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ON A GENERALIZATION OF QF-3′ RINGS*)

Dedicated to Professor Kiiti Morita for the celebration of his sixtieth birthday.

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A ring R with identity is called left QF-3′ if the injective hull $E(R)$ of the left R-module R is torsionless. This class of rings and the other related generalizations of quasi-Frobenius rings have been studied by a number of authors.

Recently, Jans [7] has given a torsion theoretic characterization of left QF-3′ rings (cf. also Kato [8] and Tsukerman [14]). The purpose of this paper is, generalizing this idea, to consider a module theoretic generalization of left QF-3′ rings. We shall say that a left R-module Q is QF-3′ if its injective hull $E(Q)$ is torsionless with respect to Q, i.e., $E(Q)$ can be embedded in a direct product of copies of Q.

The main theorem of §1 will give some equivalent conditions for Q to be QF-3′.

In §2, we shall discuss basic properties of QF-3′ R-modules and study a relation between QF-3′ R-modules and cogenerators for R-mod.

We shall treat, in §3, QF-3′ R-modules with zero singular submodule. We shall give some results relating the notions of Q-torsionless R-modules and non-singular R-modules. In particular we shall show that, if Q is faithful, these notions coincide if and only if Q is QF-3′ and has zero singular submodule. We shall also give another characterization of a QF-3′ R-module with zero singular submodule making use of its injective submodules.

After completed this paper, we found that the similar results were obtained by Bican [2] and wrought a slight change in the paper.

Throughout this paper, R will denote an associative ring with identity and R-mod the category of unital left R-modules and R-homomorphisms. We shall deal only with left R-modules and so R-modules will mean unital left R-modules. $E(M)$ will always denote the injective hull of a left R-module M and $r_M(*)$ the right annihilator of $*$ in M.

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1. Preliminaries

A subfunctor r of the identity functor of R-mod is called a preradical of R-mod. It is called idempotent if $r(r(M))=r(M)$ and a radical if $r(M/r(M))=0$ for all R-modules M. To each preradical r we associate two classes of R-modules, namely

$$\mathbf{T}(r)=\{M \mid r(M)=M\} \text{ and } \mathbf{F}(r)=\{M \mid r(M)=0\} .$$

In case a preradical r is idempotent and is a radical, the pair $(\mathbf{T}(r), \mathbf{F}(r))$ forms a torsion theory for R-mod in the sense of [5].

In the class of all preradicals of R-mod, there is a partial ordering in which $r_1 \leqq r_2$ means that $r_1(M) \subset r_2(M)$ for all R-modules M. For each preradical r there exists a largest idempotent preradical $\hat{r}$ smaller than or equal to r and a smallest radical $\bar{r}$ larger than or equal to r. It is easy to see that $\mathbf{T}(\hat{r})=\mathbf{T}(r)$ and $\mathbf{F}(\bar{r})=\mathbf{F}(r)$. Moreover, if r is idempotent, then so is $\bar{r}$, and $\hat{r}$ is a radical if r is.

Let Q be an R-module and let us define

$$k_Q(M)=\bigcap_{f \in \mathrm{Hom}_R(M,Q)} \mathrm{Ker}(f)$$

for each R-module M. Then k_Q is a radical of R-mod such that $k_Q(Q)=0$. Moreover it is a unique maximal one of those preradicals r of R-mod for which $r(Q)=0$, and $k_{E(Q)}$ is a unique maximal one of those left exact radicals r of R-mod for which $r(Q)=0$. As is well-known, every left exact radical of R-mod is of the form k_E for some injective R-module E. For example, we can take E as the direct product of injective hulls of all cyclic torsion-free R-modules (e.g., see [11]).

Since $k_Q \leqq k_{Q'}$ for each submodule Q' of Q and $k_{E(Q)}$ is idempotent, we have $k_{E(Q)} \leqq \hat{k}_Q \leqq k_Q$.

The class $\mathbf{T}(k_Q)$ coincides with the class $\{M \mid \mathrm{Hom}_R(M, Q)=0\}$ and is closed under taking homomorphic images, direct sums and extensions. So this is a torsion class in R-mod and the corresponding torsion-free class coincides with $\mathbf{F}(\hat{k}_Q)$. On the other hand, the class of R-modules $\mathbf{F}(k_Q)$ is not a torsion-free class in general. This is closed under taking submodules and direct products, but not extensions in general (see e.g. [16, Example B]). As is well-known, Q is a cogenerator for $\mathbf{F}(k_Q)$. Moreover, an R-module M is in $\mathbf{F}(k_Q)$ if (and only if) it can be embedded in a direct product of copies of Q. However, for simple R-modules we have

Proposition 1.1. *A simple R-module S is in $\mathbf{F}(k_Q)$ if and only if there exists an R-monomorphism of S into Q.*

The proof is easy and so we will omit it.

As was mentioned above, $\mathbf{F}(k_Q)$ is not closed under taking extensions.

The following proposition shows when it is closed under taking extensions. Evidently this is the case if Q is injective.

Proposition 1.2. *The following conditions on an R-module Q are equivalent:*

(1) $\mathbf{F}(k_Q)$ *is closed under taking extensions, i.e., it becomes a torsion-free class.*

(2) $k_Q=\hat{k}_Q$, *i.e.,* k_Q *is idempotent.*

(3) $\mathbf{F}(k_Q)=\mathbf{F}(\hat{k}_Q)$.

Bican [2] has obtained the same result independently, and so we will omit the proof.

The class $\mathbf{T}(k_Q)$ is a torsion class, but it is not, in general, closed under taking submodules (e.g., see [16]). Concerning this, we have

Proposition 1.3. *For an R-module Q, the following conditions are equivalent:*

(1) $\mathbf{T}(k_Q)$ *is closed under taking submodules.*

(2) $\hat{k}_Q=k_{E(Q)}$, *i.e.,* $\hat{k}_Q$ *is left exact.*

(3) $\mathbf{T}(k_Q)=\mathbf{T}(k_{E(Q)})$.

Proof. (1)$\Rightarrow$(2). Suppose that $\mathbf{T}(k_Q)$ is closed under taking submodules. Then, since $\hat{k}_Q\leqq k_Q$, $\hat{k}_Q(Q)=0$ and, since the corresponding torsion-free class $\mathbf{F}(\hat{k}_Q)$ of $\mathbf{T}(k_Q)$ is closed under taking injective hulls, we have $\hat{k}_Q(E(Q))=0$. So $\hat{k}_Q\leqq k_{E(Q)}$ and hence $\hat{k}_Q=k_{E(Q)}$.

(2)$\Rightarrow$(3) is clear and since $k_{E(Q)}$ is left exact, (3) implies (1).

This proposition was also proved in Bican [2] by a different method.

Combining this with Proposition 1.2, we have

Theorem 1.4. *The following conditions on an R-module Q are equivalent:*

(1) $(\mathbf{T}(k_Q),\ \mathbf{F}(k_Q))$ *forms a hereditary torsion theory for* R-mod.

(2) $\mathbf{T}(k_Q)$ *is closed under taking submodules and* $\mathbf{F}(k_Q)$ *is closed under taking extensions.*

(3) $k_Q=k_{E(Q)}$.

(4) k_Q *is left exact.*

(5) $\mathbf{F}(k_Q)$ *is closed under taking injective hulls.*

(6) $\mathbf{F}(k_Q)$ *is closed under taking essential extensions.*

(7) $\mathbf{F}(k_Q)$ *contains an injective R-module M with* $k_M(Q)=0$.

(8) $E(Q)\in\mathbf{F}(k_Q)$.

(9) $\mathbf{T}(k_Q)=\mathbf{T}(k_{E(Q)})$ and $\mathbf{F}(k_Q)=\mathbf{F}(k_{E(Q)})$.

Proof. Here we show only that (7) implies (8). The proof of the other is easy. Since $k_M(Q)=0$, $Q\subset\prod M$, a direct product of copies of M, and hence $E(Q)\subset\prod M$. $\mathbf{F}(k_Q)$ is closed under taking direct products and submodules and so we have $E(Q)\in\mathbf{F}(k_Q)$.

The equivalence of (3), (4) and (8) was also proved in Bican [2]. In case $Q=R$, the equivalence of these conditions, except for (1), (4), (5) and (9), was shown by Colby and Rutter [4], Jans [7], and Kato [8].

2. QF-3′ R-modules

Recall that a ring R is left QF-3′ if the injective hull of the R-module R is torsionless, i.e., $k_R(E(R))=0$. Recently, Jans [7] has shown that R is left QF-3′ if and only if $\mathbf{F}(k_R)$ is closed under taking extensions and $\mathbf{T}(k_R)$ is closed under taking submodules (cf. also Kato [8] and Tsukerman [14]). From this point of view we now make the following definition.

DEFINITION. An R-module Q is called QF-3′ if Q satisfies each one of the conditions of Theorem 1.4.

It follows from this definition that every injective R-module is QF-3′. The following example pointed out by Tsukerman without proof shows that there exist non-injective QF-3′ R-modules.

EXAMPLE 2.1. Every direct sum of injective R-modules is QF-3′.

To see this, let $Q=\sum_{\lambda\in\Lambda}\oplus Q_\lambda$ be a direct sum of injective R-modules. Then $\mathbf{F}(k_{Q_\lambda})\subset\mathbf{F}(k_Q)$ for all λ and hence $\prod_{\lambda\in\Lambda}Q_\lambda\in\mathbf{F}(k_Q)$. Since $Q\subset E(Q)\subset\prod_{\lambda\in\Lambda}Q_\lambda$, $E(Q)\in\mathbf{F}(k_Q)$ and thus Q is QF-3′.

Proposition 2.2. (1) *Every direct product of* QF-3′ *R-modules is* QF-3′.
(2) *Every direct sum of* QF-3′ *R-modules is* QF-3′.

Proof. (1) Let $Q=\prod_{\lambda\in\Lambda}Q_\lambda$ be a direct product of QF-3′ R-modules. Then $\mathbf{F}(k_{Q_\lambda})\subset\mathbf{F}(k_Q)$ for all λ and hence $\prod_{\lambda\in\Lambda}E(Q_\lambda)\in\mathbf{F}(k_Q)$. Since $Q\subset E(Q)\subset\prod_{\lambda\in\Lambda}E(Q_\lambda)$, $E(Q)\in\mathbf{F}(k_Q)$ and thus Q is QF-3′. The proof of (2) is similar to that of (1) and so it will be omitted.

It should be noted that, as we shall show later, direct summands of a QF-3′ R-module need not be QF-3′ in general.

Proposition 2.3. *Every essential extension of a* QF-3′ *R-module is* QF-3′.

Proof. Suppose that Q is QF-3′ and Q' is an essential extension of Q. Then we can assume that $Q\subset Q'\subset E(Q)$ and hence we have $k_{E(Q)}\leqq k_{Q'}\leqq k_Q$. By Theorem 1.4, $k_Q=k_{Q'}$ and thus Q' is QF-3′ again by Theorem 1.4.

It follows from this that every rational extension of a QF-3′ R-module is also QF-3′. This appeared in [13] for left QF-3′ rings.

Let Q be an R-module. As is easily seen, Q is faithful if and only if $k_Q(R)=0$ and this is so if and only if $k_Q\leqq k_R$. On the other hand, Q is torsionless if and only if $k_R(Q)=0$, or equivalently, $k_R\leqq k_Q$. Therefore if Q is both faithful and torsionless, then we have $k_Q=k_R$. Applying Theorem 1.4, we have

Theorem 2.4. *For a ring R, the following conditions are equivalent:*
(1) *R is a left* QF-3′ *ring.*
(2) *The R-module R is* QF-3′.
(3) *There exists a* QF-3′ *R-module Q which is both faithful and torsionless.*

Proposition 2.5. *Let Q be an R-module.*

(1) *If Q is a* QF-3′ *R-module with non-zero socle, then the injective hull of every simple submodule of Q is isomorphic to a submodule of Q.*

(2) *If the injective hull of every cyclic submodule of Q is isomorphic to a submodule of Q, then Q is* QF-3′.

Proof. (1) Let S be a simple submodule of Q. Take $x(\neq 0)$ in $E(S)$. Then there exists $ax(\neq 0)$ in $Rx\cap S$. S is in $\mathbf{F}(k_Q)$ and so $E(S)$ is in $\mathbf{F}(k_Q)$ by Theorem 1.4. We can find an R-homomorphism f: $E(S)\to Q$ such that $f(ax)\neq 0$. Hence we have $f(S)\neq 0$ and f must be a monomorphism.

(2) Take $x(\neq 0)$ in $E(Q)$. There exists ax $(\neq 0)$ in $Rx\cap Q$. By assumption, $E(Rax)\subset Q$, and the inclusion mapping $Rax\to E(Rax)$ can be extended to an R-homomorphism f: $E(Q)\to Q$ such that $f(x)\neq 0$, which shows that Q is QF-3′.

Clearly, for a direct sum Q of injective R-modules, the injective hull of every cyclic submodule is isomorphic to a submodule of Q and so (2) above gives another proof of Example 2.1.

As an immediate consequence of this proposition, we have at once

Corollary 2.6. *For an R-module Q with non-zero socle, the following conditions are equivalent:*
(1) *Q is indecomposable and* QF-3′.
(2) *$Q=E(S)$ for every simple submodule S of Q.*
(3) *$Q=E(Q')$ for every non-zero submodule Q' of Q.*

Proposition 2.7. *Let Q be an R-module. If every cyclic submodule of Q is* QF-3′, *then Q is itself* QF-3′.

Proof. Take $x(\neq 0)$ in $E(Q)$ and claim that there exists an R-homomorphism f^*: $E(Q)\to Q$ with $f^*(x)\neq 0$. Choose an element a in R such that $ax(\neq 0)$ is in Q. Then we have $Rax\subset E(Rax)\subset E(Q)$ and $E(Q)=E(Rax)\oplus Q_1$ for some submodule Q_1 of $E(Q)$. By assumption, Rax is QF-3′ and so there exists an R-homomorphism f: $E(Rax)\to Rax$ such that $f(ax)\neq 0$. Then it is easy to see that the composition f^*: $E(Q)\to Q$ of f and the projection mapping $E(Q)\to E(Rax)$ has the desired property.

As a direct consequence of this, we see that if every cyclic R-module is QF-3′, then every R-module is also QF-3′. This was proved by Tsukerman

[14] under the assumption that R is left hereditary.

Recall that an R-module Q is a cogenerator for R-mod if $\mathbf{F}(k_Q)=R$-mod. Therefore, a cogenerator for R-mod is necessarily QF-3'. We now consider the question of when a QF-3' R-module becomes a cogenerator for R-mod. To do this we shall prove

Proposition 2.8. *For an R-module Q ($\neq 0$), the following conditions are equivalent:*

(1) *Q contains a copy of every simple R-module.*

(2) *Every simple R-module belongs to* $\mathbf{F}(k_Q)$.

(3) *For every simple R-module S,* $\mathrm{Hom}_R(S, Q)\neq 0$.

(4) *For every maximal left ideal $\mathfrak{m}$ of R, $r_Q(\mathfrak{m})\neq 0$.*

(5) *For every proper left ideal $\mathfrak{m}$ of R, $r_Q(\mathfrak{m})\neq 0$.*

(6) *For every non-zero finitely generated R-module M,* $\mathrm{Hom}_R(M, Q)\neq 0$.

(7) *For every non-zero cyclic R-module M,* $\mathrm{Hom}_R(M, Q)\neq 0$.

(8) *$E(Q)$ is a cogenerator for R-*mod.

(9) *Every non-zero injective R-module M with $k_M(Q)=0$ is a cogenerator for R-*mod.

Proof. We shall show only (7)⇒(8)⇒(9)⇒(1).

(7)⇒(8). Let M be an R-module. Take $x(\neq 0)$ in M. Then by assumption there exists a non-zero R-homomorphism $f\colon Rx\to Q$. Since $E(Q)$ is injective, it can be extended to an R-homomorphism $f'\colon M\to E(Q)$ and $f'(x)=f(x)\neq 0$. This shows that $E(Q)$ is a cogenerator for R-mod.

(8)⇒(9). Since $k_M(Q)=0$, $Q\subset \prod M$, a direct product of copies of M, and hence $E(Q)\subset \prod M$. Then we have $R\text{-mod}=\mathbf{F}(k_{E(Q)})\subset \mathbf{F}(k_{\Pi M})$. It follows that $\prod M$ is a cogenerator for R-mod and so is M by [12, Lemma 1].

(9)⇒(1). Since $E(Q)$ is a cogenerator for R-mod, for every simple R-module S, there exists a non-zero R-homomorphism $f\colon S\to E(Q)$. S is simple, so f must be a monomorphism. Since $f(S)\cap Q\neq 0$, $f(S)\cap Q=f(S)$ and hence $f(S)$ is contained in Q.

An R-module satisfying (1) and (8) was called lower distinguished by Azumaya [1] and a quasi-cogenerator by Morita [10] respectively.

Generalizing results due to Kato [8], Jans [6] and Sugano [12], we have

Theorem 2.9. *The following conditions on an R-module Q are equivalent:*

(1) *Q is a cogenerator for R-*mod.

(2) *Q is* QF-3' *and contains a copy of every simple R-module.*

(3) $\sum_{\lambda\in\Lambda}\oplus E(S_\lambda)\in \mathbf{F}(k_Q)$, *where $\{S_\lambda\}_{\lambda\in\Lambda}$ be a complete set of representatives for the isomorphism classes of simple R-modules.*

(4) *There exists a cogenerator for R-*mod *contained in* $\mathbf{F}(k_Q)$.

(5) *Every R-module M with $k_M(Q)=0$ is a cogenerator for R-*mod.

(6) *Q is faithful* QF-3′ *and* $\mathbf{F}(k_Q)$ *is closed under taking homomorphic images.*

Proof. (1)⇒(2)⇒(3) follow from Proposition 1.1 and Theorem 1.4 and (3)⇒(4) and (5)⇒(6) are easy.

(4)⇒(5). Let N be a cogenerator for R-mod contained in $\mathbf{F}(k_Q)$ and let M be an R-module with $k_M(Q)=0$. Then we have $k_M \leqq k_Q \leqq k_N$ and $\mathbf{F}(k_N)=R$-mod. Hence $\mathbf{F}(k_M)=R$-mod as desired.

(6)⇒(1). By assumption, there exists a class $\mathbf{T}$ of R-modules such that $(\mathbf{T}(k_Q), \mathbf{F}(k_Q), \mathbf{T})$ forms a 3-fold torsion theory for R-mod in the sense of [9]. It follows from Lemma 2.1 of [9] that $k_Q(M)=k_Q(R)\cdot M$ for each R-module M. Hence it results that $\mathbf{F}(k_Q)=R$-mod since Q is faithful.

3. Non-singular QF-3′ R-modules

In case the singular submodule $Z(Q)=0$, we can give a simple criterion for Q being QF-3′.

Theorem 3.1. *Let Q be an R-module with $Z(Q)=0$. Then Q is* QF-3′ *if and only if $\mathbf{T}(k_Q)$ is closed under taking submodules.*

This was also obtained by the same method in Bican [2] and we will omit the proof.

As is well-known, the functor Z of R-mod which assigns to each R-module M its singular submodule $Z(M)$ is a left exact preradical of R-mod. It is to be noted that, for this preradical, $\mathbf{F}(Z)$ is nothing but the torsion-free class of the so-called Goldie torsion theory. We shall now give other characterizations of non-singular QF-3′ R-modules by means of the functor Z. To do this, we first prove the following which appeared in Colby and Rutter [4] for the case $Q=R$.

Proposition 3.2. *The following conditions on an R-module Q are equivalent:*

(1) $Z(Q)=0$.

(2) $\mathbf{F}(k_Q)\subset\mathbf{F}(Z)$.

(3) $\mathbf{T}(Z)\subset\mathbf{T}(k_Q)$.

Proof. (1)⇒(2). Since $Z\leqq k_Q$, we have $\mathbf{F}(k_Q)\subset\mathbf{F}(Z)$.

(2)⇒(3). Let M be in $\mathbf{T}(Z)$. Take f in $\mathrm{Hom}_R(M, Q)$ and x in M. Then, since $\mathrm{Ann}_R(x)$ is essential in R, so is $\mathrm{Ann}_R(f(x))$ and hence $f(x)$ is in $Z(Q)$. But by assumption (2) $Z(Q)=0$ and this implies that M is contained in $\mathbf{T}(k_Q)$.

(3)⇒(1). Since Z is an idempotent preradical, $Z(Q)$ is in $\mathbf{T}(Z)$ and hence is in $\mathbf{T}(k_Q)$. This shows that $\mathrm{Hom}_R(Z(Q), Q)=0$ and $Z(Q)=0$.

Lemma 3.3. *Let Q be a faithful R-module. Then we have*
(1) $\mathbf{T}(k_{E(Q)})\subset\mathbf{T}(Z)$, *and*
(2) $\mathbf{F}(Z)\subset\mathbf{F}(k_{E(Q)})$.

Proof. (1) For every R-module M in $\mathbf{T}(k_{E(Q)})$ and every element x in M, we shall claim that $\mathrm{Ann}_R(x)$ is essential in R. Suppose that $\mathfrak{m}$ is a non-zero left ideal in R such that $\mathrm{Ann}_R(x)\cap\mathfrak{m}=0$. Define f: $\mathfrak{m}x\to R$ such that $f(ax)=a$ for $a\in\mathfrak{m}$. Clearly this is a well defined R-homomorphism. Let a be a non-zero element of $\mathfrak{m}$. Then there exists an R-homomorphism g: $R\to E(Q)$ such that $g(a)\neq 0$ since $E(Q)$ is faithful. The composition map $g\circ f$: $\mathfrak{m}x\to E(Q)$ can be extended to an R-homomorphism h: $M\to E(Q)$ and $h(ax)=g(f(ax))=g(a)\neq 0$. Thus we have $\mathrm{Hom}_R(M, E(Q))\neq 0$, but this is a contradiction. Similarly we can show that (2) holds.

It follows from Lemma 3.3 that, if Q is faithful and non-singular, then $E(Q)$ is a cogenerator for $\mathbf{F}(Z)$. However, we can show that this is also true for more general QF-3′ R-modules.

Theorem 3.4. *For a faithful R-module Q, the following conditions are equivalent:*
(1) *Q is* QF-3′ *and* $Z(Q)=0$.
(2) $\mathbf{T}(k_Q)=\mathbf{T}(Z)$.
(3) $\mathbf{F}(k_Q)=\mathbf{F}(Z)$.
(4) $k_Q=Z$.
(5) *Q is a cogenerator for* $\mathbf{F}(Z)$.

Proof. (1)⇒(2) and (1)⇒(3) follow from Proposition 3.2 and Lemma 3.3.

(2)⇒(1). By Proposition 3.2, $Z(Q)=0$. Since $\mathbf{T}(Z)$ is closed under taking submodules, so is $\mathbf{T}(k_Q)$. Therefore, Q is QF-3′ by Theroem 3.1.

(3)⇒(1). By Proposition 3.2, $Z(Q)=0$. Since $\mathbf{F}(Z)$ is closed under taking injective hulls, so is $\mathbf{F}(k_Q)$. Therefore, Q is QF-3′ by Theorem 1.4.

(4)⇒(1) follows from Theorem 1.4 since Z is left exact. So we assume (2) and also (3). By Proposition 3.2, $Z(Q)=0$ and we have $Z\leqq k_Q$. $\mathbf{F}(k_Q)=\mathbf{F}(Z)$ is closed under taking extensions and so by Proposition 1.2 k_Q is idempotent. For each R-module M, $k_Q(M)\in\mathbf{T}(k_Q)=\mathbf{T}(Z)$ and $k_Q(M)=Z(k_Q(M))\subset Z(M)$. Therefore we have $k_Q\leqq Z$.

(3)⇄(5). The fact that Q is a cogenerator for $\mathbf{F}(Z)$ means that $Z(Q)=0$ and $\mathbf{F}(Z)\subset\mathbf{F}(k_Q)$, or equivalently $\mathbf{F}(Z)=\mathbf{F}(k_Q)$ by Proposition 3.2. This completes the proof of the theorem.

In [4], it was given a similar characterization, except for (4) and (5), of non-singular left QF-3 rings in case these are semi-primary, and (4) may be viewed as a generalization of a result of [15].

In Proposition 2.3 we have shown that every essential extension of a QF-3′

R-module is QF-3′. However, in case it is non-singular, we have

Corollary 3.5. *Let Q be a faithful* QF-3′ *R-module and let Q' be a non-singular R-module such that $Q \subset Q'$. Then Q' is also* QF-3′.

Proof. By Proposition 3.2 and Theorem 3.4, $\mathbf{F}(Z)=\mathbf{F}(k_Q)\subset\mathbf{F}(k_{Q'})\subset\mathbf{F}(Z)$. Hence we have $\mathbf{F}(k_{Q'})=\mathbf{F}(Z)$ and Q' is QF-3′.

As another corollary to this theorem, we have

Corollary 3.6. *For a ring R with $Z({}_RR)=0$ and its maximal ring of left quotients Q, the following conditions are equivalent:*

(1) *Every non-singular R-module is torsionless, i.e., R is a cogenerator for* $\mathbf{F}(Z)$.

(2) *R is a left* QF-3′ *ring.*

(3) *${}_RQ$ is torsionless.*

Recently, Cateforis [3] has given a necessary and sufficient condition for a non-singular R-module to be a cogenerator for $\mathbf{F}(Z)$. The following theorem is motivated by his Theorem 1.1, and provides alternative characterizations of non-singular QF-3′ R-modules to that given in Theorem 3.4.

Theorem 3.7. *For a non-singular R-module Q, the following conditions are equivalent:*

(1) *Q is faithful and* QF-3′.

(2) *Q contains non-zero injective submodules and the sum Q^* of all such injective submodules is faithful.*

(3) *There exists a faithful submodule Q_0 of Q such that Q_0 contains the injective hull of every one of its finitely generated submodules.*

Before proving the theorem, we shall quote Lemma 0.2 of [3] and give its proof for the sake of completeness.

Lemma 3.8. *If A is an injective R-module and B is a non-singular R-module, then, for every R-homomorphism $f\colon A\to B$, both* $\mathrm{Ker}(f)$ *and* $\mathrm{Im}(f)$ *are injective.*

Proof. Since A is injective, we can assume that $\mathrm{Ker}(f)\subset E(\mathrm{Ker}(f))\subset A$. Take $x(\neq 0)$ in $E(\mathrm{Ker}(f))$ and $a(\neq 0)$ in R. If $ax=0$, then a is in $Ra\cap \mathrm{Ann}_R(f(x))$. If $ax\neq 0$, then we can find $bax(\neq 0)$ in $Rax\cap\mathrm{Ker}(f)$ for some b in R. Since $f(bax)=0$ and $ba\neq 0$, $Ra\cap\mathrm{Ann}_R(f(x))\neq 0$. At any rate, we have $Ra\cap\mathrm{Ann}_R(f(x))\neq 0$ and hence $\mathrm{Ann}_R(f(x))$ is essential in R. $f(x)$ is then in $Z(B)=0$. Therefore, x is in $\mathrm{Ker}(f)$ which shows that $\mathrm{Ker}(f)=E(\mathrm{Ker}(f))$.

Proof of Theorem 3.7. (1)⇒(2). By assumption, $\mathrm{Hom}_R(E(Q),Q)\neq 0$

and so by Lemma 3.8 Q contains certainly non-zero injective submodules. Moreover $k_{Q^*}(E(Q))=k_Q(E(Q))$ again by Lemma 3.8. Hence we have $k_{Q^*}(Q)=0$ which implies that $k_{Q^*}\leqq k_Q$ and Q^* is faithful. (Moreover in this case $k_Q=k_{Q^*}$ holds.)

(2)⇒(3). For every finite family $\{M_1, M_2, \cdots, M_n\}$ of non-zero injective submodules of Q, $\sum_{i=1}^n M_i$ is a homomorphic image of an injective R-module $\sum_{i=1}^n \oplus M_i$ and so by Lemma 3.8 it is also injective. It follows from this that Q^* contains the injective hull of every one of its finitely generated submodules.

(3)⇒(1). By Proposition 2.5 Q_0 is QF-3′. Q_0 is faithful and Q is non-singular, so by Corollary 3.5 Q is also QF-3′. (Here we shall point out that $k_Q=k_{Q_0}$ holds. To see this it is sufficient to show that $k_{Q_0}(Q)=0$. Take $x(\neq 0)$ in $E(Q)$. Then $\mathrm{Ann}_R(x)$ is not essential in R so we can find $a(\neq 0)$ in R such that $Ra\cap \mathrm{Ann}_R(x)=0$. Since ax is a non-zero element of $E(Q)$, there exists some $bax(\neq 0)$ in $Rax\cap Q$. ba is a non-zero element in R and Q_0 is faithful and so for some x_0 in Q_0 we have $bax_0\neq 0$. Then the mapping f: $Rbax\to Rbax_0$ given by $f(rbax)=rbax_0$, for r in R, is a well-defined R-homomorphism. By assumption, $E(Rbax_0)\subset Q_0$ and so f has an extension f^*: $E(Q)\to Q_0$ and $f^*(x)\neq 0$. Thus $k_{Q_0}(E(Q))=0$ and $k_{Q_0}(Q)=0$.)

To illustrate the theorem, we shall give some examples.

EXAMPLE 3.9. Let R be the ring of 2×2 upper triangular matrices over a field K. Then it is a faithful non-singular left module over itself. It has only one non-zero injective left ideal, namely

$$\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix},$$

and this is also a faithful R-module. Hence R is a QF-3′ R-module with

$$R^* = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}.$$

There is no faithful left ideal of R properly contained in R^*, so we have $R_0=R^*$. Moreover $R=R^*\oplus R'$, where

$$R' = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$$

and is not QF-3′.

EXAMPLE 3.10. Let R be as above and Q the ring of all 2×2 matrices over K. Then Q is also a faithful non-singular R-module and is QF-3′ since $Q=E({}_RR)$. In this case, $Q=Q^*$ and we may take for Q_0, for example, as

$$\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}, \text{ or } Q = \begin{pmatrix} K & K \\ K & K \end{pmatrix}.$$

Hence the submodule Q_0 in the theorem is not uniquely determined within isomorphisms.

REMARK. Let Q be a faithful, non-singular QF-3′ R-module. Then there exist faithful submodules Q^* and Q_0 of Q with properties mentioned in Theorem 3.7. As was pointed out in the proof of the theorem, $k_{Q^*}=k_{Q_0}=k_Q$ hold and hence by Theorem 1.4 both Q^* and Q_0 are also QF-3′. These, as well as Q and $E(Q)$, are faithful, non-singular QF-3′ R-modules. Clearly Q^* includes Q_0 and moreover it is a unique maximal one of those submodules of Q which contain the injective hull of every one of its finitely generated submodules. Since each injective submodule of Q is that of Q^*, we can conclude that Q^* coincides with the sum of all non-zero injective submodules of Q^*, i.e., $(Q^*)^*=Q^*$.

Let us suppose furthermore that every direct sum of non-singular injective R-modules is injective. For example, we may take a finite dimensional ring R in the sense that it contains no infinite direct sum of submodules. Then Q^* is itself injective and hence Q can be decomposed into a direct sum of submodules Q^* and Q': $Q=Q^*\oplus Q'$. Since Q^* is a unique maximal non-zero injective submodule of Q, if $Q'\neq 0$, then Q' does not contain any non-zero injective submodule of Q. Therefore by Lemma 3.8 $\mathrm{Hom}_R(E(Q'), Q')=0$. This shows that Q' can not be QF-3′.

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