Sumioka, T.
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A CHARACTERIZATION OF THE TRIANGULAR MATRIX RINGS OVER QF RINGS

TAKESHI SUMIOKA

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In [4], Harada proved that a ring R is a right QF-3 and semi-primary hereditary ring if and only if R is a direct sum of rings whose basic rings are the rings of triangular matrices over division rings. We consider an analogous result to the above one for a right QF-3 semi-primary ring with some injective properties. By Zaks [8], the ring R of triangular matrices of degree $n\geqq 2$ over a QF ring has the injective dimension one both as right and left R-modules, and moreover it is easy to see that the ring R is a QF-3 ring whose maximal right quotient ring is a QF ring. It is our purpose to show that for a basic indecomposable semi-primary ring, the converse is also true.

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Throughout this paper we shall assume that every ring R has an identity element 1, and every R-module is unitary. The notations M_R and $_RM$ are used to underline the fact that M is a right or a left R-module, respectively. For a ring R, a right (resp. left) R-module M is called a *minimal faithful module* if M is a faithful R-module and every faithful right (resp. left) R-module contains an isomorphic image of M as a direct summand. A ring R is called a *right (resp. left) QF-3 ring* if R has a minimal faithful right (resp. left) module, and R is called a *QF-3 ring* if R is both a right and left QF-3 ring.

For a semi-primary ring R, the following conditions are equivalent, (see Jans [5]).

(1) R is a right QF-3 ring.

(2) R has a faithful projective injective right ideal.

Let S be a ring which contains a ring R as a subring. Then S is called a *right (resp. left) quotient ring* of R if S is a rational extension of R as a right (resp. left) R-module. By R' we denote the maximal right quotient ring of R. If R is a QF-3 ring, then the maximal right quotient ring of R coincides with the maximal left quotient ring of R, (see Tachikawa [7]).

Lemma 1. *Let S be a QF ring. Let R be a right QF-3 ring such that R* *is a subring of S and an essential submodule of S as a right R-module, and let eR be a minimal faithful right ideal of R, where e is an idempotent in R. Then we have* $S=\mathrm{Hom}_{eRe}(eR, eR)$*, and S is injective both as right and left R-modules. Therefore, S is the maximal left quotient ring of R.*

Proof. Since eR is R-injective and eR is an essential submodule of an R-module eS, we have $eR=eS$, and hence eS is right S-faithful because of assumptions that eR is right R-faithful and R_R is an essential submodule of S_R. Therefore, $eRe=eSe$ is a QF ring and $eR=eS$ is a finitely generated projective left eRe-module, (see Curtis and Reiner [3]), which shows that $\mathrm{Hom}_{eRe}(eR, eR)$ is right R-injective since eR is right R-injective. Moreover we have $S\cong\mathrm{Hom}_{eSe}(eS, eS)=\mathrm{Hom}_{eRe}(eR, eR)$ because S is QF. Therefore S is right R-injective. Moreover, since eR is left eRe-injective, S is also left R-injective, (see Cartan and Eilenberg [2], Chap. VI, Prop. 1.4). On the other hand, $\mathrm{Hom}_{eRe}(eR, eR)$ is a left quotient ring of R, and consequently S is the maximal left quotient ring of R. This completes the proof.

Let R be a right QF-3 ring whose maximal right quotient ring R' is QF. Then, by Lemma 1, R' is a injective hull of R both as right and left R-modules and R' is the maximal left quotient ring of R.

For a ring 7? we consider the following conditions:

(C_1) R is a right QF-3 ring whose maximal right quotient ring R' is QF, and the injective dimension of R as a right R-module is one.

(C_2) R satisfies the condition C_1, and moreover the injective dimension of R as a left R-module is one.

We study a semi-primary ring R satisfying the condition C_2. Now let R be a semi-primary ring, and let $1=\sum e_{ij}$ be a decomposition of 1 into a sum of orthogonal primitive idempotents e_{ij} such that $e_{ij}R\cong e_{kl}R$ if and only if $i=k$. If we write $e=\sum e_{i1}$, then eRe is a basic ring.

The following lemma is obtained from Morita [6], the proof of Theorem 1.1.

Lemma *2.* (Morita) *Let R be a right QF-3 semi-primary ring with a minimal faithful right ideal eR, where e is an idempotent in R. If eRe is a left artinian ring and eR is a finitely generated left eRe-module, then R is left QF-3.*

By Lemma 1 and Lemma 2, we have

Corollary 1. *Let R be a right QF-3 semi-primary ring with minimal faithful right ideal eR. Then the following conditions are equivalent.*

(1) *There exists a QF ring S such that R is a subring of S and* R_R *is an essential submodule of* S_R.

(2) The maximal right quotient ring R' *is QF.*

(3) *The double centralizer* $\mathrm{Hom}_{eRe}(eR, eR)$ *of* eR_R *is QF.*

Moreover if R satisfies these equivalent conditions, we have $S=R'=\mathrm{Hom}_{eRe}(eR, eR)$ *and R is (left) QF-3.*

Proposition 1. *Let R be a semi-primary ring, and let eRe be a basic ring of R where e is an idempotent in R. Then R satisfies the condition* C_1 *(resp.* C_2*) if and only if eRe satisfies the condition* C_1 *(resp.* C_2*).*

Proof. Suppose that R is a right QF-3 ring with a minimal faithful right ideal $e'R$ of R. Then we may assume that e' is an idempotent in eRe since eR is R-faithful. On the other hand, eRe is Morita equivalent to R with respect to eR. Therefore, $e'R\otimes_R Re\simeq e'Re$ is a minimal faithful right ideal of eRe, and so eRe is right QF-3. If R' is QF, then $e'Re'$ is QF and $e'Re$ is a finitely generated faithful projective left $e'Re'$-module and so $\mathrm{Hom}_{e'Re'}(e'Re, e'Re)$ is QF. Thus, by Corollary 1, the maximal right quotient ring of eRe is QF. Using the fact that ${}_R Re$ is a generator, it is similarly proved that if eRe is a right QF-3 ring whose maximal right quotient ring is QF, then so is R. The remaining assertions are easily showed, because eRe is Morita equivalent to R with respect to eR.

From Proposition 1, we may assume our ring is basic.

Lemma 3. *Let R be a semi-primary basic ring satisfying the condition* C_1 *and let e be an idempotent such that eR is a minimal faithful right ideal in R, and set* $f=1-e$. *Then* $fRe=0$.

Proof. As in the proof of Lemma 1, eRe is a QF ring and $eR=eR'$ is left eRe-projective. And moreover eR' is a minimal faithful right ideal in R'. Consider an exact sequence

$$0 \to R \to R' \to R'/R \to 0\,.$$

Then this sequence induces an exact sequence

$$0 \to \mathrm{Hom}_R(eR, R) \to \mathrm{Hom}_R(eR, R') \to \mathrm{Hom}_R(eR, R'/R) \to 0\,.$$

Since R'/R is a finitely generated injective right R-module and eR is a projective left tf/fe-module, $\mathrm{Hom}_R(eR, R'/R)$ is finitely generated right eRe-injective, which implies that $\mathrm{Hom}_R(eR, R'/R)$ is right eRe-projective, (see Curtis and Reiner [3]). Therefore the exact sequence above splits, and consequently we have a right eRe-isomorphism $Re\oplus M\simeq R'e$ with some right eRe-module M, since $\mathrm{Hom}_R(eR, R)\simeq Re$ and $\mathrm{Hom}_R(eR, R')\simeq R'e$. This isomorphism induces an isomorphism $Re\otimes_{eRe}eR\oplus M_{eRe}eR\simeq R'e\otimes_{eRe}eR$ as right R-modules. On the other hand, we have $R'e\simeq \mathrm{Hom}_{eRe}(eR, eRe)$ and moreover $\mathrm{Hom}_{eRe}(eR, eRe)\otimes_{eRe}eR\simeq \mathrm{Hom}_{eRe}(eR, eR)\simeq R'$, by Auslander and Goldman [1]. Therefore $R'e\otimes_{eRe}eR\simeq R'$

and so that $Re\otimes_{eRe}eR\cong ReR$. Thus, the fact that R' is right R-injective implies that ReR is also right R-injective.

Now suppose $fRe\neq 0$. Then there exists a primitive idempotent f_i such that $f_iRe\neq 0$ and f_iR is a direct summand fR. Since f_iReR is right R-injective and f_iR is an indecomposable right R-module, we have $f_iReR=f_iR$. Therefore f_iR is right R-injective and f_iR is isomorphic to a direct summand of $eR'=eR$. This is a contradiction, because R is a basic ring. Thus we have $fRe=0$.

Lemma 4. *Let R be a ring and let M and N be right R-modules such that N is a submodule of M. Let f be an idempotent in R such that fR is left fRf-projective. If M/N is right R-injective then Mf/Nf is also right fRf-injective.*

Proof. Since $\mathrm{Hom}_R(fR, M/N)\cong Mf/Nf$, Lemma 4 is immediate.

Corollary 2. *Let R, e and f be as in Lemma 3. Then $fR'f$ is a right fRf-injective hull of fRf and fRf has the injective dimension $\leqq 1$ as a right fRf-module. Moreover $fR'f$ is a QF ring.*

Proof. Since $fRe=0$, fR is left fRf-projective. It is obvious that fR' and fR'/fR are right R-injective. Therefore, by Lemma 4, $fR'f$ and $fR'f/fRf$ are right fRf-injective. Moreover, fRf is an essential submodule of a right fRf-module $fR'f$, because $fRe=0$ and fR is an essential submodule of a right R-module fR'. Thus, $fR'f$ is a right fRf-injective hull of fRf. Next, setting $S=eR'+fR'f$, clearly S is a ring. Since R is right QF-3 and $R\subset S\subset R'$, S is also right QF-3 and $R'=S'$. Hence, by Lemma 1, R' is right S-injective, and consequently $fR'f$ is a QF ring, by Lemma 4.

Lemma 5. *Let S be a QF ring such that $fSe=0$, where e and f are idempotents in S and $e+f=1$. Then, we have $eSf=0$.*

Proof. For a right R-module M_R, we denote the socle of M_R by $\mathrm{Soc}(M_R)$. Suppose that $eSf\neq 0$. Then, there exist idempotents e_i and f_j such that $\mathrm{Soc}(e_iS_S)f_j\neq 0$, and $e_i+e'=e$ and $f_j+f'=f$ are sums of orthogonal idempotents, because eSf is a right ideal of S. By our assumption, S is a QF ring, which implies that $\mathrm{Soc}(e_iS)$ is a simple right S-module. Hence $\mathrm{Soc}(e_iS)_S\cong(S/f_jN)_S$, where N is the Jacobson radical of S.

On the other hand, by Lemma 4, fSf is also a QF ring, hence there exists an idempotent f_k in fSf such that $f_jSf/f_jNf\cong\mathrm{Soc}(f_kSf)_{fSf}$ as right fSf-modules. But we have $fSe=0$ and this shows that $f_jS/f_jN\cong\mathrm{Soc}(f_kS_S)$ as right S-modules, and consequently $\mathrm{Soc}(e_iS_S)\cong\mathrm{Soc}(f_kS_S)$. Thus we have $e_iS_S\cong f_kS_S$, since these are injective hulls of $\mathrm{Soc}(e_iS_S)$ and $\mathrm{Soc}(f_kS_S)$, respectively. This contradicts the assumption $fSe=0$.

Lemma 6. *Let R, e and f be as in Lemma 3 and moreover let R be an* *indecomposable ring. Then Rf is a faithful left R-module, therefore fR' is a faithful right R'-module.*

Proof. By Corollary 1, R is a left QF-3 ring. Let Rg be a minimal faithful left ideal of R where g is an idempotent. Then $Rg=R'g$ and $R'g$ is a minimal faithful left ideal of R', hence $gR'\cong eR'$. Since $fRe=0$ and $f\neq 0$, ${}_RRg$ is not isomorphic to any submodule of ${}_RRe$. If ${}_RRg$ is isomorphic to a submodule of ${}_RRf$, then Rf is clearly a faithful left R-module.

Now suppose that ${}_RRg$ is not isomorphic to any submodule of ${}_RRf$. Then, we may assume that $g=e''+f'$ and $e=e''+e'$ are sums of (non-zero) idempotents e'', f' and e'', e', respectively, where $e', e''\in eRe$ and $f'\in fRf$. Since eR' is a direct sum of non-isomorphic indecomposable right ideals of R', we have $f'R'\cong e'R'$. On the other hand, $fR'e''=fRe''=0$ and in particular $f'R'e''=0$, because $R'e''$ is a direct summand of a left R'-module Rg. Hence, we have $e'R'e''=0$ and consequently $(1-e'')R'e''=0$. Therefore, by Lemma 5, we have $e''R'(1-e'')=0$. But e'' is in R. It follows that R is decomposable as a ring, which is a contradiction. Thus, Rg is isomorphic to a submodule of Rf and Rf is a faithful left R-module. Then, obviously, fR' is a faithful right R'-module.

Lemma 7. *Let R be a ring, and let e and f be idempotents in R such that $fRe=0$ and $e+f=1$. Let M and N be left R-modules such that N is a submodule of M and $eM=eN$. If M/N is left R-injective, then fM/fN is also left fRf-injective.*

Proof. For each $x\in R$ and $fm+fN\in fM/fN$, we set $x(fm+fN)=fxfm+fN$. Then, since $fRe=0$, fM/fN is regarded as a left R-module, which, restricting its operation to fRf, coincides with the original left fRf-module fM/fN. Next, define a map $M/N\to fM/fN$ by corresponding each $m+N\in M/N$ to $fm+fN\in fM/fN$. Then, this map is obviously a left R-isomorphism, because of our assumptions $fRe=0$ and $eM=eN$. Therefore, fM/fN is left R-injective.

Now we show that fM/fN is left fRf-injective. Let I be an arbitrary left ideal of fRf, and let φ be an arbitrary fRf-homomorphism of I into fM/fN. Then, using the map φ, we define a map ψ of a left R-ideal RI into a left R-module fM/fN as follows: for each $x\in RI$, set $(x)\psi=(fx)\varphi$. Then it is clear that ψ is an R-homomorphism. Since fM/fN is left R-injective, there exists an element $fm+fN$ in fM/fN such that for each $x\in RI$, $(fx)\varphi=x(fm+fN)$. In particular, for any element a in I, $(a)\varphi=a(fm+fN)$ which implies fM/fN is left fRf-injective.

Lemma 8. *Let R be a semi-primary basic indecomposable ring satisfying the condition C_2 and let e be an idempotent such that eR is a minimal faithful right ideal of R, and set $f=1-e$. Then fRf is a (right and left) QF-3 ring.*

Proof. Let Rg be a minimal faithful left ideal of R where g is an idempotent. Then, by Lemma 6, we may assume that g is in fRf. Thus, fRg is a faithful projective left ideal of fRf. On the other hand, we have $Rg={}_RR'g\simeq{}_RR'e$, hence $fRg\simeq fR'e$ as left fRf-modules. However, by the assumption C_2 and Lemma 7, $fR'e/fRe$ is left fRf-injective. Therefore, $fR'e$ is left fRf-injective, since $fRe=0$. Consequently fRf is a left QF-3 semi-primary ring with a minimal faithful left ideal fRg, because of $fRg_i\not\simeq fRg_j$ for $i\neq j$. Moreover, gRg is a right artinian ring and fRg is a finitely generated right gRg-module. Thus, by Lemma 2, fRf is a right QF-3 ring.

Corollary 3. *Let R, e and f be as in Lemma 8. Then fRf has the injective dimension $\leqq 1$ as a left fRf-module.*

Proof. By Corollary 2, Lemma 8 and Lemma 1, $fR'f$ is left fRf-injective. On the other hand, by Lemma 7, $fR'f/fRf$ is left fRf-injective. Therefore, fRf has the injective dimension $\leqq 1$ as a left fRf-module.

Corollary 4. *Let R, e and f be as in Lemma 8. Then fRf is an indecomposable ring.*

Proof. Setting $S=fRf$ and $T=fR'f$, then S is a right QF-3 ring and T is a QF ring by Lemma 8 and Corollary 2 (or Lemma 6), respectively. If $f'S$ is a minimal faithful right ideal of S, $f'S=f'T$ is a minimal faithful right ideal of T, where f' is an idempotent. However, T is Morita equivalent to R' with respect to fR' and so $f'R'$ is a minimal faithful right ideal of R'. Now suppose that S is a decomposable ring, i.e. there exist orthogonal central idempotents g and h in S such that $f=g+h$. Then we may assume that $f'=g'+h'$ for some (non-zero) idempotents $g'\in gRg$ and $h'\in hRh$. Because of $f'S=f'T=f'R'f$, we have $g'R'h=g'Sh=0$ and $h'R'g=h'Sg=0$. Since $f'R'\simeq eR'$, there exist orthogonal idempotents e' and e'' in eRe such that $e'R'\simeq g'R'$, $e''R'\simeq h'R'$ and $e=e'+e''$. Then, noting $fRe=0$, we have $(e'+g)R(e''+h)=0$ and $(e''+h)R(e'+g)=0$. This contradicts the indecomposability of R. Thus S is an indecomposable ring.

Theorem 1. *Let R be an indecomposable basic ring. If R is a right QF-3 semi-primary ring whose maximal right quotient ring is QF and R has the injective dimension 1 both as right and left R-modules, then R is isomorphic to the ring of triangular matrices of degree $n\geqq 2$ over a QF ring. Therefore, R is (right and left) artinian. The converse is also true.*

Proof. Let e be an idempotent such that eR is a minimal faithful right ideal in R. Set $e_1=e$, $f_1=1$ and $f_2=f_1-e_1=1-e_1$. Now assume that there exist idempotents $e_1, \cdots, e_{i-1}$ and $f_1, \cdots, f_i$, which satisfy the following conditions:

(1) $\begin{cases} \{e_j\} \text{ is mutually orthogonal.} \\ f_i = 1-(e_1+\cdots+e_{i-1}), \text{ in particular } f_1=1. \\ f_iR' \text{ is right } R\text{-faithful.} \\ f_iRf_i \text{ is an indecomposable basic semi-primary ring and an essential} \\ \text{submodule of } f_iR'f_i \text{ as a right } f_iRf_i\text{-module.} \end{cases}$

(2) f_iRf_i satisfies the condition C_2.

(3) $e_{i-1}Rf_{i-1}=e_{i-1}R'f_{i-1}$, $e_{i-1}R'\cong eR'$ and $f_iRe_{i-1}=0$.

Let e_i be an idempotent such that e_iRf_i is a minimal faithful right ideal in f_iRf_i and set $f_{i+1}=f_i-e_i$. Then, from (1) and (2), we can easily show that the idempotents $e_1, \cdots, e_i$ and $f_1, \cdots, f_{i+1}$ satisfy the conditions (1) and (3) on which we replace i by $i+1$. And moreover $f_{i+1}Rf_{i+1}$ is either a ring satisfying the condition C_2 or a QF ring.

Thus, for some $n\geqq 2$, there exist idempotents $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ such that f_nRf_n is a QF ring and these idempotents satisfy the conditions (1) and (3) for each i, $2\leqq i\leqq n+1$, where $f_{n+1}=0$. On the other hand eRe is a QF ring. It follows that R is isomorphic to the ring of triangular matrices of degree n over a QF ring eRe. Therefore, R is a (right and left) artinian.

Conversely, we assume that R is isomorphic to the ring of triangular matrices of degree $n\geqq 2$ over a QF ring. Then, by Zaks [8], R has the injective dimension 1 both as right and left R-modules. Moreover, it is obvious that R is a (right) QF-3 ring whose maximal right quotient ring of R is QF.

Now let R be a semi-primary ring, and let $R=R_1\oplus\cdots\oplus R_n$ a decomposition of R into indecomposable rings R_i with basic rings e_iRe_i. If R is a right QF-3 ring whose maximal right quotient ring is QF and R has the injective dimension $\leqq 1$ both as right and left R-modules, then so does R_i, or equivalently e_iRe_i, for each i (see Proposition 1). And moreover the converse is also true. Therefore, by Theorem 1, we have

Corollary 5. *Let R be a ring. Then R is a right QF-3 semi-primary ring whose maximal right quotient ring is QF and R has the injective dimension $\leqq 1$ both as right and left R-modules if and only if R is a direct sum of rings whose basic rings are isomorphic to the rings of triangular matrices over QF rings.*

OSAKA CITY UNIVERSITY

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