PRODUCTS OF TORSION THEORIES AND APPLICATIONS TO COALGEBRAS

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1. Introduction

Throughout this note $R$ is a ring with 1. We shall write $/ \leq R$ if $/$ is a right ideal of $R$. A non-empty set of right ideals $\Gamma$ of $R$ is called a Gabriel filter if it satisfies

T1. If $I \in \Gamma$ and $r \in R$, then $(I:r) \in \Gamma$.

T2. If $/$ is a right ideal and there exists $J \in \Gamma$ such that $(I:r) \in \Gamma$ for every $r \in J$, then $I \in \Gamma$.

It is well-known [4] that there is a one to one correspondence between Gabriel filters of $R$ and hereditary torsion theories for the category of right $R$-modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs $\{(R_i, \Gamma_i), \Gamma_i: \text{Gabriel filter of } R_i\}$, $\Gamma_0=\{D \leq \pi R_i \mid D \supseteq \sum D_i \in \Gamma_i\}$ is a Gabriel filter of the product ring $\pi R_i$, furthermore the ring of right quotient of $\pi R_i$ with respect to $\Gamma_0$ is isomorphic to the product of rings of right quotient of $R_i$ with respect to $\Gamma_i$. This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets $\Gamma_1=\{D \leq \pi R_i \mid D \supseteq \pi D_i \in \Gamma_i\}$ and $\Gamma_2=\{D \leq \pi R_i \mid D \supseteq \pi D_i \in \Gamma_i \text{ and almost all } D_i=R_i\}$ will be studied. $\Gamma_1$ does not always satisfy T2. A necessary and sufficient condition for $\Gamma_1$ to be a Gabriel filter is given. It follows that $\Gamma_1$ is a better notion of products of perfect torsion theories. However $\Gamma_2$ is a Gabriel filter of $\pi R_i$, and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra $(C, \Delta, \varepsilon)$ over a field $K$, there exists a natural algebra structure on its dual space $C^*=\text{Hom}_K(C, K)$ induced by the diagonal map $\Delta$ and every left comodule $(M, \phi_M)$ over $C$ can be defined as a right $C^*$-module by $mc^*=(c^* \otimes 1)\phi_M(m), m \in M, c^* \in C^*$. Moreover a right $C^*$-module $M$ is called a rational module if it is a left comodule $(M, \phi_M)$ over $C$ and its right $C^*$-module structure is derived in the way described above. With these observations we can embed the category of left $C$-comodules $\mathcal{C}M$ as a full subcategory, into the category of right $C^*$-modules $\mathcal{M}_C$. A subspace $/ \leq C^*$ is called cofinite.
closed if $I = V^\perp$ for some finite-dimensional subspace $V$ of $C$.

We assume the reader is familiar with torsion theories of modules and elementary coalgebra theories. The terminology and notation are those of Stenstrom [4] and Sweedler [5].

2. Some properties

In this section we derive some properties of $\Gamma_1$ and $\Gamma_2$. For convenience, we write a pair $(R_i, \Gamma_i)$ as $\Gamma_i$ is a Gabriel filter of $R_i$. The following are easily proved.

**Lemma 1.** If $I$ is a right ideal of $R$ and there exists $J \in \Gamma$ such that $(I: r) \in \Gamma$ for $r$ runs through a family of generators of $J$, then $I \in \Gamma$.

**Lemma 2.** $\Gamma_1, \Gamma_2$ satisfy T1.

**Proposition 1.** If $\{(R_i, \Gamma_i)_{i \in I}\}$ is a family of pairs and each $\Gamma_i$ has a cofinal family of $n$-generated right ideals (for a fixed integer $n$), then $\Gamma_i = \{D \leq \pi R_i | D \supset \pi D_i, D_i \in \Gamma_i, \text{ all } i \in I\}$ is a Gabriel filter of $\pi R_i$. Moreover $(\pi R_i)_{\Gamma_i} \cong (\pi R_i)_{\Gamma_1}$. 

**Proof.** It only has to check T2 for $\Gamma_1$. Let $T \leq \pi R_i$ and $D \in \Gamma_1$ such that $(T: r) \in \Gamma_1$ for every $r \in D$. We can assume $D = \pi D_i, D_i \in \Gamma_1$, and each $D_i$ has $n$ generators; $x^1_i, \ldots, x^n_i$. Construct $n$ elements of $\pi D_i$ as $x^1 = (x^1_i), \ldots, x^n = (x^n_i)$, then we have $(T: x^i) \in \Gamma_1$. Therefore for each $j = 1, \ldots, n$, there is $\pi D_i^{(j)}$ such that $x^j \pi D_i^{(j)} \subseteq T$. However, for fixed $i$ the finite sum $J_i = \sum_{j=1}^n x^j \pi D_i^{(j)} \in \Gamma_1$, by Lemma 1 and $\pi J_i = \sum_{j=1}^n \pi x^j \pi D_i^{(j)} = \sum_{j=1}^n \pi x^j \pi D_i^{(j)}$. This shows that $\pi J_i \subseteq T \in \Gamma_1$.

Next we find an isomorphism from $\pi(R_i)_{\Gamma_i}$ to $(\pi R_i)_{\Gamma_1}$. Let $([f_i]) \in (\pi R_i)_{\Gamma_1}$, where $f_i \in \text{Hom}_{R_i}(D_i, R_i/([t_i(D_i)])$ and $[f_i]$ is its equivalent class in $(R_i)_{\Gamma_1}$, and define a $\pi R_i$-homomorphism $\alpha$ from $\pi(D_i)$ to $\pi R_i/\pi R_i$ as $f_i((d_i)) = ([f_i(d_i)])$. Since $t(\pi R_i) = t(\pi R_i), \pi R_i/\pi R_i \cong \pi R_i/\pi R_i$, we have a well-defined map $\alpha$ from $\pi(R_i)_{\Gamma_1}$ to $(\pi R_i)_{\Gamma_1}$, as $\alpha([f_i])=1$, if $f_i$ and $f'_i$ agree on $D_i$ for each $i$, then the corresponding $f$ and $f'$ agree on $\pi D_i$. It is routine to check that $\alpha$ is a one to one ring-homomorphism. Let $f_1: \pi D_i \rightarrow \pi R_i/\pi R_i$: a $\pi R_i$-homomorphism, $D \in \Gamma_1$ and define $f_i = \pi_i f e_i$, where $e_i$ is the ith-inclusion, $\pi_i$ is the ith-projection. Then $\alpha([f_i]) = [f_i]$. Thus $\alpha$ is an isomorphism.

Note. (1) We agree that $n$ generators of right ideals are not necessary distinct.

(2) In proposition 1, $\Gamma_1$ also has a cofinal family of $n$-generated right ideals.
Proposition 2. If \{\{R_i, \Gamma_i\}, i \in I\} is a family of pairs, then \(\Gamma_2 = \{I \leq \pi R_i | I \supseteq \pi D_i, D_i \in \Gamma_i \text{ and almost all } D_i = R_i\}\) is a Gabriel filter of \(\pi R_i\).

Proof. Similarly it only has to check \(T_2\) for \(\Gamma_2\). Let \(I \leq \pi R_i\) and \(D \in \Gamma_2\) such that \((j,d) \in \Gamma_2\) for all \(d \in D\). We can assume \(D = \pi D_i, D_i \in \Gamma_i\) and except for \(D_i, k = I, \ldots, n\), all other \(D_i\) are equal to \(R_i\). Let \(e \in \pi D_i\) be an element with \(i_k\)-th component \(= 0\), other component \(= 1\). It follows that there is a right ideal of the form \(\pi J_i\) with \(J_i \in \Gamma_i\) and almost all \(J_i = R_i\) such that \(I \supseteq e \pi J_i\).

Also for each \(d \in D_i\), there exists a right ideal \(J^{(i)}_i \in \Gamma_i\) such that \(I \supseteq e_i(d_i J^{(i)}_i)\), where \(e_i\) is the \(i\)-th inclusion. Now take \(H_{i_k} = \sum d_i J^{(i)}_i\), the sum runs through all elements of \(D_i\). We have \(H_{i_k} \in \Gamma_i\) and

\[(*) I \supseteq e \pi J_i + e_i(H_{i_1}) + \cdots + e_i(H_{i_n}).\]

However the right side of \((*)\) is of the form \(\pi J_i\) with \(J_i \in \Gamma_i\) and almost all \(J_i = R_i\). Thus \(I \in \Gamma_2\).

3. Products of perfect torsion theories

For a fixed ring \(R\) with a perfect Gabriel filter \(\Gamma\), we will investigate the notion of their products.

The following two theorems (Chapt. 13, [4]) motivate our definition.

Theorem A. The following properties of a pair \((R, \Gamma)\) are equivalent:

1. \(\ker(M \to M \otimes_R R_i) = \psi(M)\) for all right \(R\)-module \(M\).
2. \(\psi(R(I)) R_i = R_i\) for every \(I \in \Gamma\).

Theorem B. If \(\phi: A \to B\) is a ring homomorphism. The following statements are equivalent:

1. \(\phi\) is an epimorphism and makes \(B\) into a flat left \(A\)-module.
2. The family \(\Gamma\) of right ideal \(I\) of \(A\) such that \(\phi(i) B = B\) is a Gabriel filter, and there exists a ring isomorphism \(\sigma: B \to A\) such that \(\sigma \phi = \psi\).
3. The following two conditions are satisfied:
   (3a) for every \(b \in B\), there exists a finite subset \(T_n = \{(s_1, b_1), \ldots, (s_n, b_n)\}\) of \(A \times B\) such that \(b \phi(s_i) \in \phi(A)\) and \(\sum_i \phi(s_i) b_i = 1\).
   (3b) if \(\phi(a) = 0\), then there exists a finite subset \(S_n = \{(s_1, b_1), \ldots, (s_n, b_n)\}\) such that \(a \phi(s_i) \in 0\) and \(\sum_i \phi(s_i) b_i = 1\).

Note. A Gabriel filter \(\Gamma\) of a ring \(R\) is called perfect if it has properties listed in Theorem A. If \(\Gamma\) is perfect, then

1. \(\Gamma\) has a cofinal family of finitely generated right ideals.
2. \(\Gamma = \{I \leq R | \psi(I) R_i = R_i\}\).

Definition. If \(\Gamma\) is a perfect Gabriel filter of \(R\), for each \(b \in R_i\), define \(\text{Ind } b = \inf \left| T_n \right|, T_n\) runs through all subsets of \(R \times R_i\) that satisfy Theorem B, 3(a).
If $\psi_r(r)=0$, define $\text{Ind } r = \text{Inf } \{ |S_n|, S_n \text{ runs through all subsets of } R \times R \}$ that satisfy Theorem B, (3b). Then let

$$\text{Ind } R_{\Gamma} = \text{Max } \{ \sup_{r \in R_{\Gamma}} (\text{Ind } b), \sup_{r \in R_{\Gamma}} (\text{Ind } r) \}.$$ 

**Theorem 3.** The following statements are equivalent for a perfect Gabriel filter $\Gamma$ of $R$.

1. $\Gamma$ has a cofinal family of $n$-generated right ideals.
2. $\Gamma = \{ I \subseteq \pi R | I \supseteq \pi D_i, D_i \in \Gamma \}$ is a Gabriel filter of $\pi R$, for any direct product of $R$.
3. $\text{Ind } R_{\Gamma}$ is finite.

Proof. (1) $\Rightarrow$ (2). By Proposition 1.

(2) $\Rightarrow$ (3). If $\Gamma$ is a Gabriel filter, then it is perfect. Suppose there is a sequence $\{ b_1, b_2, \cdots, b_n \mid b_n \in R_{\Gamma} \}$ such that $\text{Ind } b_n > \text{Ind } b_{n-1}$. Consider the countable product $\pi R$ of $R$ and the element $x=(b_1, b_2, \cdots)$. Then we have $s_1, \cdots, s_t \in \pi R \times \pi R$, such that $\psi_\Gamma(s_i) \in \pi R$ and $\sum \psi_\Gamma(s_i) = 1 \cdot x$. Projecting to each component, $\text{Ind } b_n \leq t$ for each $n$. This is a contradiction. Similarly we can prove that $\sup_{r \in R_{\Gamma}} \{ \text{Ind } r \}$ is finite.

(3) $\Rightarrow$ (1). If $\text{Ind } R_{\Gamma}$ is finite, then any direct product $\pi R_{\Gamma}$ of $R_{\Gamma}$ satisfies Theorem B, (3). So the product $\pi R_{\Gamma}$ is a ring of right quotient of $\pi R$ with respect to this perfect Gabriel filter $\Gamma = \{ D \subseteq \pi R | d_{\Gamma} = \pi R_{\Gamma} \}$. Applying the well-ordering theorem to the family $\Gamma$, the right ideal $\pi D_i$ is in $\Gamma$. So $\pi D_i$ contains a $n$-generated right ideal $J \in \Gamma$. For each $i$, the $i$-th projection of $J_i$ is contained in $D_i$. Since $\psi_\Gamma(J_i) = \pi R_{\Gamma} \subseteq \Gamma$, this shows that $\Gamma$ has a cofinal family of $n$-generated right ideals.

EXAMPLE. Let $Z$ be the ring of integers, $\Gamma = \{ \text{all non-zero ideals of } Z \}$, take a countable product $\pi Z$ of $Z$, then $\Gamma = \{ I \subseteq \pi Z | I \supseteq \sum D_i, D_i \in \Gamma \}$ is not a perfect Gabriel filter. However $\Gamma = \{ / \subseteq \pi Z \mid I \supseteq I \} \Gamma \} = \text{finite}$.

**4. Applications to coalgebras**

In this section we consider a subfunctor of the identity for the category of right $C^*$-module $\mathcal{M}_{C^*}$ and study when this subfunctor defines a hereditary torsion theory. The main effect is to classify some types of cocommutative coalgebras. If $C$ is a coalgebra, for a right $C^*$-module $M$ there is a unique maximal rational submodule $M_{\text{rat}}$ of $M$. Actually $M_{\text{rat}} = \{ m \in M | \text{Ann}(m) \text{ is cofinite} \}$ closed in $C^*$. There are some properties of $\mathcal{M}_{C^*}$.

1. If $(M, \phi_M)$ is a left $C$-comodule, $M$ can be considered as a right $C^*$-module by $mc^*=(c^* \otimes 1)\phi_M(m)$. Then $(M_{C^*})_{\text{rat}} = M$. 

(2) Direct sum of rational $C^*$-modules is rational.
(3) $(C^{**})^{rat} = C$.
(4) For a submodule $N$ of a $C^*$-module $M$, $N^{rat} = N \cap M^{rat}$.
(5) Homomorphic image of a rational module is rational.

So we have a subfunctor $rat$ of the identity on $\mathcal{M}_{C^*}$ just assigned each $C^*$-module $M$ the maximal rational submodule $M^{rat}$ and each homomorphism $f: M \to N$ the restriction map $\mathcal{f}: M^{rat} \to N^{rat}$.

**DEFINITION.** A coalgebra $C$ is said to have torsion rat functor if $rat$ is a left exact radical of $\Pi_C$.

Note. If $C$ has the torsion rat functor, then
(1) the category of left $C$-comodules or equivalently of rational right $C^*$-modules is the torsion class.
(2) the corresponding Gabriel filter is
$$\Gamma = \{I \subseteq C^* \mid I \text{ is cofinite closed in } C^*\}.$$  

**EXAMPLE.** Let $V$ be an infinite dimensional vector space and $C = C(V)$ denote the connected coalgebra $K \otimes V$ with
$$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \forall v \in V$$
$$\varepsilon(1) = 1$$
$$\varepsilon(v) = 0 \quad \forall v \in V.$$  

Take a basis $\{v_i \mid i \in I\}$ of $V$ and let $\{v_i^* \mid i \in I\}$ be its dual independent set in $V^*$. Extending this set to a basis $\{v_i^* \mid i \in I\}$ of $V^*$. We construct a linear map $f$ from $C^*$ to $K$ as
$$f(v_i^*) = 1 \text{ if } i \in I$$
$$f(1) = 1,$$
this element $f \in C = C^{**_{rat}}$, however $fv^* = f(v^*)1 \in C$ for any $v^* \in V^*$. So $(C^{**}(C^{**_{rat}})^{rat}) = 0$.

The following proposition is proved in [2, p. 521].

**Proposition.** Suppose $C$ is a coalgebra and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of right $C^*$-modules with $M'$ and $M''$ rational. If the annihilator of each $m'' \in M''$ is a finitely generated right ideal, then $M$ is rational.

Note. From the proposition, we see that if $C^*$ is a right Noetherian, then $C$ has the torsion rat functor. In particular the universal cocommutative pointed irreducible coalgebra $B(V)$ over a finite dimensional vector space $V$ has the torsion rat functor.
Proposition 4. If D is a subcoalgebra of C, then D has the torsion rat functor provided C has.

Proof. There exists a ring epimorphism \( \pi: C^* \to D^* \). Every \( D^* \)-module \( M \) is a \( C^* \)-module by \( mc^* = m \pi(c^*) \). Thus \( (M_D)^{\text{rat}} = (M_C)^{\text{rat}} \) and \( (M_D^*/M_D^{\text{rat}})^{\text{rat}} = (M_C^*/M_C^{\text{rat}})^{\text{rat}} = 0 \).

Corollary 5. For any pointed irreducible cocommutative coalgebra C, it has the torsion rat functor if and only if its space of primitive elements \( P(C) \) is finite-dimensional.

Proof. If \( P(C) \) is infinite-dimensional, the connected sub-coalgebra \( D = K \oplus P(C) \) of C does not have the torsion rat functor. Conversely if \( P(C) \) is finite-dimensional there is an inclusion map from C to the universal cocommutative pointed irreducible coalgebra over \( P(C) \). So by Proposition 4 C has the torsion rat functor.

Theorem 6. (*) If \( \{ C_i | i \in I \} \) is a family of coalgebras and \( C_i \) has the torsion rat functor for each \( i \in I \). Then the direct sum \( C = \bigoplus_{i} C_i \) has the torsion rat functor.

Proof. Let \( \Gamma_i = \{ D_i \subseteq C | D_i \text{ is cofinite closed in } C^*_i \} \), and \( \Gamma = \{ D \subseteq C^* = \pi C^* | D \text{ is cofinite closed in } C^* \} \). By proposition 2 \( \Gamma_2 = \{ I \subseteq \pi C^* | I \supseteq \pi D_i, D_i \subseteq \Gamma_i \text{ and almost all } D_i = C_i^* \} \) is a Gabriel filter of \( C^* = \pi C^* \). Hence it is sufficient to show that \( \Gamma = \Gamma_2 \). If \( D \in \Gamma \), then \( D = V^\perp \) for a finite dimensional subspace \( V \subseteq C \), and so \( V \subseteq C_1 \oplus \cdots \oplus C_n \) for some \( n \).

For each \( i \), let \( V_i \) be the projection of \( V \) to \( C_i \). Then \( V_i \) is a finite-dimensional subspace, almost all \( V_i = 0 \) and \( V \subseteq \pi V_i \). Hence we have \( \pi V_i \subseteq V^\perp = D \subseteq \Gamma_2 \). Conversely suppose \( I \in \Gamma_2 \), since \( I \) contains a cofinite closed subspace \( \pi D_i \), so \( I \) is also cofinite closed. Thus \( \Gamma = \Gamma_2 \).

Corollary 7. Over an algebraically closed field, a cocommutative coalgebra has the torsion rat functor if and only if each space of primitives of its irreducible components is finite-dimensional.

Proof. Over an algebraically closed field, a cocommutative coalgebra is a direct sum of its pointed irreducible components. So by Theorem 6 and Corollary 5, we have this result.

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(*) This theorem also appeared in [1], here we use the notion of products of torsion theories to give a different proof.
References
