{Tor}\ \Omega_L^*)\}^{Z_p}$$

is equal to

$$\Omega_L^*[[\prod_{i=0}^{p-1}[r^i]_F(X)]]/([q]_F(X).\ X^m, (\mathrm{Tor}\ \Omega_L^*)\cdot X)\,.$$

Proof. As remarked above, $[r^p]_F(X)-X\in([q]_F(X))$ and so $[r]_F\{\prod_{i=0}^{p-1}[r^i]_F(X)\}=\prod_{i=1}^{p}[r^i](X)\equiv\prod_{i=0}^{p-1}[r^i](X) \bmod ([q]_F(X))$. Hence every power series in $\prod_{i=0}^{p-1}[r^i]_F(X)$ represents a Z_p-invariant class in $\bar{\Omega}_L^*(L^{m-1}(p))$.

Conversely, suppose a power series $G(X)=\sum_{j=0}^{\infty}a_jX^j\in\Omega_L^*[[X]]$ represents a Z_p-invariant class of $\bar{\Omega}_L^*(L^{m-1}(p))$. Then the class of $G_1(X)=G(X)-a_0$ is also Z_p-invariant. So

$$G_1([r]_F(X))-G_1(X)=a_1(r-1)X+\cdots\in([q]_F(X),\ X^m,\ (\mathrm{Tor}\ \Omega_L^*)\cdot X)\,.$$

Thus, if $m>1$, $a_1\in q\cdot\Omega_L^*$ ($\supset\mathrm{Tor}\ \Omega_L^*$) by virtue of the hypothesis that $(r-1, q)=1$. Put $a_1=q\cdot\bar{a}_1$ and $G_2(X)=G_1(X)-\bar{a}_1[q]_F(X)$. Then $G_2(X)\equiv G_1(X) \bmod ([q]_F(X))$, $G_2(X)\in(X^2)$ and $G_2(X)$ also represents a Z_p-invariant class. In this way we inductively obtain, if $m>p$, series $G_1(X),\cdots,G_p(X)$ such that $G_1(X)\equiv\cdots\equiv G_p(X) \bmod ([q]_F(X))$, $G_j(X)\in(X^j)$ and that $G_j(X)$ represents a Z_p-invariant class $(j=1, 2,\cdots,p)$. (Notice that, if $m\leqslant p$, this procedure terminates at $G_m(X)$ and that $G(X)\equiv a_0+G_m(X)\equiv a_0 \bmod ([q]_F(X), X^m)$ as desired.) Now that

$$\prod_{i=0}^{p-1}[r^i]_F(X)=(-1)^{p-1}(nq+1)X^p+\cdots$$

for some $n\geqslant 1$. We put

$$G_{p+1}(X)=G_p(X)-(-1)^{p-1}a_p^{(p)}\{\prod_{i=0}^{p-1}[r^i]_F(X)\\ -(-1)^{p-1}[n]_F([q]_F(X))\}\,,$$

where $a_p^{(p)}$ denotes the coefficient of X^p in $G_p(X)$. Then $G_p(X)\equiv G_{p+1}(X)+b_1\{\prod_{i=0}^{p-1}[r^i]_F(X)\} \bmod ([q]_F(X))$, $G_{p+1}(X)\in(X^{p+1})$ and $G_{p+1}(X)$ represents a Z_p-invariant class of $\bar{\Omega}_L^*(L^{m-1}(p))$. And then we again obtain inductively $G_{p+2}(X)$, $\cdots$, $G_{2p}(X)$ if $m>2p$, and so on.

So we obtain a sequence of power series $G(X)=G_0(X), G_1(X),\cdots,G_j(X)$, $\cdots$ $(j\leqslant m)$ such that for $pk+1\leqslant j<p(k+1)+1$

$$G_j(X) \equiv G(X) - \{a_0 + \sum_{h=1}^{k} b_h (\prod_{i=0}^{p-1} [r^i]_F(X))^h\} \bmod ([q]_F(X)),$$

$G_j(X) \in (X^j)$ and $G_j(X)$ represents a Z_p-invariant class of $\tilde{\Omega}_L^*(L^{m-1}(p))$ for every $j \leqslant m$. This proves the lemma for a finite value of m. For the case $m = \infty$, we remark that $a_0 + \sum_{j=1}^{\infty} b_j (\prod_{i=0}^{p-1} [r^i]_F(X))^j$ is a convergent power series since $\prod_{i=0}^{p-1} [r^i]_F(X) \in (X^p)$, and this convergent series gives rise to the same class as that of $G(X)$ in $\Omega_L^*(L^\infty(p)) \cong \Omega_L^*[[X]]/([q]_F(X), (\mathrm{Tor}\ \Omega_L^*)\cdot X)$. This completes the proof of the lemma. Q.E.D.

Let $\Omega_L^*((\prod_{i=0}^{p-1} e(\eta^{r^i})))$ denote the Ω_L^*-subalgebra of $\Omega_L^*(BZ_q)$ generated by homogeneous power series in $\prod_{i=0}^{p-1} e(\eta^{r^i})$.

Corollary 5.3. *The cononical homomorphism*

$$\varprojlim_m \pi_m \colon \Omega_L^*((\prod_{i=0}^{p-1} e(\eta^{r^i}))) \to \varprojlim_m \Omega_L^*(\prod_{i=0}^{p-1} e(\eta_{q;m}^{r^i}))$$

is an isomorphism.

Proof. (Z. Yoshimura) Observe that the Euler class $e(\eta_{q;m}^{r^i})$ corresponds to $[r^i]_F(X)$ with respect to the isomorphism

$$\tilde{\Omega}_L^*(L^{m-1}(q)) \cong \Omega_L^*[[X]]/([q]_F(X), X^m, (\mathrm{Tor}\ \Omega_L^*)\cdot X)$$

for $1 \leqslant m \leqslant \infty$. Therefore in the following commutative diagram

$$\begin{array}{ccccccc}
0 \to \Omega_L^*((\prod_{i=0}^{p-1} e(\eta^{r^i}))) & \xrightarrow{\iota} & \Omega_*^L(BZ_q) & \xrightarrow{1-[r]_F} & \Omega_L^*(BZ_q) \\
\downarrow & & \downarrow\cong & & \downarrow\cong \\
0 \to \varprojlim \Omega_L^*(\prod_{i=0}^{p-1} e(\eta_{q;m}^{r^i})) & \xrightarrow{\varprojlim \iota} & \varprojlim \tilde{\Omega}_L^*(L^{mp-1}(q)) & \xrightarrow{\varprojlim (1-[r]_F)} & \varprojlim \tilde{\Omega}_L^*(L^{mp-1}(q)),
\end{array}$$

the horizontal sequences are exact by 5.2. The corollary follows by diagram chasing. Q.E.D.

Now we can prove our theorem stated in 1.

Proof of Theorem. Consider Milnor's short exact sequence

$$0 \to \varprojlim_m{}^1 \Omega_L^{*-1}(M(m, m)) \to \Omega_L^*(\varinjlim_m M(m, m))$$
$$\to \varprojlim_m \Omega_L^*(M(m, m)) \to 0$$

(Milnor [10]).

It is easy to see that Lemmas 4.2 and 5.1 imply $\varprojlim_m^1 \Omega_L^{*-1}(M(m, m))=0$ and $\varprojlim_m \Omega_L^*(M(m, m))=\varprojlim_m \Omega_L^*(e(\nu_{q,p\,;\,m,m}))\oplus \varprojlim_m \pi^*\Omega_L^*(L^{m-1}(p))$.

$$\begin{aligned}\text{So } \Omega_L^*(\varinjlim M(m, m)) &\simeq \varprojlim \Omega_L^*(M(m, m)) \\ &\simeq \varprojlim \Omega_L^*(e(\nu_{q,p\,;\,m,m}))\oplus\varprojlim \pi^*\Omega_L^*(L^{m-1}(p)) \\ &\simeq \varprojlim \Omega_L^*(e(\nu_{q,p\,;\,m,m}))\oplus\pi^*\Omega_L^*(\varinjlim L^{m-1}(p))\,.\end{aligned}$$

Denote by $\nu_{q,p}$ the direct limit bundle $\varprojlim_m \nu_{q,p\,;\,m,m}$ over $\varprojlim_m M(m, m)=BZ_{q,p}$ and by $\Omega_L^*((e(\nu_{q,p})))$ the graded subalgebra of $\Omega_L^*(BZ_{q,p})$ generated by homogeneous power series in $e(\nu_{q,p})$. Consider the following diagram:

$$(5.4)\qquad \begin{array}{ccccc} \Omega_L^*(Z_q)\supset\Omega_L^*((\prod_{i=0}^{p-1} e(\eta^{r^i}))) & \xleftarrow[\simeq]{(\varprojlim i\circ f)^*} & & & \Omega_L^*((e(\nu_{q,p}))) \\ \simeq\Big\downarrow\varprojlim\pi_m \quad \simeq\Big\downarrow\varprojlim\pi_m & & \varprojlim f^*\circ i^* & & \text{mono}\Big\downarrow\varprojlim\pi_m \\ \varprojlim\Omega_L^*(L^{mp-1}(q))\supset\varprojlim\Omega_L^*(\prod_{i=0}^{p-1} e(\eta_{q\,;\,m}^{r^i})) & \xleftarrow[\text{mono}]{} & & & \varprojlim\Omega_L^*(e(\nu_{q,p\,;\,m,m}))\,. \end{array}$$

By 5.3, $\varprojlim \pi_m$ in the middle is an isomorphism of Ω_L^*-algebras. And $\varprojlim \pi_m$ in the right is a monomorphism because it factors throught the isomorphism $\Omega_L^*(\varprojlim M(m, m))\simeq\varprojlim \Omega_L^*(e(\nu_{q,p\,;\,m,m}))\oplus\varprojlim \pi^*\Omega_L^*(L^{m-1}(p))$ and because $\Omega_L^*((e(\nu_{q,p})))$ is mapped trivially on the second summand by 4.6. So the commutativity of the diagram implies the injectivity of $(\varprojlim i\circ f)^*$ in the upper right. Since $(\varprojlim \pi_m)\circ(\varprojlim i\circ f)^* e(\nu_{q,p})=(\varprojlim f^*\circ i^*)\circ(\varprojlim \pi_m)\, e(\nu_{q,p})=(\varprojlim f^*\circ i^*)$ $\varprojlim e(\nu_{q,p\,;\,m,m})=\varprojlim \prod_{i=0}^{p-1} e(\eta_{q\,;\,m}^{r^i})=(\varprojlim \pi_m)\{\prod_{i=0}^{p-1} e(\eta^{r^i})\}$, the injectiveity of $\varprojlim \pi_m$ in the middle implies $(\varprojlim i\circ f)^* e(\nu_{q,p})=\prod_{i=0}^{p-1} e(\eta^{r^i})$. Therefore $(\varprojlim i\circ f)^*$ is an epimorphism, and hence an isomorphism.

Consequently, the diagram chasing asserts that

$$\varprojlim \pi_m\colon \Omega_L^*((e(\nu_{q,p}))) \simeq \varprojlim \Omega_L^*(e(\nu_{q,p\,;\,m,m}))$$

and

$$\varprojlim f^*\circ i^*\colon \varprojlim \Omega_L^*(e(\nu_{q,p\,;\,m,m})) \simeq \varprojlim \Omega_L^*(\prod_{i=0}^{p-1} e(\eta_{q\,;\,m}^{r^i}))\,.$$

This completes the proof of Theorem.

Appendix. A generalization and examples

For simplicity, we have assumed in the present article and the preceding one [15], that p is a prime integer and so that r is a primitive p-th root of unity mod q.

In fact, these assumptions are not necessary if we assume in turn that $(p, q)=1$ and that

$$(1-r^j, q) = 1, (1 \leqslant i \leqslant p_0-1)$$

where r is a primitive p_0-th root of unity mod q. (Hence $p_0|p$.) Of course $r, p_0 \geqslant 2$ for otherwise $Z_{q,p}$ is a cyclic group. (Notice that the condition $(1-r, q)=1$ requires q odd.) Under these hypotheses, we should replace $\prod_{j=0}^{p-1}$ by $\prod_{j=0}^{p_0-1}$ in (2) of Theorem of this article (but $[p]_F(Y)$ should remain as it is), and β_{2pk-1} by β_{2p_0k-1} in (2) of Theorem 2.10 [15] (but $\tilde{\Omega}^L_*(Z_p)$ should remain as it is), etc.

Notice that for an odd prime q such that $(p, q)=1$ with r a primitive p_0-th root of unity mod q, the above assumptions are always satisfied.

Here are some examples which satisfy the above conditions, and hence, for which the analogous theorems as in the present and preceding articles hold:

Example 1. $\{x^{5^2}=y^{2^2}=1, yxy^{-1}=x^7\}$ with $p_0=p=2^2$.

Example 2. $D(4a, 2q+1)=\{x^{2q+1}=y^{4a}=1, yxy^{-1}=x^{-1}\}$ with $p_0=p/2a=2$. (Notice that $(2q+1, a)=1$ is assumed.) According to Wolf [16], the group $D(4a, 2q+1)$ acts freely and orthogonally on $S=\{(z_1, z_2)\in C^2;\ |z_1|^2+|z_2|^2=1\}$ by

$$x\cdot(z_1, z_2) = ((\rho_{2q+1})^{(2a)^{-1}}z_1,\ (\rho_{2q+1})^{-(2a)^{-1}}z_2)$$

and

$$y\cdot(z_1, z_2) = (z_2,\ (\rho_{2a})^{(2a+1)^{-1}}z_1)$$

where $\rho_j=\exp(2\pi\sqrt{-1}/j)$ $(j=2q+1, 2a)$, and $(2a)^{-1}$ and respectively $(2q+1)^{-1}$ are arbitrary integers such that $2a\cdot(2a)^{-1}\equiv 1 \bmod 2q+1$ and respectively such that $(2q+1)\cdot(2q+1)^{-1}\equiv 1 \bmod 2a$. Let us denote the bordism class of this $D(4a, 2q+1)$-action by $[L, S^3]$. The generalized version of [15] Th. 4.5 asserts that

$$\begin{aligned}\Omega^U_3(D(4a, 2q+1)) = {} & Z_{2q+1}(i_*\beta_3)\oplus Z_{8a}(s_*[T_{(4a,1)}, S^3]) \\ & \oplus Z_{2a}([CP(1)]s_*[T_{(4a,1)}, S^1]+2s_*[T_{(4a,1)}, S^3]) .\end{aligned}$$

(For the notations, see [15].) The calculations show that

$$\begin{aligned}[L, S^3] = {} & aq\cdot i_*\beta_3+(2q+1+\varepsilon\cdot 4a)s_*[T_{(4a,1)}, S^3] \\ & +(-2q+\varepsilon'\cdot 2a)\ [CP_1]s_*[T_{(4a,1)}, S^1]\end{aligned}$$

with $\varepsilon, \varepsilon'=0$ or 1. Therefore

$$\begin{aligned}\Omega^U_3(D(4a, 2q+1)) = {} & Z_{(2q+1)\cdot 8a}([L, S^3]) \\ & \oplus Z_{2a}([CP(1)]s_*[T_{(4a,1)}, S^1]+(4k+2)s_*[T_{(4q,1)}, S^3] \\ & \text{for some } k\geqslant 0 .\end{aligned}$$

OSAKA UNIVERSITY

References

[1] M.F. Atiyah: *Bordism and cobordism,* Proc. Cambridge Philos. Soc. **57** (1961), 200–208.

[2] H. Cartan: *Espace avec groupes d'opérateurs* (II), Séminaire H. Cartan, 1950/51, exposé 12.

[3] P.E. Conner and E.E. Floyd: Differentiable Periodic Maps, Academic Press, New York; Springer-Verlag, Berlin, 1964.

[4] P.E. Conner and E.E. Floyd: The Relation of Cobordism to K-Theories, Lecture Notes in Math. 28, Springer-Verlag, Berlin and New York, 1966.

[5] P.E. Conner and L. Smith: *On the complex bordism of finite complexes,* Inst. Hautes Etudes Sci. Publ. Math. (Paris) **37** (1969), 117–221.

[6] M. Kamata: *The structure of the bordism group $U_*(BZ_p)$,* Osaka J. Math. **7** (1970), 409–416.

[7] M. Kamata: *On complex cobordism groups of classifying spaces for dihedral groups,* Osaka J. Math. **11** (1974), 367–378.

[8] M. Kamata and H. Minami: *Bordism groups of dihedral groups,* J. Math. Soc. Japan **25** (1973), 334–341.

[9] C. Lazarov: *Actions of groups of order pq,* Trans. Amer. Math. Soc. **173** (1972), 215–230.

[10] J. Milnor: *On axiomatic homology theory,* Pacific J. Math. **12** (1962), 337–341.

[11] J. Milnor: *On the cobordism ring Ω^* and a complex analogue* (1), Amer. J. Math. **82** (1960), 505–521.

[12] D. Quillen: *On the formal group laws of unoriented and complex cobordism theory,* Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[13] D. Quillen: *Elementary proofs of some results of cobordism theory using Steenrod operations,* Advances in Math. **7** (1971), 29–56.

[14] K. Shibata: *Oriented and weakly complex bordism algebra of free periodic maps,* Trans. Amer. Math. Soc. **177** (1973), 199–220.

[15] K. Shibata: *Oriented and weakly complex bordism of free metacyclic actions,* Osaka J. Math. **11** (1974), 171–180.

[16] J. Wolf: Spaces of Constant Curvature, McGraw-Hill, New York, 1968.