Hirashima, Y. and Ōshima, H.
Osaka J. Math.
13 (1976), 157–161

A NOTE ON STABLE JAMES NUMBERS OF PROJECTIVE SPACES

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(Received February 10, 1975)

In [4] and [5], the second named author has evaluated the stable James numbers

$$d_F(n) = \#[\text{cokernel of } \{FP^n, FP^1\} \xrightarrow{i^*} \{FP^1, FP^1\}]$$

of the n-dimensional F-projective spaces FP^n for $F=C$ (complex numbers) or H (quaternions), and especially decided $d_C(n)=k_s^{n,2}$ for $n\leqq 4$, the odd components of $d_C(n)$ and $d_H(m)$ for $m\leqq 2$. The purpose of this note is to prove the following theorem.

We will use the notations introduced in [4] and [5] without notice.

Theorem. (i) $d_C(5)=2^2\cdot 3\cdot 5$, $v_2(d_C(6))\leqq 3$,
(ii) $d_C(7)=d_C(8)$, $d_C(15)=d_C(16)$,
(iii) $v_2(d_C(9))\leqq v_2(d_C(7))+1\leqq 5$,
(iv) $d_H(3)=2^3\cdot 3^2\cdot 5$, $d_H(4)=2^5\cdot 3^2\cdot 5\cdot 7$, $d_H(5)=2^{5+\varepsilon}\cdot 3^2\cdot 5^2\cdot 7$ *and* $d_H(6)=2^{5+\varepsilon+\tau}\cdot 3^3\cdot 5^2\cdot 7\cdot 11$, *where* $0\leqq\varepsilon\leqq 1$ *and* $0\leqq\tau\leqq 2$.

Proof. (ii) follows from [4] for the odd components and the following fact for the 2-component that

$$(\#)\qquad \text{if } v_2(|G_{2j-1}|)=0\,, \quad \text{then} \quad v_2(d_C(j))=v_2(d_C(j+1))\,,$$

where $|G_{2j-1}|=\min\{k>0;\, kG_{2j-1}=0\}$. Then (ii) follows, since $G_{13}=Z_3$ [6] and $v_2(|G_{29}|)=0$ [3]. We shall prove (#). Let $f\colon CP^j\to S^2=CP^1$ be a stable map such that the degree of the composition $S^2\subset CP^j\xrightarrow{f}S^2$ is $d_C(j)$. Then $f\circ p\colon S^{2j+1}\xrightarrow{p}CP^j\xrightarrow{f}S^2$ represents an element of the $(2j-1)$-stem G_{2j-1} and so that $d_C(j+1)$ is a divisor of $|G_{2j-1}|d_C(j)$, where p denotes a natural projection. Hence $v_2(d_C(j+1))\leqq v_2(|G_{2j-1}|)+v_2(d_C(j))$. But $v_2(d_C(j))\leqq v_2(d_C(j+1))$, since $d_C(j+1)$ is a multiple of $d_C(j)$ [4]. Therefore (#) follows.

Recall that $K(FP^n)=Z[\mu_F(n)]/\mu_F(n)^{n+1}$, where $\mu_F(n)$ indicates the stable bundle of the underlying complex vector bundle of the canonical F-line bundle over FP^n.

For $f\in\{FP^n, FP^1\}$, we consider the following commutative diagram

$$\begin{array}{ccc} K(FP^n) & \overset{f^*}{\longleftarrow} & K(FP^1) \\ \psi^2\downarrow & & \downarrow\psi^2 \\ K(FP^n) & \overset{f^*}{\longleftarrow} & K(FP^1) \end{array}$$

where ψ^2 denotes the Adams operation. Let $f^*(\mu_F(1))=\sum_{i=1}^n a_i\mu_F(n)^i$ and $d_F=2$ (if $F=C$) or 4 (if $F=H$). Then

$$f^*\psi^2(\mu_F(1))=f^*(d_F\mu_F(1))=d_F\sum_{i=1}^n a_i\mu_F(n)^i$$

and this equals

$$\psi^2 f^*(\mu_F(1))=\sum_{i=1}^n a_i\{\psi^2(\mu_F(n))\}^i=\sum_{i=1}^n a_i\{\mu_F(n)^2+d_F\mu_F(n)\}^i$$
$$=\sum_{i=1}^n\sum_{j=1}^n a_i\binom{i}{j-i}d_F{}^{2i-j}\mu_F(n)^j,$$

since $\psi^2(\mu_C(n))=\{1+\mu_C(n)\}^2-1=\mu_C(n)^2+2\mu_C(n)$ and $\psi^2(\mu_H(n))=\mu_H(n)^2+4\mu_H(n)$ (see e.g. [2]). Comparing the coefficients of $\mu_F(n)^j$, we have

$$(1)_F \qquad d_F a_j=\sum_{i=1}^n a_i\binom{i}{j-i}d_F{}^{2i-j}.$$

Notice that $(1)_C$ implies $a_1=(-1)^{j+1}ja_j$ if $F=C$ (cf. [4]).

For $f\in\{FP^n, FP^1\}$, we consider the following commutative diagram of cofibrations

$$\begin{array}{ccccccc} S^{(n+1)d_F-1} & \overset{p}{\longrightarrow} & FP^n & \longrightarrow & FP^{n+1} & \longrightarrow & S^{(n+1)d_F} \\ \downarrow = & & \downarrow f & & \downarrow \bar f & & \downarrow = \\ S^{(n+1)d_F-1} & \overset{g}{\longrightarrow} & FP^1 & \overset{j}{\longrightarrow} & C_g & \overset{h}{\longrightarrow} & S^{(n+1)d_F} \end{array}$$

where p denotes the canonical projection, $g=f\circ p$ and C_g the mapping cone of g. Then we have the commutative diagram of the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \tilde K(FP^n) & \longleftarrow & \tilde K(FP^{n+1}) & \longleftarrow & \tilde K(S^{(n+1)d_F}) & \longleftarrow & 0 \\ & & \uparrow f^* & & \bar f^*\uparrow & & =\uparrow & & \\ 0 & \longleftarrow & \tilde K(FP^1) & \overset{j^*}{\longleftarrow} & \tilde K(C_g) & \overset{h^*}{\longleftarrow} & \tilde K(S^{(n+1)d_F}) & \longleftarrow & 0. \end{array}$$

Let $x\in\tilde K(C_g)$ be such that $j^*(x)=\mu_F(1)=g_C{}^{d_F/2}$. Let $y=h^*(g_C{}^{(n+1)d_F/2})$ and let $\bar f^*(x)=\sum_{i=1}^{n+1} a_i\mu_F(n+1)^i$. Then

$$f^*(\mu_F(1))=\sum_{i=1}^n a_i\mu_F(n)^i$$

and

$$\psi_F^2(x) = d_F x + \lambda y \qquad \text{for some } \lambda \in Z$$

and

$$e(g) = \frac{\lambda}{(2^{nd_F/2}-1)2^{d_F/2}} = \frac{\lambda}{(d_F{}^n-1)d_F} \in Q/Z\,,$$

where e denotes the e-invariant (see e.g. [1]). Now

$$\bar{f}^*\psi_F^2(x) = \bar{f}^*(d_F x+\lambda y) = d_F \sum_{i=1}^{n+1} a_i \mu_F(n+1)^i + \lambda \mu_F(n+1)^{n+1}$$

and this equals

$$\psi_F^2\bar{f}^*(x) = \psi_F^2(\sum_{i=1}^{n+1} a_i \mu_F(n+1)^i) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_i \binom{i}{j-i} d_F{}^{2i-j}\mu_F(n+1)^j\,.$$

Comparing the coefficients of $\mu_F(n+1)^{n+1}$, we have

$$d_F a_{n+1} + \lambda = \sum_{i=1}^{n+1} a_i \binom{i}{n+1-i} d_F{}^{2i-n-1}$$

and so

$$(2)_F \qquad e(g) = \frac{\lambda}{(d_F{}^n-1)d_F} = \frac{\sum_{i=1}^{n} a_i \binom{i}{n+1-i} d_F{}^{2i-n-1}}{(d_F{}^n-1)d_F}\,.$$

Consider the case with $F=C$ and $n=4$. Suppose that the degree of the composition $S^2 \subset CP^4 \xrightarrow{f} S^2$ is $d_C(4)=12$ [4]. Then $(1)_C$ implies

$$a_1 = 12\,,\quad a_2 = -6\,,\quad a_3 = 4 \quad \text{and} \quad a_4 = -3\,,$$

and then by $(2)_C$

$$e(g) = -\frac{12}{5}$$

and so

$$e(5g) = 0\,.$$

But e: $G_7 \to Q/Z$ is monomorphic [1, §7] , so that

$$5g = 0\,.$$

Then there exists f': $CP^5 \to S^2$ such that the composition $CP^4 \subset CP^5 \xrightarrow{f'} S^2$ coincides with $5f$. Clearly the degree of the composition $S^2 \subset CP^5 \xrightarrow{f'} S^2$ is $5d_C(4)=2^2\cdot 3\cdot 5$ [4]. Hence $d_C(5)$ is a divisor of $2^2\cdot 3\cdot 5$. But, by [4], $d_C(5)$ is a multiple of $2^2\cdot 3\cdot 5$. Therefore we have that $d_C(5)=2^2\cdot 3\cdot 5$. This implies the half of (i).

By the same method as the proof of (ii), we have that $d_C(6)$ is a divisor of $2d_C(5)$, since $G_9=Z_2+Z_2+Z_2$ [6]. Hence we have

$$v_2(d_C(6))\leqq v_2(2d_C(5)) = 3 .$$

This completes the proof of (i).

When $F=C$ and $n=8$, the same computations as the proof of (i) imply that

$$3e(g) = 0 .$$

But the kernel of e: $G_{15}\to Q/Z$ is Z_2 [1, §7], so that

$$2\cdot 3g = 0$$

and then $d_C(9)$ is a divisor of $2\cdot 3d_C(8)$. Then (ii) and [4, Th. A] imply

$$v_2(d_C(9))\leqq v_2(2\cdot 3d_C(7)) = v_2(d_C(7))+1\leqq 5 .$$

This implies (iii).

By the same computations as the proof of (i) using the fact $d_H(2)=2^3\cdot 3$ [4], we have

$$d_H(3) = 2^3\cdot 3^2\cdot 5 .$$

And when $F=H$ and $n=3$, we have

$$e(g) = \frac{3^2}{2\cdot 7} .$$

But $e=2e'_R$: $G_{11}\to Q/Z$ and the real e-invariant e'_R is monomorphic in this case [1, §7] so that

$$2^2\cdot 7g = 0$$

and then $d_H(4)$ is a divisor of $2^2\cdot 7d_H(3)=2^5\cdot 3^2\cdot 5\cdot 7$. But, by [5], $d_H(4)$ is a multiple of $2^5\cdot 3^2\cdot 5\cdot 7$. Thus

$$d_H(4) = 2^5\cdot 3^2\cdot 5\cdot 7 .$$

By the same methods as aboves, we can prove the remaining parts of (iv).

Q.E.D.

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