A CHARACTERIZATION OF THE TRIANGULAR MATRIX RINGS OVER QF RINGS

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(Received May 8, 1974)

In [4], Harada proved that a ring $R$ is a right QF-3 and semi-primary hereditary ring if and only if $R$ is a direct sum of rings whose basic rings are the rings of triangular matrices over division rings. We consider an analogous result to the above one for a right QF-3 semi-primary ring with some injective properties. By Zaks [8], the ring $R$ of triangular matrices of degree $n \geq 2$ over a QF ring has the injective dimension one both as right and left $R$-modules, and moreover it is easy to see that the ring $R$ is a QF-3 ring whose maximal right quotient ring is a QF ring. It is our purpose to show that for a basic indecomposable semi-primary ring, the converse is also true.

The author wishes to express his appreciation to Professor M. Harada who has carefully checked my original manuscript and has given the author valuable advices.

Throughout this paper we shall assume that every ring $R$ has an identity element 1, and every $R$-module is unitary. The notations $M_R$ and $_RM$ are used to underline the fact that $M$ is a right or a left $R$-module, respectively. For a ring $R$, a right (resp. left) $R$-module $M$ is called a minimal faithful module if $M$ is a faithful $R$-module and every faithful right (resp. left) $R$-module contains an isomorphic image of $M$ as a direct summand. A ring $R$ is called a right (resp. left) QF-3 ring if $R$ has a minimal faithful right (resp. left) module, and $R$ is called a QF-3 ring if $R$ is both a right and left QF-3 ring.

For a semi-primary ring $R$, the following conditions are equivalent, (see Jans [5]).

1. $R$ is a right QF-3 ring.
2. $R$ has a faithful projective injective right ideal.

Let $S$ be a ring which contains a ring $R$ as a subring. Then $S$ is called a right (resp. left) quotient ring of $R$ if $S$ is a rational extension of $R$ as a right (resp. left) $R$-module. By $R'$ we denote the maximal right quotient ring of $R$. If $R$ is a QF-3 ring, then the maximal right quotient ring of $R$ coincides with the maximal left quotient ring of $R$, (see Tachikawa [7]).

Lemma 1. Let $S$ be a QF ring. Let $R$ be a right QF-3 ring such that $R$
is a subring of \( S \) and an essential submodule of \( S \) as a right \( R \)-module, and let \( eR \) be a minimal faithful right ideal of \( R \), where \( e \) is an idempotent in \( R \). Then we have \( S = \text{Hom}_{eRe}(eR, eR) \), and \( S \) is injective both as right and left \( R \)-modules. Therefore, \( S \) is the maximal left quotient ring of \( R \).

Proof. Since \( eR \) is \( R \)-injective and \( eR \) is an essential submodule of an \( R \)-module \( eS \), we have \( eR = eS \), and hence \( eS \) is right \( S \)-faithful because of assumptions that \( eR \) is right \( R \)-faithful and \( R \) is an essential submodule of \( S \). Therefore, \( eRe = eSe \) is a QF ring and \( eRe = eS \) is a finitely generated projective left \( eRe \)-module, (see Curtis and Reiner [3]), which shows that \( \text{Hom}_{eRe}(eR, eR) \) is right \( R \)-injective since \( eR \) is right \( R \)-injective. Moreover we have \( S = \text{Hom}_{eSe}(eS, eS) = \text{Hom}_{eRe}(eR, eR) \) because \( S \) is QF. Therefore \( S \) is right \( R \)-injective. Moreover, since \( eR \) is left \( R \)-injective, \( S \) is also left \( R \)-injective, (see Cartan and Eilenberg [2], Chap. VI, Prop. 1.4). On the other hand, \( \text{Hom}_{eRe}(eR, eR) \) is a left quotient ring of \( R \), and consequently \( S \) is the maximal left quotient ring of \( R \). This completes the proof.

Let \( R \) be a right QF-3 ring whose maximal right quotient ring \( R' \) is QF. Then, by Lemma 1, \( R' \) is an injective hull of \( R \) both as right and left \( R \)-modules and \( R' \) is the maximal left quotient ring of \( R \).

For a ring \( R \) we consider the following conditions:

\((C_1) \) \( R \) is a right QF-3 ring whose maximal right quotient ring \( R' \) is QF, and the injective dimension of \( R \) as a right \( R \)-module is one.

\((C_2) \) \( R \) satisfies the condition \( C_1 \), and moreover the injective dimension of \( R \) as a left \( R \)-module is one.

We study a semi-primary ring \( R \) satisfying the condition \( C_2 \). Now let \( R \) be a semi-primary ring, and let \( 1 = \sum e_{ij} \) be a decomposition of \( 1 \) into a sum of orthogonal primitive idempotents \( e_{ij} \) such that \( e_{ij}R = e_{kk}R \) if and only if \( i = k \). If we write \( e = \sum e_{ii} \), then \( eRe \) is a basic ring.

The following lemma is obtained from Morita [6], the proof of Theorem 1.1.

**Lemma 2.** (Morita) Let \( R \) be a right QF-3 semi-primary ring with a minimal faithful right ideal \( eR \), where \( e \) is an idempotent in \( R \). If \( eRe \) is a left artinian ring and \( eR \) is a finitely generated left \( eRe \)-module, then \( R \) is left QF-3.

By Lemma 1 and Lemma 2, we have

**Corollary 1.** Let \( R \) be a right QF-3 semi-primary ring with minimal faithful right ideal \( eR \). Then the following conditions are equivalent.

1. There exists a QF ring \( S \) such that \( R \) is a subring of \( S \) and \( R \) is an essential submodule of \( S \).
2. The maximal right quotient ring \( R' \) is QF.
(3) The double centralizer \( \text{Hom}_{eRe}(eR, eR) \) of \( eRe \) is QF.
Moreover if \( R \) satisfies these equivalent conditions, we have \( S=R'=\text{Hom}_{eRe}(eR, eR) \) and \( R \) is (left) QF-3.

**Proposition 1.** Let \( R \) be a semi-primary ring, and let \( eRe \) be a basic ring of \( R \) where \( e \) is an idempotent in \( R \). Then \( R \) satisfies the condition \( C_1 \) (resp. \( C_2 \)) if and only if \( eRe \) satisfies the condition \( C_1 \) (resp. \( C_2 \)).

Proof. Suppose that \( R \) is a right QF-3 ring with a minimal faithful right ideal \( e'R \) of \( R \). Then we may assume that \( e' \) is an idempotent in \( eRe \) since \( eR \) is \( R \)-faithful. On the other hand, \( eRe \) is Morita equivalent to \( R \) with respect to \( eR \). Therefore, \( e'R \otimes_{eRe} e'R \) is a minimal faithful right ideal of \( eRe \), and so \( eRe \) is right QF-3. If \( R' \) is QF, then \( e'Re \) is QF and \( e'Re \) is a finitely generated faithful projective left \( eRe \)-module and so \( \text{Hom}_{eRe}(e'Re, e'Re) \) is QF. Thus, by Corollary 1, the maximal right quotient ring of \( eRe \) is QF. Using the fact that \( RRe \) is a generator, it is similarly proved that if \( eRe \) is a right QF-3 ring whose maximal right quotient ring is QF, then so is \( R \). The remaining assertions are easily showed, because \( eRe \) is Morita equivalent to \( R \) with respect to \( eR \).

From Proposition 1, we may assume our ring is basic.

**Lemma 3.** Let \( R \) be a semi-primary basic ring satisfying the condition \( C_1 \) and let \( e \) be an idempotent such that \( eR \) is a minimal faithful right ideal in \( R \), and set \( f=1-e \). Then \( fRe=0 \).

Proof. As in the proof of Lemma 1, \( eRe \) is a QF ring and \( eR=eR' \) is left \( eRe \)-projective. And moreover \( eR' \) is a minimal faithful right ideal in \( R' \). Consider an exact sequence

\[
0 \to R 
\to R' 
\to R'/R 
\to 0 .
\]

Then this sequence induces an exact sequence

\[
0 \to \text{Hom}_R(eR, R) \to \text{Hom}_R(eR, R') \to \text{Hom}_R(eR, R'/R) \to 0 .
\]

Since \( R'/R \) is a finitely generated injective right \( R \)-module and \( eR \) is a projective left \( R \)-module, \( \text{Hom}_R(eR, R'/R) \) is finitely generated right \( eRe \)-injective, which implies that \( \text{Hom}_R(eR, R'/R) \) is right \( eRe \)-projective, (see Curtis and Reiner [3]). Therefore the exact sequence above splits, and consequently we have a right \( eRe \)-isomorphism \( Re \oplus M \approx R'e \) with some right \( eRe \)-module \( M \), since \( \text{Hom}_R(eR, R) \approx Re \) and \( \text{Hom}_R(eR, R') \approx R'e \). This isomorphism induces an isomorphism \( Re \otimes_{eRe} eR \oplus M \otimes_{eRe} eR \approx R'e \otimes_{eRe} eR \) as right \( R \)-modules. On the other hand, we have \( R'e \approx \text{Hom}_{eRe}(eRe, eR) \) and moreover \( \text{Hom}_{eRe}(eRe, eR) \otimes_{eRe} eR \approx \text{Hom}_{eRe}(eR, eR) \approx R' \), by Auslander and Goldman [1]. Therefore \( R'e \otimes_{eRe} eR \approx R' \).
and so that $Re \otimes eRe = ReR$. Thus, the fact that $R'$ is right $R$-injective implies that $ReR$ is also right $R$-injective.

Now suppose $fRe \neq 0$. Then there exists a primitive idempotent $f_1$ such that $f_1Re \neq 0$ and $f_1R$ is a direct summand $fR$. Since $f_1ReR$ is right $R$-injective and $f_1R$ is an indecomposable right $R$-module, we have $f_1ReR = f_1R$. Therefore $f_1R$ is right $R$-injective and $f_1R$ is isomorphic to a direct summand of $eR' = eR$. This is a contradiction, because $R$ is a basic ring. Thus we have $fRe = 0$.

**Lemma 4.** Let $R$ be a ring and let $M$ and $N$ be right $R$-modules such that $N$ is a submodule of $M$. Let $f$ be an idempotent in $R$ such that $fR$ is left $R^f$-projective. If $M/N$ is right $R$-injective then $Mf/Nf$ also right $R^f$-injective. Moreover $fR'$ is a $QF$ ring.

**Proof.** Since $\text{Hom}_R(fRM/N) \approx Mf/Nf$, Lemma 4 is immediate.

**Corollary 2.** Let $R$, $e$ and $f$ be as in Lemma 3. Then $fR'$ is a right $fR^f$-injective hull of $fRf$ and $fR'$ has the injective dimension $\leq 1$ as a right $fRf$-module. Moreover $fR'$ is a $QF$ ring.

**Proof.** Since $fRe = 0$, $fR$ is left $fR^f$-projective. It is obvious that $fR'$ and $fR'^fR$ are right $R$-injective. Therefore, by Lemma 4, $fR'$ and $fR'^fR$ are right $fR^f$-injective. Moreover, $fR'$ is an essential submodule of a right $fR^f$-module $fR'^f$, because $fRe = 0$ and $fR$ is an essential submodule of a right $R$-module $fR'$. Thus, $fR'^f$ is a right $fR^f$-injective hull of $fR'$. Next, setting $S = eR' + fR'^f$, clearly $S$ is a ring. Since $R$ is right QF-3 and $R \subseteq S \subseteq R'$, $S$ is also right QF-3 and $R' = S'$. Hence, by Lemma 1, $R'$ is right $S$-injective, and consequently $fR'$ is a QF ring, by Lemma 4.

**Lemma 5.** Let $S$ be a QF ring such that $fSe = 0$, where $e$ and $f$ are idempotents in $S$ and $e + f = 1$. Then, we have $eSf = 0$.

**Proof.** For a right $R$-module $M_R$, we denote the socle of $M_R$ by $\text{Soc}(M_R)$. Suppose that $eSf \neq 0$. Then, there exist idempotents $e_i$ and $f_i$ such that $\text{Soc}(e_iS) \neq 0$, and $e_i + e' = e$ and $f_i + f' = f$ are sums of orthogonal idempotents, because $eSf$ is a right ideal of $S$. By our assumption, $S$ is a QF ring, which implies that $\text{Soc}(eS)$ is a simple right $S$-module. Hence $\text{Soc}(eS)_S \approx (S/ffN)_S$ where $N$ is the Jacobson radical of $S$.

On the other hand, by Lemma 4, $fSf$ is also a QF ring, hence there exists an idempotent $f_1$ in $fSf$ such that $f_1Sf_1Nf \approx \text{Soc}(f_1Sf_1N)$ as right $fSf$-modules. But we have $fSe = 0$ and this shows that $f_1Sf_1N \approx \text{Soc}(f_1Sf_1S)$ as right $S$-modules, and consequently $\text{Soc}(eS)_S \approx \text{Soc}(f_1Sf_1S)$. Thus we have $eSf \approx fSfS$ since these are injective hulls of $\text{Soc}(eS)_S$ and $\text{Soc}(fSfS)_S$. This contradicts the assumption $fSe = 0$.

**Lemma 6.** Let $R$, $e$ and $f$ be as in Lemma 3 and moreover let $R$ be an
indecomposable ring. Then $\mathfrak{R}$ is a faithful left $\mathcal{R}$-module, therefore $\mathfrak{fR}'$ is a faithful right $\mathcal{R}'$-module.

Proof. By Corollary 1, $\mathcal{R}$ is a left QF-3 ring. Let $\mathfrak{Rg}$ be a minimal faithful left ideal of $\mathcal{R}$ where $g$ is an idempotent. Then $\mathfrak{Rg}=\mathfrak{R'}g$ and $\mathfrak{R'g}$ is a minimal faithful left ideal of $\mathfrak{R'}$, hence $\mathfrak{gR'}\cong \mathfrak{eR'}$. Since $f\mathcal{R}e=0$ and $f\neq 0$, $\mathfrak{Rg}$ is not isomorphic to any submodule of $\mathfrak{R}e$. If $\mathfrak{Rg}$ is isomorphic to a submodule of $\mathfrak{Rf}$, then $\mathfrak{Rf}$ is clearly a faithful left $\mathfrak{R}$-module.

Now suppose that $\mathfrak{Rg}$ is not isomorphic to any submodule of $\mathfrak{Rf}$. Then, we may assume that $g=e''+ef$ and $e=e''+e'$ are sums of (non-zero) idempotents $e''$, $e'$, respectively, where $e', e''\in e\mathcal{R}e$ and $f\in f\mathcal{R}f$. Since $\mathfrak{eR'}$ is a direct sum of non-isomorphic indecomposable right ideals of $\mathfrak{R'}$, we have $f'\mathfrak{R'}\cong e'\mathfrak{R'}$. On the other hand, $f\mathcal{R'}e''=f\mathcal{R}e''=0$ and in particular $f'\mathcal{R'}e''=0$, because $\mathfrak{R'}e''$ is a direct summand of a left $\mathfrak{R}'$-module $Kg$. Hence, we have $e'\mathcal{R'}e''=0$ and consequently $(1-e'')\mathfrak{R'}e''=0$. Therefore, by Lemma 5, we have $e''\mathcal{R'}(1-e'')=0$. But $e''$ is in $\mathcal{R}$. It follows that $\mathfrak{R}$ is decomposable as a ring, which is a contradiction. Thus, $\mathfrak{Rg}$ is isomorphic to a submodule of $\mathfrak{Rf}$ and $\mathfrak{Rf}$ is a faithful left $\mathfrak{R}$-module. Then, obviously, $\mathfrak{fR'}$ is a faithful right $\mathfrak{R}'$-module.

Lemma 7. Let $\mathcal{R}$ be a ring, and let $e$ and $f$ be idempotents in $\mathcal{R}$ such that $f\mathcal{R}e=0$ and $e+f=1$. Let $M$ and $N$ be left $\mathcal{R}$-modules such that $N$ is a submodule of $M$ and $eM=eN$. If $M/N$ is left $\mathcal{R}$-injective, then $fM/fN$ is also left $f\mathcal{R}f$-injective.

Proof. For each $x\in \mathcal{R}$ and $f+nfN\in fM/fN$, we set $x(f+nfN)=fxf+nfN$. Then, since $f\mathcal{R}e=0$, $fM/fN$s regarded as a left $\mathcal{R}$-module, which, restricting its operation to $f\mathcal{R}f$, coincides with the original left $f\mathcal{R}f$-module $fM/fN$. Next, define a map $M/N\rightarrow fM/fN$ by corresponding each $m+N\in M/N$ to $fm+nfN\in fM/fN$. Then, this map is obviously a left $\mathcal{R}$-isomorphism, because of our assumptions $f\mathcal{R}e=0$ and $eM=eN$. Therefore, $fM/fN$ is left $\mathcal{R}$-injective.

Now we show that $fM/fN$s left $f\mathcal{R}f$-injective. Let $/ be an arbitrary left ideal of $f\mathcal{R}f$, and let $\varphi$ be an arbitrary $f\mathcal{R}f$-homomorphism of $/ into fM/fN$. Then, using the map $\varphi$, we define a map $\psi$ of a left $\mathcal{R}$-ideal $RI$ into a left $\mathcal{R}$-module $fM/fN$s follows: for each $x\in RI$, set $(x)\psi=(fx)\varphi$. Then it is clear that $\psi$ is an $\mathcal{R}$-homomorphism. Since $fM/fN$ is left $\mathcal{R}$-injective, there exists an element $fm+nfN$ in $fM/fN$ such that for each $x\in RI$, $(fx)\varphi=x(fm+nfN)$. In particular, for any element $a$ in $/$, $(a)\varphi=a(fm+nfN)$ which implies $fM/fN$s left $f\mathcal{R}f$-injective.

Lemma 8. Let $\mathcal{R}$ be a semi-primary basic indecomposable ring satisfying the condition $C_2$ and let $e$ be an idempotent such that $e\mathcal{R}$ is a minimal faithful right ideal of $\mathcal{R}$, and set $f=1-e$. Then $f\mathcal{R}$ is a (right and left) QF-3 ring.
Proof. Let $Rg$ be a minimal faithful left ideal of $R$ where $g$ is an idempotent. Then, by Lemma 6, we may assume that $g$ is in $fRf$. Thus, $fRg$ is a faithful projective left ideal of $fRf$. On the other hand, we have $Rg=Rg \cong RRe$, hence $fRg$ is a left $fR$-module. However, by the assumption $C_2$ and Lemma 7, $fR'e$ is a left $fRf$-injective. Therefore, $fR'e$ is left $fRf$-injective, since $fRe=0$. Consequently, $fRf$ is a left QF-3 semi-primary ring with a minimal faithful left ideal $fRg$, because of $fRg \cong fRg_i$ for $i \neq j$. Moreover, $gRg$ is a right artinian ring and $fRg$ is a finitely generated right $gRg$-module. Thus, by Lemma 2, $fRf$ is a right QF-3 ring.

**Corollary 3.** Let $R$, $e$ and $f$ be as in Lemma 8. Then $fRf$ has the injective dimension $\leq 1$ as a left $fRf$-module.

Proof. By Corollary 2, Lemma 8 and Lemma 1, $fRf$ is left $fRf$-injective. On the other hand, by Lemma 7, $fR'jfR'$ is left $fRf$-injective. Therefore, $fRf$ has the injective dimension $\leq 1$ as a left $fRf$-module.

**Corollary 4.** Let $R$, $e$ and $f$ be as in Lemma 8. Then $fRf$ is an indecomposable ring.

Proof. Setting $S=fRf$ and $T=fR'f$, then $S$ is a right QF-3 ring and $T$ is a QF ring by Lemma 8 and Corollary 2 (or Lemma 6), respectively. If $f'S$ is a minimal faithful right ideal of $S$, $f'S=f'T$ is a minimal faithful right ideal of $T$, where $f'$ is an idempotent. However, $T$ is Morita equivalent to $R'$ with respect to $fR'$ and so $f'R'$ is a minimal faithful right ideal of $R'$. Now suppose that $S$ is a decomposable ring, i.e. there exist orthogonal central idempotents $g$ and $h$ in $S$ such that $f=g+h$. Then we may assume that $f'=g'+h'$ for some (non-zero) idempotents $g' \in gRg$ and $h' \in hRh$. Because of $f'S=f'T=fR'$, we have $g'R'h=g'Sh=0$ and $h'R'g=h'Sg=0$. Since $f'R' \cong eR'$, there exist orthogonal idempotents $e'$ and $e''$ in $eRe$ such that $e'R' \cong g'R'$, $e''R' \cong h'R'$ and $e=e'+e''$. Then, noting $fRe=0$, we have $(e'+g)R(e''+h)=0$ and $(e''+h)R(e'+g)=0$. This contradicts the indecomposability of $R$. Thus $S$ is an indecomposable ring.

**Theorem 1.** Let $R$ be an indecomposable basic ring. If $R$ is a right QF-3 semi-primary ring whose maximal right quotient ring is QF and $R$ has the injective dimension 1 both as right and left $R$-modules, then $R$ is isomorphic to the ring of triangular matrices of degree $n \geq 2$ over a QF ring. Therefore, $R$ is (right and left) artinian. The converse is also true.

Proof. Let $e$ be an idempotent such that $eR$ is a minimal faithful right ideal in $R$. Set $e_1=e$, $f_1=1$ and $f_i=f_i-e_i=1-e_i$. Now assume that there exist idempotents $e_1, \ldots, e_{i-1}$ and $f_1, \ldots, f_i$, which satisfy the following conditions:
\begin{itemize}
\item \( \{e_j\} \) is mutually orthogonal.
\item \( f_i = 1 - (e_1 + \cdots + e_{i-1}) \), in particular \( f_i = 1 \).
\item \( f_i R' \) is right \( R \)-faithful.
\item \( f_i R' \) is an indecomposable basic semi-primary ring and an essential submodule of \( R' f_i \) as a right \( f_i R_i \)-module.
\end{itemize}

Let \( e_i \) be an idempotent such that \( e_i R_i \) is a minimal faithful right ideal in \( R_i R_i \) and set \( f_{i+1} = f_i - e_i \). Then, from (1) and (2), we can easily show that the idempotents \( e_1, \ldots, e_i \) and \( f_i \), \( f_i \) satisfy the conditions (1) and (3) on which we replace \( i \) by \( i + 1 \). And moreover \( f_{i+1} R_{i+1} \) is either a ring satisfying the condition \( C_2 \) or a QF ring.

Thus, for some \( n \geq 2 \), there exist idempotents \( e_1, \ldots, e_n \) and \( f_1, \ldots, f_n \) such that \( f_n R_n \) is a QF ring and these idempotents satisfy the conditions (1) and (3) for each \( i \), \( 2 \leq i \leq n+1 \), where \( f_{n+1} = 0 \). On the other hand \( e_i R_i \) is a QF ring. It follows that \( R \) is isomorphic to the ring of triangular matrices of degree \( n \) over a QF ring \( e_i R_i \). Therefore, \( R \) is a (right and left) artinian.

Conversely, we assume that \( R \) is isomorphic to the ring of triangular matrices of degree \( n \geq 2 \) over a QF ring. Then, by Zaks [8], \( R \) has the injective dimension 1 both as right and left \( R \)-modules. Moreover, it is obvious that \( R \) is a (right) QF-3 ring whose maximal right quotient ring of \( R \) is QF.

Now let \( R \) be a semi-primary ring, and let \( R = R_1 \oplus \cdots \oplus R_n \) a decomposition of \( R \) into indecomposable rings \( R_i \) with basic rings \( e_i R_i \). If \( R \) is a right QF-3 ring whose maximal right quotient ring is QF and \( R \) has the injective dimension \( \leq 1 \) both as right and left \( R \)-modules, then so does \( R_i \), or equivalently \( e_i R_i \), for each \( i \) (see Proposition 1). And moreover the converse is also true. Therefore, by Theorem 1, we have

**Corollary 5.** Let \( R \) be a ring. Then \( R \) is a right QF-3 semi-primary ring whose maximal right quotient ring is QF and \( R \) has the injective dimension \( \leq 1 \) both as right and left \( R \)-modules if and only if \( R \) is a direct sum of rings whose basic rings are isomorphic to the rings of triangular matrices over QF rings.

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References


