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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper conditions are obtained under which all solutions of certain real non-autonomous nonlinear differential equations tend to zero as $t\to\infty$.

Theorem 1 is concerned with the system of differential equations;

(1.1) $$\dot{x}=A(t)x+f(t,x)$$

where x, f are n-dimensional vectors, $A(t)$ is a bounded continuously differentiable $n\times n$ matrix for $t\geqq 0$, and $f(t, x)$ is continuous in (t, x) for $t\geqq 0$, $||x||<\infty$, here $||\ ||$ denotes the Euclidean norm.

Theorem 2 is concerned with the differential equation of the third order;

(1.2) $$\dddot{x}+a(t)f(x,\dot{x},\ddot{x})\ddot{x}+b(t)g(x,\dot{x})+c(t)h(x)=p(t,x,\dot{x},\ddot{x})$$

where $a(t)$, $b(t)$, $c(t)$ are positive continuously differentiable and f, g, h, p are continuous real-valued functions depending only on the arguments shown, and the dots indicate the differentiation with respect to t.

The asymptotic property of solutions of third order differential equations has received a considerable amount of attention during the past two decades, particularly when (1.2) is autonomous. Many of these results are summarized in [11].

A few authors have studied non-autonomous third order differential equations. K. E. Swick [13] considered the following equations

(1.3) $$\dddot{x}+p(t)\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0,$$

(1.4) $$\dddot{x}+f(x,\dot{x},t)\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0,$$

with the assumption that $q(t)$, $r(t)$ are positive, bounded and monotone decreasing.

In [6], the author studied the asymptotic behavior of the solutions of the equation

(1.5) $\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})\dot{x}+c(t)h(x)=e(t)$

under the assumptions that $|a'(t)|$, $|b'(t)|$, $|c'(t)|$ and $|e(t)|$ are integrable and suitable conditions on f, g, h. Here we assume the condition

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} \{|a'(s)|+b'_+(s)+|c'(s)|\}\, ds<\gamma$$

where γ is a sufficiently small positive constant.

Conditions on $p(t, x, \dot{x}, \ddot{x})$ are also relaxed, for example $p(t, x, y, z)$ may be unbounded for $x^2+y^2+z^2$.

Recently H. O. Tejumola [15] established conditions under which all solutions of equations of the form

(1.6) $\dddot{x}+f(x,\dot{x},\ddot{x})\ddot{x}+g(x,\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x})$

tend to zero as $t\to\infty$. Theorem 2 develops Tejumola's result [15; Theorem 1] to the non-autonomous equations of the form (1.2).

The main tools used in this work are Theorem A and Liapunov functions. Theorem A would be especially convenient to study the non-autonomous differential equations.

2. Auxiliary theorem

Consider a system of differential equations

(2.1) $\dot{x}=F(t,x)$

where x and F are n-dimensional vectors.

Theorem A. *Suppose that $F(t,x)$ of (2.1) is continuous in $I\times R^n$ ($I=[0,\infty)$) and that there exists a Liapunov function $V(t,x)$, continuously differentiable in $I\times R^n$, satisfying the following conditions;*

(i) $a(\|x\|)\leqq V(t,x)\leqq b(\|x\|)$, *where $a(r)\in CIP$ (the family of continuous and increasing positive definite functions), $a(r)\to\infty$ as $r\to\infty$ and $b(r)\in CIP$,*

(ii) $$\dot{V}_{(2.1)}(t,x)=\limsup_{h\to 0+}\frac{1}{h}\{V(t+h,x+hF(t,x))-V(t,x)\}$$
$$\leqq -cV(t,x)+\lambda_1(t)V(t,x)+\lambda_2(t)(1+V(t,x)),$$

where $c>0$ is a constant and $\lambda_i(t)\geqq 0$ ($i=1,2$) are continuous functions satisfying

(2.2) $$\limsup_{(t,v)\to(\infty,\infty)}\frac{1}{v}\int_t^{t+v}\lambda_1(s)\,ds<c,$$

(2.3) $$\int_t^{t+1} \lambda_2(s)\,ds \to 0 \quad \text{as} \quad t \to \infty\,.$$

Then, any solution $x(t)$ of (2.1) *is uniform-bounded and satisfies* $x(t)\to 0$ *as* $t\to\infty$.

The following is an immediate consequence of Theorem A.

Corollary. *Under the assumptions in Theorem A, if*

(ii)′ $\dot{V}_{(2.1)}(t, x) \leqq L(t)V(t, x) + \lambda_2(t)(1 + V(t, x))$,
where $L(t)$ is a continuous function satisfying

(2.2)′ $$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} L(\tau)\,d\tau < 0\,,$$

then any solution $x(t)$ of (2.1) *is uniform bounded and satisfies* $x(t)\to 0$ *as* $t\to\infty$.

REMARK. The condition (2.3) can be replaced by

(2.3)′ $$\int_t^{t+\tau_0} \lambda_2(\tau)\,d\tau \to 0 \quad \text{as} \quad t \to \infty$$
where $\tau_0 > 0$ is an arbitrary constant.

3. Proof of Theorem A

Proof of Theorem A. Let $\varepsilon > 0$ be chosen such that $\frac{c}{2} > \varepsilon$ and

(3.1) $$\limsup_{(t,s-t)\to(\infty,\infty)} \frac{1}{s-t}\int_t^s \lambda_1(\tau)\,d\tau \leqq c - 3\varepsilon\,.$$

Let $T_0 > 0$ such that

$$\frac{1}{s-t}\int_t^s \lambda_1(\tau)\,d\tau \leqq \limsup_{(t,s-t)\to(\infty,\infty)} \frac{1}{s-t}\int_t^s \lambda_1(\tau)\,d\tau + \varepsilon \qquad \text{for} \quad t \geqq T_0$$
and $s \geqq T_0 + t$. T_0 does not depend on t and s.

Using (3.1) and above inequality, we have

(3.2) $$\frac{1}{s-t}\int_t^s \lambda_1(\tau)\,d\tau \leqq c - 2\varepsilon \quad \text{for} \quad t \geqq T_0 \quad \text{and} \quad s \geqq T_0 + t\,.$$

Let $K > 0$ be a constant satisfying $\exp\{cT_0 + \int_0^{T_0} \lambda_1(\tau)\,d\tau\} \leqq K$. Then it is easy to show that for all $s \geqq t \geqq 0$, we have

(3.3) $$e^{-c(s-t)+\int_t^s \lambda_1(\tau)d\tau} \leqq K e^{-2\varepsilon(s-t)}\,.$$

Now we consider the function $U(t,x)$ defined by

$$(3.4)\qquad U(t,x)=V(t,x)e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}\,e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds,$$

and show that

$$(3.5)\qquad e^{-c}\ a(\|x\|)\leqq U(t,x)\leqq\frac{K}{\varepsilon}b(\|x\|)\qquad\text{for all}\quad t\geqq 0\quad\text{and}\quad x\in R^n.$$

To verify the right-hand side inequality in (3.5), we use the inequality (3.3). Then we obtain

$$(3.6)\qquad e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds\leqq e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}Ke^{-2\varepsilon(s-t)}ds$$
$$=\frac{K}{\varepsilon}.$$

A short calculation shows the left-hand side inequality in (3.5), i.e.

$$(3.7)\qquad e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds > e^{-\varepsilon t}\int_t^{t+1}e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds$$
$$\geqq e^{-\varepsilon t}\int_t^{t+1}e^{\varepsilon t}e^{-c(t+1-t)+\int_t^t\lambda_1(\tau)d\tau}ds$$
$$=e^{-c}.$$

Therefore we have

$$(3.8)\qquad e^{-c}\ V(t,x)\leqq U(t,x)\leqq\frac{K}{\varepsilon}V(t,x)$$

and using the hypothesis (i) of Theorem A, we obtain (3.5).

From (3.4) it follows that

$$\dot U_{(2.1)}(t,x)$$
$$=\dot V_{(2.1)}(t,x)e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds$$
$$-\ \varepsilon V(t,x)e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds$$
$$+\{c-\lambda_1(t)\}\ V(t,x)e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds-V(t,x)$$
$$\leqq\ \{-cV(t,x)+\lambda_1(t)V(t,x)+\lambda_2(t)(1+V(t,x))\}\ e^{-\varepsilon t}\int_t^\infty e^{\varepsilon s}e^{-c(s-t)+\int_t^s\lambda_1(\tau)d\tau}ds$$
$$-\varepsilon U(t,x)+\{c-\lambda_1(t)\}U(t,x)-V(t,x).$$

Using (3.6), we obtain

(3.9) $\qquad \dot{U}_{(2.1)}(t,x) \leqq -\varepsilon U(t,x)+\frac{K}{\varepsilon}\lambda_2(t)(1+V(t, x))\,.$

From (3.8) and (3.9), we have

(3.10) $\qquad \dot{U}_{(2.1)}(t,x) \leqq \{-\varepsilon+\frac{K}{\varepsilon}e^{c}\lambda_2(t)\}\,U(t, x)+\frac{K}{\varepsilon}\lambda_2(t)\,.$

Set

$$W(t)= U(t, x(t))$$

where $x(t)$, $x(t_0)=x_0$, is any solution of (2.1). Then the inequality (3.10) implies

$$\frac{d}{dt}W(t) \leqq \{-\varepsilon+\frac{K}{\varepsilon}e^{c}\lambda_2(t)\}\,W(t)+\frac{K}{\varepsilon}\lambda_2(t)\,.$$

This immediately gives

$$W(t)\leqq \quad . \ . \quad {}^{t}\, g(\tau)d\tau+\int_{t_0}^{t} e^{\int_s^t g(\tau)d\tau}h(s)\,ds$$

where $g(t)=-\varepsilon+\frac{K}{\varepsilon}e^{c}\lambda_2(t)$ and $h(t)=\frac{K}{\varepsilon}\lambda_2(t)$

Using the hypothesis (2.3), we can choose a constant $\tilde{T}>0$ so that

$$\frac{Ke^{c}}{\varepsilon}\cdot\frac{1}{t-t_0}\int_{t_0}^{t}\lambda_2(\tau)\,d\tau\leq\frac{\varepsilon}{2} \qquad \text{for} \quad t\geqq 1+t_0 \quad \text{and} \quad t_0\geqq \tilde{T}\,.$$

Let $\tilde{K}>0$ be a constant satisfying $\exp\left\{\frac{\varepsilon}{2}(1+\tilde{T})+\frac{K e^{c}}{\varepsilon}\int_0^{\tilde{T}}\lambda_2(\tau)\,d\tau\right\}\leqq\tilde{K}$.

Then for all $t\geqq t_0\geqq 0$ we have

(3.11) $\qquad U(t,x(t)) \leqq \frac{K\tilde{K}}{\varepsilon}\{b(||x_0||)\,e^{-\varepsilon(t-t_0)/2}+\int_{t_0}^{t} e^{-\varepsilon(t-s)/2}\lambda_2(s)\,ds\}\,.$

Using the left-hand side of (3.5), we find that all the solutions of (2.1) are uniform-bounded.

Furthermore the condition (2.3) implies that

$$U(t,x(t))\to 0 \quad \text{as} \quad t\to\infty\,.$$

Therefore by the inequality (3.5) we have

$$x(t) \to 0 \text{ as } t \to \infty\,.$$

Q.E.D.

Proof of Corollary. Let $c>0$ be an arbitrary positive constant. By assumption (ii)′, we have

$$\dot{V}_{(2.2)}(t, x) \leqq -cV(t,x)+\{c+L(t)\}V(t, x)+\lambda_2(t)(1+V(t, x)) .$$

It now follows from (2.2)′ that

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{V}\int_t^{t+v} \{c+L(\tau)\}dr < c ,$$

which establishes the assumption of Theorem A, and thus the proof is completed.

Q.E.D.

4. Theorems

Let $A(t)$ satisfy the hypothesis (i) of the following Theorem 1 and $P(t)$ be a solution of the matrix equation

(4.1) $$A^T(t)P(t)+P(t)A(t) = -I .$$

Notice that $P(t)$ is bounded for bounded $A(t)$.

The following propositions are due to J. R. Dickerson [2].

Proposition A. *$x^T P(t)x \geqq C||x||^2$, where C is a positive constant.*

Proposition B. *$|x^T\dot{P}(t)x| \leqq 2||\dot{A}(t)||\ ||P(t)||\ x^T P(t)x$, where $\dot{P}(t)$ and $\dot{A}(t)$ denote the time derivative of matrices $P(t)$ and $A(t)$ respectively.*

Theorem 1. *Suppose that the following conditions are satisfied;*

(i) *there exists a positive constant τ_0 such that the real parts of all the eigenvalues of $A(t) \leqq -\tau_0 < 0$ for all $t \geqq 0$,*

(ii) $$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} ||\dot{A}(s)||\, ds < \frac{1}{2P_1^2}$$

where $P_1 = \limsup_{t\to\infty} ||P(t)||$,

(iii) $||f(t, x)|| \leqq \gamma(t)(1+||x||)$ *where $\gamma(t)$ is a non-negative continuous function on* $[0, \infty)$,

(iv) $$\int_t^{t+1} \gamma(s)\,ds \to 0 \quad as \quad t\to\infty \qquad (i= 1, 2) .$$

Then, all solutions $x(t)$ of (1.1) *are uniform-bounded and satisfy $x(t)\to 0$ as $t\to\infty$.*

REMARK. It may be shown by examples [16] that the smallness of $||\dot{A}(t)||$ is essential, even if the condition (i) is satisfied.

Next, we consider the equation (1.2) and assume that $g(x, y)$, $g_x(x,y)$, $f(x, y, z)$, $f_x(xy, z)$ and $f_z(xy, z)$ are continuous for all $(x, y, z)\in R^3$ and $h(x)$

is continuously differentiable for all $x \in R^1$.

Theorem *2. Suppose that $a(t)$, $b(t)$, $c(t)$ are continuously differentiable on $[0, \infty)$ and $g(x, 0)=h(0)=0$ and the following conditions are satisfied;*

(i) $A \geqq a(t) \geqq a_0>0,\quad B \geqq b(t) \geqq b_0>0,\quad C \geqq c(t) \geqq c_0>0$ *for* $t \in I=[0, \infty)$,

(ii) $h(x)/x \geqq \delta>0 \qquad (x \neq 0)$,

(iii) $\bar{f} \geqq f(x, y, z) \geqq f_0>0$ *for all* (x, y, z) *and* $\bar{g} \geqq \dfrac{g(x, y)}{y} \geqq g_0>0$ *for all* $y \neq 0$ *and* x,

(iv) $yf_x(x, y, 0) \leqq 0.\quad yf_z(x, y, z) \geqq 0,\quad g_x(x, y) \leqq 0$ for all $(x, y, z) \in R^3$,

(v) $\dfrac{a_0 b_0 f_0 g_0}{C}>h_1 \geqq h'(x)$,

(vi) $\delta>\dfrac{\mu_2}{4c_0}\left\{A(\bar{f}-f_0)+\dfrac{B}{\mu_1}(\bar{g}-g_0)\right\}$

where μ_1 and μ_2 are arbitrarily fixed constants satisfying

$$\frac{Ch_1}{b_0 g_0}<\mu_1<a_0 f_0,\quad 0<\mu_2<\frac{a_0 b_0 f_0 g_0-Ch_1}{Af_0}$$

(vii) $\displaystyle\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\{|a'(s)|+b'_+(s)+|c'(s)|\}\,ds<\gamma$,

where γ is a small positive constant whose magnitude depends only on the constants appeared in (i)~(vi), *and $b'_+(t)=max(b'(t), 0)$,*

(viii) $|p(t, x, y, z)| \leqq p(t)\{1+(x^2+y^2+z^2)^{1/2}\}+\Delta(x^2+y^2+z^2)^{1/2}$

where Δ is a positive constant and $p(t)$ is a non-negative continuous function,

(ix) $\displaystyle\int_t^{t+1} p(s)\,ds \to 0 \quad as \quad t \to \infty$.

Then there exists a finite constant $\varepsilon=\varepsilon(A, a_0, B, b_0, C, c_0, \delta, \bar{f}\, f_0, \bar{g}, g_0\, h_1)>0$ such that if $\Delta \leqq \varepsilon$ then every solution $x(t)$ of (1.2) *is uniform-bounded and satisfies*

$$x(t) \to 0,\quad \dot{x}(t) \to 0,\quad \ddot{x}(t) \to 0 \quad as \quad t \to \infty .$$

REMARK. It should be pointed out that in the special case $f \equiv 1$ (so that the assumption (iv) is automatically satisfied) Theorem 2 reduces to the author's earlier result [7; Theorem 2]. Also in another special case in which

$a(t)f(x, y, z) = \text{fl}$, $b(t)g(x,y) \equiv by$ and $c(t)h(x) = cx$ in (1.2) (so that all the conditions (ii)~(iv) and (vi) are trivially fulfilled) the hypothesis (i) and (v) reduce to

$$a>0, \quad b>0, \quad c>0, \quad ab-c>0$$

which is the Routh-Hurwitz criterion for the asymptotic stability in the large of the zero solution of the equation

$$\dddot{x}+a\ddot{x}+b\dot{x}+cx=0 .$$

5. Proof of theorems

Proof of Theorem 1. We consider the Liapunov function

$$V(t,x) = x^T P(t) x . \tag{5.1}$$

By virtue of Proposition A and the boundedness of $P(t)$, there exist positive constants C and P_2 such that

$$C\|x\|^2 \leqq V(t,x) \leqq P_2\|x\|^2 . \tag{5.2}$$

A simple calculation shows that

$$\begin{aligned}\dot{V}_{(1.1)}(t, x) &= \dot{x}^T P(t) x + x^T P(t)\dot{x} + x^T \dot{P}(t) x \\ &= -x^T x + f^T(t,x) P(t) x + x^T P(t) f(t,x) + x^T \dot{P}(t) x .\end{aligned}$$

Applying Proposition B to the function $x^T \dot{P}(t) x$, we obtain

$$\begin{aligned}\dot{V}_{(1.1)}(t, x) \leqq & -\|x\|^2 + 2\|f(t, x)\| \cdot \|P(t)\| \cdot \|x\| \\ & + 2\|\dot{A}(t)\| \cdot \|P(t)\| \cdot x^T P(t) x\end{aligned}$$

Using (5.1), (5.2) and (iii) of Theorem 1, we have

$$\begin{aligned}\dot{V}_{(1.1)}(t, x) \leqq & \left\{-\frac{1}{\|P(t)\|} + 2\|P(t)\|\,\|\dot{A}(t)\|\right\} V(t, x) \\ & + 2\|P(t)\|\,\gamma(t)\left\{\left(\frac{V(t, x)}{C}\right)^{1/2} + \frac{V(t, x)}{C}\right\} .\end{aligned}$$

We'll show that

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\left\{-\frac{1}{\|P(\tau)\|} + 2\|P(\tau)\|\,\|\dot{A}(\tau)\|\right\} d\tau < 0.$$

Let ρ_0 be a positive number such that

$$\rho_0 = \frac{1}{2P_2^2} - \limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} \|\dot{A}(\tau)\|\, d\tau .$$

Given $\varepsilon>0$, there exists a positive number T such that $||P(\tau)||<P_1+\varepsilon$ for all $\tau\geqq T$. This implies also

$$\frac{-1}{||P(\tau)||}<\frac{-1}{P_1+\varepsilon}<-\frac{1}{P_1}+\frac{\varepsilon}{P_1^2} \qquad \text{for all} \quad \tau\geqq T.$$

Using these inequalities, we have

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\left\{-\frac{1}{||P(\tau)||}+2||P(\tau)||\,||\dot{A}(\tau)||\right\}d\tau$$
$$<\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v}\left\{-\frac{1}{P_1}+\frac{\varepsilon}{P_1^2}+2(P_1+\varepsilon)||\dot{A}(\tau)||\right\}d\tau$$
$$=-2P_1\rho_0+\varepsilon\left\{\frac{1}{P_1^2}+\limsup_{(t,v)\to(\infty,\infty)}\frac{2}{v}\int_t^{t+v}||\dot{A}(\tau)||\,d\tau\right\}.$$

If ε is chosen so that

$$\varepsilon\left\{\frac{1}{P_1^2}+\limsup_{(t,v)\to(\infty,\infty)}\frac{2}{v}\int_t^{t+v}||\dot{A}(\tau)||\,d\tau\right\}<P_1\rho_0,$$

then we have

$$\limsup_{(t,v)\to(\infty,\infty)}\frac{}{v}\int_t^{t+v}\left\{-\frac{}{||P(\tau)||}+2||P(\tau)||\,||\dot{A}(\tau)||\right\}d\tau<-P_1\rho_0<0.$$

Hence, the assumptions of Corollary hold and the proof of Theorem 1 is completed. Q.E.D.

Proof of Theorem 2. The equation (1.2) is equivalent to the system

$$\left\{\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ z &= -a(t)f(x.y,\ z)z-b(t)g(x,y)-c(t)h(x)+p(t,x,\ y,\ z). \end{aligned}\right. \tag{5.3}$$

We consider the Liapunov function

$$V(t,x,\ y,\ z)=V_1(t,x,\ y,\ z)+V_2(t,x,\ y,\ z)+V_3(t,x,\ y,\ z) \tag{5.4}$$

where V_1, V_2 and V_3 are defined by

$$\begin{aligned} 2V_1 = {} & 2\mu_1 c(t)\int_0^x h(\xi)\,d\xi+2c(t)h(x)y+2b(t)\int_0^y g(x,\eta)\,d\eta \\ & +2\mu_1 a(t)\int_0^y f(x,\ \eta,\ 0)\eta\,d\eta+2\mu_1 yz+z^2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} 2V_2 = {} & \mu_2 b(t)g_0x^2+2a(t)f_0c(t)\int_0^x h(\xi)\,d\xi+a^2(t)f_0^2y^2 \\ & -\mu_2y^2+2b(t)\int_0^y g(x,\ \eta)\,d\eta+z^2+2\mu_2a(t)f_0xy \\ & +2\mu_2xz+2a(t)f_0yz+2c(t)h(x)y, \end{aligned} \tag{5.6}$$

$$(5.7) \qquad 2V_3 = 2a^2(t)f_0\int_0^y f(x,\ \eta, 0)\eta d\eta - a^2(t)f_0^2y^2\,,$$

and $\mu_1>0$, $\mu_2>0$ are two arbitrarily fixed constants such that

$$\frac{Ch_1}{b_0g_0}<\mu_1<a_0f_0, \quad 0<\mu_2<\frac{a_0b_0f_0g_0-Ch_1}{Af_0}, \quad \delta>\frac{\mu_2}{4r_0}\left\{A(\bar{f}-f_0)+\frac{B}{\mu_1}(\bar{g}-g_0)\right\}.$$

We shall prove the following two properties of V:

$$(5.8) \qquad D_1\cdot(x^2+y^2+z^2)\le V(t,\ x,\ y,\ z)\leqq D_2\cdot(x^2+y^2+z^2)$$

for all $(x, y,\ z)\in R^3$ and

$$(5.9) \qquad \dot{V}_{(5.3)}\leqq -D_3\cdot(x^2+y^2+z^2)+D_4\cdot(|a'(t)|+b'_+(t)+|c'(t)|)\cdot(x^2+y^2+z^2) \\ +D_5\cdot(x^2+y^2+z^2)^{1/2}\cdot|p(t,\ x,\ y,\ z)|$$

along any solution $(x(t), y(t), z(t))$ of (5.3), where $D_1\sim D_5$ are certain positive constants.

At first we verify (5.8).

From the inequality $\frac{Ch_1}{b_0g_0}<\mu_1<a_0f_0$ there exists a positive number δ_0 such that $\mu_1b_0g_0(1-\delta_0)>Ch_1$ and $a_0f_0(1-\delta_0)>\mu_1$. Thus we have

$$\begin{aligned}
2V_1 &= 2\mu_1c(t)\int_0^x\left\{1-\frac{h'(\xi)}{h_1}\right\}h(\xi)d\xi+\mu_1c(t)\left\{\frac{h(x)}{\sqrt{h_1}}+\frac{\sqrt{h_1}}{\mu_1}y\right\}^2 \\
&\quad +(\mu_1y+z)^2+\frac{2}{\mu_1}\int_0^y\{\mu_1b(t)g(x,\ \eta)-c(t)h_1\eta\}d\eta \\
&\quad +2\mu_1\int_0^y\{a(t)f(x,\ \eta,\ 0)-\mu_1\}\eta d\eta \\
&= 2\mu_1c(t)\int_0^x\left\{1-\frac{h'(\xi)}{h_1}\right\}h(\xi)d\xi+\frac{\delta_0\mu_1c(t)}{h_1}h^2(x) \\
&\quad +\mu_1c(t)\left\{\sqrt{\frac{1-\delta_0}{h_1}}h(x)+\frac{1}{\mu_1}\sqrt{\frac{h_1}{1-\delta_0}}y\right\}^2+\delta_0z^2 \\
&\quad +\left(\sqrt{1-\delta_0}z+\frac{\mu_1}{\sqrt{1-\delta_0}}y\right)^2+\frac{2}{\mu_1}\int_0^y\left\{\mu_1b(t)g(x,\ \eta)-\frac{c(t)h_1}{1-\delta_0}\eta\right\}d\eta \\
&\quad +2\mu_1\int_0^y\left\{a(t)f(x,\eta,\ 0)-\frac{\mu_1}{1-\delta_0}\right\}\eta d\eta \\
&\geqq \frac{\delta_0\mu_1c_0\delta^2}{h_1}x^2+\left\{b_0g_0-\frac{Ch_1}{\mu_1(1-\delta_0)}+\mu_1a_0f_0-\frac{\mu_1^2}{1-\delta_0}\right\}y^2+\delta_0z^2\,.
\end{aligned}$$

Denoting $2D_6=\min\left\{\frac{\delta_0\mu_1c_0\delta^2}{h_1}, b_0g_0-\frac{Ch_1}{\mu_1(1-\delta_0)}+\mu_1a_0f_0-\frac{\mu_1^2}{1-\delta_0},\ \delta_0\right\}$, we have

$V_1(t, x, y, z) \geqq D_6(x^2+y^2+z^2)$.

It is easy to see that there exists a positive number D_7 such that

$$V_1(t,x, y, z) \leqq D_7 \ (x^2+y^2+z^2)$$

Hence we obtain

$$D_6 \ (x^2+y^2+z^2) \leqq V_1(t,x, y, z) \leqq D_7 \ (x^2+y^2+z^2) \,.$$

As before we have

$$\begin{aligned} 2V_2 = & \ \mu_2[g_0 b(t)-\mu_2]x^2+\mu_2{}^2x^2+a^2(t)f_0{}^2y+z^2 \\ & + 2\mu_2 a(t)f_0 xy + 2\mu_2 xz + 2a(t)f_0 yz \\ & +2a(t)f_0 c(t)\int_0 h(\xi)\,d\xi+2b(t)\int_0^y g(x, \eta)\,d\eta-\mu_2 y^2+2c(t)h(x)y \\ = & \ \mu_2[g_0 b(t)-\mu_2]x^2+[\mu_2 x+a(t)f_0 y+z]^2 \\ & +a(t)f_0 c(t)\left\{2\int_0^x h(\xi)\,d\xi-\frac{1}{h_1}h^2(x)\right\}+\frac{h_1 c(t)}{a(t)}\left\{\frac{\sqrt{f_0}\,a(t)h(x)}{h_1}+\frac{y}{\sqrt{f_0}}\right\}^2 \\ & +2\int_0^y\left\{\frac{b(t)g(x, \eta)}{\eta}-\frac{h_1 c(t)}{a(t)f_0}-\mu_2\right\}\eta\,d\eta \,. \end{aligned}$$

We find easily that

$$[g_0 b(t)-\mu_2]>0 \,,$$

$$\left\{2\int_0^x h(\xi)\,d\xi-\frac{1}{h_1}h^2(x)\right\} = \frac{2}{h_1}\int_0^x h(\xi)\{h_1-h'(\xi)\}\,d\xi \geqq 0,$$

$$\left\{\frac{b(t)g(x, \eta)}{\eta}-\frac{h_1 c(t)}{a(t)f_0}-\mu_2\right\}> 0 \,.$$

Then we have a positive number D_8 such that

$$0 \leqq V_2(t, x, y, z) \leqq D_8\cdot(x^2+y^2+z^2).$$

We can see also that

$$2V_3=2\dot{a}^2(t)f_0\int_0^y\{f(x, \eta, 0)-f_0\}\eta\,d\eta \,,$$

$$0 \leqq 2V_3(t, x, y, z) \leqq A^2 f_0(\bar{f}-f_0)y^2 \,.$$

Therefore there exist positive numbers D_1 and D_2 such that

$$D_1(x^2+y^2+z^2) \leqq V(t, x, y, z) \leqq D_2\cdot(x^2+y^2+z^2) \,. \tag{5.8}$$

Next we prove the inequality (5.9). Along any solution $(x(t), y(t), z(t))$ of (5.3), we have

$$\dot{V}_{1(5.3)} = -\left[\frac{\mu_1 b(t)g(x, y)}{y} - c(t)h'(x)\right]y^2 - [a(t)f(x, y, z) - \mu_1]z^2$$
$$+b(t)y\int_0^y g_x(x, \eta)\,d\eta + \mu_1 a(t)y\int_0^y f_x(x, \eta, 0)\eta d\eta$$
$$-\mu_1 a(t)[f(x, y, z) - f(x, y, 0)]yz$$
$$+\mu_1 c'(t)\int_0^x h(\xi)\,d\xi + c'(t)h(x)y + b'(t)\int_0^y g(x, \eta)\,d\eta$$
$$+\mu_1 a'(t)\int_0^y f(x, \eta, 0)\eta d\eta + (\mu_1 y + z)p(t, x, y, z),$$

$$\dot{V}_{2(5.3)} = -\mu_2 c(t)xh(x) - \mu_2 b(t)\left[\frac{g(x, y)}{y} - g_0\right]xy$$
$$-\left[a(t)f_0 b(t)\frac{g(x, y)}{y} - c(t)h'(x) - \mu_2 a(t)f_0\right]y^2$$
$$-a^2(t)f_0[f(x, y, z) - f_0]yz + b(t)y\int_0^y g_x(x, \eta)\,d\eta$$
$$-a(t)[f(x, y, z) - f_0]z^2 - \mu_2 a(t)[f(x, y, z) - f_0]xz$$
$$+\frac{1}{2}\mu_2 b'(t)g_0 x^2 + f_0[a'(t)c(t) + a(t)c'(t)]\int_0^x h(\xi)\,d\xi$$
$$+a(t)a'(t)f_0^2 y^2 + b'(t)\int_0^y g(x, \eta)\,d\eta + \mu_2 a'(t)f_0 xy$$
$$+a'(t)f_0 yz + c'(t)h(x)y + [z + \mu_2 x + a(t)f_0 y]p(t, x, y, z)$$

and

$$\dot{V}_{3(5.3)} = a^2(t)f_0 y\int_0^y f_x(x, \eta, 0)\eta d\eta + a^2(t)f_0[f(x, y, 0) - f_0]yz$$
$$+2a(t)a'(t)f_0\int_0^y [f(x, \eta, 0) - f_0]\eta d\eta\,.$$

Thus we obtain

$$\dot{V}_{(5.3)} = -W(t, x, y, z) + 2b(t)y\int_0^y g_x(x, \eta)\,d\eta$$
$$+a(t)[\mu_1 + a(t)f_0]y\int_0^y f_x(x, \eta, 0)\eta d\eta$$
$$+\{\mu_1 c'(t) + f_0[a'(t)c(t) + a(t)c'(t)]\}\int_0^x h(\xi)\,d\xi$$
$$+2c'(t)h(x)y + 2b'(t)\int_0^y g(x, \eta)\,d\eta + \mu_1 a'(t)\int_0^y f(x, \eta, 0)\eta d\eta$$
$$+\frac{1}{2}\mu_2 b'(t)g_0 x^2 + \mu_2 a'(t)f_0 xy + a'(t)f_0 yz$$
$$+2a(t)a'(t)f_0\int_0^y f(x, \eta, 0)\eta d\eta$$

$$+\{\mu_2 x+[\mu_1+a(t)f_0]y+2z\}\cdot p(t,\ x, y,\ z)$$

where

$$\begin{aligned}
W= {} & \mu_2 c(t)\, xh(x)+[a(t)f(x,\ y,\ z)-\mu_1]\, z^2+a(t)[f(x,\ y,\ z)-f_0]\, z^2 \\
& +\{[\mu_1 b(t)g_0-c(t)\, h'(x)]+[a(t)\, b(t)f_0 g_0-c(t)\, h'(x)-\mu_2 a(t)f_0]\}\, y^2 \\
& +a(t)\, b(t)f_0\left[\frac{g(x,\ y)}{y}-g_0\right]y^2+\mu_1 b(t)\left[\frac{g(x,\ y)}{y}-g_0\right]y^2 \\
& +\mu_2 b(t)\left[\frac{g(x,\ y)}{y}-g_0\right]xy+\mu_1 a(t)[f(x,\ y,\ x)-f(x,\ y,\ 0)]\, yz \\
& +a^2(t)f_0[f(x,\ y,\ z)-f(x,\ y,\ 0)]\, yz \\
& +\mu_2 a(t)[f(x,\ y,\ z)-f_0]\, xz \\
\geqq {} & \mu_2 c_0\delta x^2+[a_0 f_0-\mu_1]\, z^2 \\
& +\{[\mu_1 b_0 g_0-Ch_1]+[a_0 b_0 f_0 g_0-Ch_1-\mu_2 Af_0]\}\, y^2 \\
& +a(t)[f(x,\ y,\ z)-f_0]\{z^2+\mu_2 xz\} \\
& +\mu_1 b(t)\left[\frac{g(x,\ y)}{y}-g_0\right]\left\{y^2+\frac{\mu_2}{\mu_1}xy\right\}+a(t)\, b(t)f_0\left[\frac{g(x,\ y)}{y}-g_0\right]y^2 \\
& +a(t)\{\mu_1+a(t)f_0\}\, [f(x,\ y,\ z)-f(x,\ y,\ 0)]\, yz.
\end{aligned}$$

Hence

$$\begin{aligned}
W \geqq {} & \mu_2\left\{c_0\delta-[f(x,\ y,\ z)-f_0]\frac{A\mu_2}{4}-\left[\frac{g(x,\ y)}{y}-g_0\right]\frac{B\mu_2}{4\mu_1}\right\}x^2 \\
& +\{[\mu_1 b_0 g_0-Ch_1]+[a_0 b_0 f_0 g_0-Ch_1-\mu_2 Af_0]\}\, y^2 \\
& +[a_0 f_0-\mu_1]\, z^2+a(t)[f(x, y,\ z)-f_0]\left(z+\frac{1}{2}\mu_2 x\right)^2 \\
& +\mu_1 b(t)\left[\frac{g(x,\ y)}{y}-g_0\right]\left(y+\frac{1}{2}\frac{\mu_2}{\mu_1}x\right)^2+a(t)\, b(t)f_0\left[\frac{g(x,\ y)}{y}-g_0\right]y^2 \\
& +a(t)\{\mu_1+a(t)f_0\}\, [f(x, y,\ z)-f(x,\ y,\ 0)]\, yz\,.
\end{aligned}$$

By the assumptions

$$\begin{aligned}
& \left\{c_0\delta-[f(x,\ y,\ z)-f_0]\frac{A\mu_2}{4}-\left[\frac{g(x,\ y)}{y}-g_0\right]\frac{B\mu_2}{4\mu_1}\right\} \\
& \quad \geqq c_0\delta-[\bar{f}-f_0]\frac{A\mu_2}{4}-[\bar{g}-g_0]\frac{B\mu_2}{4\mu_1} \\
& \quad = c_0\delta-\frac{\mu_2}{4}\left\{A(\bar{f}-f_0)+\frac{B}{\mu_1}(\bar{g}-g_0)\right\}>0\,, \\
& [\mu_1 b_0 g_0-Ch_1]>\frac{Ch_1}{b_0 g_0}\cdot b_0 g_0-Ch_1=0\,, \\
& [a_0 b_0 f_0 g_0-Ch_1-\mu_2 Af_0]
\end{aligned}$$

$$> a_0b_0f_0g_0 - Ch_1 - \frac{a_0b_0f_0g_0 - Ch_1}{Af_0} \cdot Af_0 = 0\,,$$

$$[a_0f_0 - \mu_1] > 0\,.$$

Applying the Mean Value Theorem, we have for $0 < |\tilde{z}| < |z|$

$$[f(x,y,\ z) - f(x,y,\ 0)]yz = f_z(x,y,\ \tilde{z})yz^2 \geqq 0\,.$$

Therefore there exists a positive number D_3 such that for all $(x, y,\ z) \in R^1$

$$W \geqq D_3 \cdot (x^2 + y^2 + z^2)\,.$$

From (ii), (iii), (iv) and (v) it is easy to show that there exists a positive number D_4 such that

$$\dot{V}_{(5.3)} \leqq -D_3(x^2+y^2+z^2) + D_4 \cdot (|a'(t)| + b'_+(t) + |c'(t)|)(x^2+y^2+z^2)$$
$$+ \{\mu_2 x + [\mu_1 + a(t)f_0]y + 2z\}\, p(t,\ x,y,\ z)\,.$$

Setting $D_9 = \max\ \{\mu_2,\ \mu_1 + Af_0, 2\}$, we have

$$\{\mu_2 x + [\mu_1 + a(t)f_0]y + 2z\}\, p(x,\ y,\ z)$$
$$\leqq D_9 \cdot (|x| + |y| + |z|)|p(t,\ x,\ y,\ z)|$$
$$\leqq \sqrt{3}\, D_9 \cdot (x^2+y^2+z^2)^{1/2} |p(t,\ x,\ y,\ z)|\,.$$

Let $D_5 = \sqrt{3}\, D_9$. Then we obtain the inequality (5.9).

We are now ready for the principal subsidiary results needed for the completion of the proof of Theorem 2. Application to (5.9) with the assumption (viii) leads to

$$\dot{V}_{(5.3)} \leqq -D_3 \cdot (x^2+y^2+z^2) + D_4 \cdot (|a'(t)| + b'_+(t) + |c'(t)|)(tf + y^2 + z^2)$$
$$+ D_5 p(t)[(x^2+y^2+z^2)^{1/2} + (x^2+y^2+z^2)]$$
$$+ \Delta D_5 \cdot (x^2+y^2+z^2)\,.$$

Let Δ be fixed, in what follows, to satisfy

(5.10) $$\Delta \leqq \frac{D_3}{2D_5}.$$

Using the inequalities (5.8) and (5.10), we have

(5.11) $$\dot{V}_{(5.3)} \leqq -\frac{D_3}{2D_2} \cdot V + \frac{D_4}{D_1}(|a'(t)| + b'_+(t) + |c'(t)|)V$$
$$+ D_5 p(t)\left[\left(\frac{V}{D_1}\right)^{1/2} + \frac{V}{D_1}\right].$$

Assume

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{V} \int_t^{t+v} \{|a'(s)| + b'_+(s) + |c'(s)|\}ds < \frac{D_1D_3}{2D_2D_4}\,.$$

Now Theorem A will be used to prove the uniform boundedness of solutions of (1.2) and that for any solution $x(t)$

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)\to 0 \quad \text{as} \quad t\to\infty .$$

This completes the proof of Theorem 2. Q.E.D.

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