ON A GENERALIZATION OF QF-3' RINGS*)

Dedicated to Professor Kiiti Morita for the celebration of his sixtieth birthday.

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A ring $R$ with identity is called left QF-3' if the injective hull $E(R)$ of the left $R$-module $R$ is torsionless. This class of rings and the other related generalizations of quasi-Frobenius rings have been studied by a number of authors.

Recently, Jans [7] has given a torsion theoretic characterization of left QF-3' rings (cf. also Kato [8] and Tsukerman [14]). The purpose of this paper is, generalizing this idea, to consider a module theoretic generalization of left QF-3' rings. We shall say that a left $R$-module $Q$ is QF-3' if its injective hull $E(Q)$ is torsionless with respect to $Q$, i.e., $E(Q)$ can be embedded in a direct product of copies of $Q$.

The main theorem of §1 will give some equivalent conditions for $Q$ to be QF-3'.

In §2, we shall discuss basic properties of QF-3' $R$-modules and study a relation between QF-3' $R$-modules and cogenerators for $R$-mod.

We shall treat, in §3, QF-3' $R$-modules with zero singular submodule. We shall give some results relating the notions of $Q$-torsionless $R$-modules and non-singular $R$-modules. In particular we shall show that, if $Q$ is faithful, these notions coincide if and only if $Q$ is QF-3' and has zero singular submodule. We shall also give another characterization of a QF-3' $R$-module with zero singular submodule making use of its injective submodules.

After completed this paper, we found that the similar results were obtained by Bican [2] and wrought a slight change in the paper.

Throughout this paper, $R$ will denote an associative ring with identity and $R$-mod the category of unital left $R$-modules and $R$-homomorphisms. We shall deal only with left $R$-modules and so $R$-modules will mean unital left $R$-modules. $E(M)$ will always denote the injective hull of a left $R$-module $M$ and $r_M(\ast)$ the right annihilator of $\ast$ in $M$.

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1. Preliminaries

A subfunctor \( r \) of the identity functor of \( \mathbf{R}-\text{mod} \) is called a preradical of \( \mathbf{R}-\text{mod} \). It is called idempotent if \( r(r(M)) = r(M) \) and a radical if \( r(M/r(M)) = 0 \) for all \( \mathbf{R} \)-modules \( M \). To each preradical \( r \) we associate two classes of \( \mathbf{R} \)-modules, namely

\[
\mathcal{T}(r) = \{ M \mid r(M) = M \} \quad \text{and} \quad \mathcal{F}(r) = \{ M \mid r(M) = 0 \}.
\]

In case a preradical \( r \) is idempotent and is a radical, the pair \( (\mathcal{T}(r), \mathcal{F}(r)) \) forms a torsion theory for \( \mathbf{R}-\text{mod} \) in the sense of [5].

In the class of all preradicals of \( \mathbf{R}-\text{mod} \), there is a partial ordering in which \( r_1 \leq r_2 \) means that \( r_1(M) \subseteq r_2(M) \) for all \( \mathbf{R} \)-modules \( M \). For each preradical \( r \) there exists a largest idempotent preradical \( \hat{r} \) smaller than or equal to \( r \) and a smallest radical \( \hat{r} \) larger than or equal to \( r \). It is easy to see that \( \mathcal{T}(\hat{r}) = \mathcal{T}(r) \) and \( \mathcal{F}(\hat{r}) = \mathcal{F}(r) \). Moreover, if \( r \) is idempotent, then so is \( \hat{r} \), and \( \hat{r} \) is a radical if \( r \) is.

Let \( \mathcal{Q} \) be an \( \mathbf{R} \)-module and let us define

\[
k_\mathcal{Q}(M) = \bigcap_{f \in \text{Hom}_\mathbf{R}(M, \mathcal{Q})} \ker(f)
\]

for each \( \mathbf{R} \)-module \( M \). Then \( k_\mathcal{Q} \) is a radical of \( \mathbf{R}-\text{mod} \) such that \( k_\mathcal{Q}(\mathcal{Q}) = 0 \). Moreover it is a unique maximal one of those preradicals \( r \) of \( \mathbf{R}-\text{mod} \) for which \( r(\mathcal{Q}) = 0 \), and \( k_{E,\mathcal{Q}} \) is a unique maximal one of those left exact radicals \( r \) of \( \mathbf{R}-\text{mod} \) for which \( r(\mathcal{Q}) = 0 \). As is well-known, every left exact radical of \( \mathbf{R}-\text{mod} \) is of the form \( k_E \) for some injective \( \mathbf{R} \)-module \( E \). For example, we can take \( E \) as the direct product of injective hulls of all cyclic torsion-free \( \mathbf{R} \)-modules (e.g., see [11]).

Since \( k_\mathcal{Q} \leq k_{\mathcal{Q}'} \) for each submodule \( \mathcal{Q}' \) of \( \mathcal{Q} \) and \( k_{E,\mathcal{Q}} \) is idempotent, we have \( k_{E,\mathcal{Q}} \leq k_\mathcal{Q} \leq k_{\mathcal{Q}} \).

The class \( \mathcal{T}(k_\mathcal{Q}) \) coincides with the class \( \{ M \mid \text{Hom}_\mathbf{R}(M, \mathcal{Q}) = 0 \} \) and is closed under taking homomorphic images, direct sums and extensions. So this is a torsion class in \( \mathbf{R}-\text{mod} \) and the corresponding torsion-free class coincides with \( \mathcal{F}(k_\mathcal{Q}) \). On the other hand, the class of \( \mathbf{R} \)-modules \( \mathcal{F}(k_\mathcal{Q}) \) is not a torsion-free class in general. This is closed under taking submodules and direct products, but not extensions in general (see e.g. [16, Example B]). As is well-known, \( \mathcal{Q} \) is a cogenerator for \( \mathcal{F}(k_\mathcal{Q}) \). Moreover, an \( \mathbf{R} \)-module \( M \) is in \( \mathcal{F}(k_\mathcal{Q}) \) if (and only if) it can be embedded in a direct product of copies of \( \mathcal{Q} \). However, for simple \( \mathbf{R} \)-modules we have

**Proposition 1.1.** A simple \( \mathbf{R} \)-module \( S \) is in \( \mathcal{F}(k_\mathcal{Q}) \) if and only if there exists an \( \mathbf{R} \)-monomorphism of \( S \) into \( \mathcal{Q} \).

The proof is easy and so we will omit it.

As was mentioned above, \( \mathcal{F}(k_\mathcal{Q}) \) is not closed under taking extensions.
The following proposition shows when it is closed under taking extensions. Evidently this is the case if \( Q \) is injective.

**Proposition 1.2.** The following conditions on an \( R \)-module \( Q \) are equivalent:

1. \( F(k_Q) \) is closed under taking extensions, i.e., it becomes a torsion-free class.
2. \( k_Q = \hat{k}_Q \), i.e., \( k_Q \) is idempotent.
3. \( F(k_Q) = F(\hat{k}_Q) \).

Bican [2] has obtained the same result independently, and so we will omit the proof.

The class \( T(k_Q) \) is a torsion class, but it is not, in general, closed under taking submodules (e.g., see [16]). Concerning this, we have

**Proposition 1.3.** For an \( R \)-module \( Q \), the following conditions are equivalent:

1. \( \hat{T}(k_Q) \) is closed under taking submodules.
2. \( k_Q = k_{E(Q)} \), i.e., \( k_Q \) is left exact.
3. \( T(k_Q) = T(k_{E(Q)}) \).

Proof. (1)\( \Rightarrow \) (2). Suppose that \( T(k_Q) \) is closed under taking submodules. Then, since \( \hat{k}_Q \leq k_Q \), \( \hat{k}_Q(Q) = 0 \) and, since the corresponding torsion-free class \( F(\hat{k}_Q) \) of \( T(k_Q) \) is closed under taking injective hulls, we have \( \hat{k}_Q(E(Q)) = 0 \). So \( k_Q \leq k_{E(Q)} \) and hence \( \hat{k}_Q = k_{E(Q)} \).

(2)\( \Rightarrow \) (3) is clear and since \( k_{E(Q)} \) is left exact, (3) implies (1).

This proposition was also proved in Bican [2] by a different method. Combining this with Proposition 1.2, we have

**Theorem 1.4.** The following conditions on an \( R \)-module \( Q \) are equivalent:

1. \( (T(k_Q), F(k_Q)) \) forms a hereditary torsion theory for \( R\text{-mod} \).
2. \( T(k_Q) \) is closed under taking submodules and \( F(k_Q) \) is closed under taking extensions.
3. \( k_Q = k_{E(Q)} \).
4. \( k_Q \) is left exact.
5. \( F(k_Q) \) is closed under taking injective hulls.
6. \( F(k_Q) \) is closed under taking essential extensions.
7. \( F(k_Q) \) contains an injective \( R \)-module \( M \) with \( k_M(Q) = 0 \).
8. \( E(Q) \subseteq F(k_Q) \).
9. \( T(k_Q) = T(k_{E(Q)}) \) and \( F(k_Q) = F(k_{E(Q)}) \).

Proof. Here we show only that (7) implies (8). The proof of the other is easy. Since \( k_M(Q) = 0 \), \( Q \subseteq \prod M \), a direct product of copies of \( M \), and hence \( E(Q) \subseteq \prod M \). \( F(k_Q) \) is closed under taking direct products and submodules and so we have \( E(Q) \subseteq F(k_Q) \).
The equivalence of (3), (4) and (8) was also proved in Bican [2]. In case $Q = R$, the equivalence of these conditions, except for (1), (4), (5) and (9), was shown by Colby and Rutter [4], Jans [7], and Kato [8].

2. QF-3' R-modules

Recall that a ring $R$ is left QF-3' if the injective hull of the $R$-module $R$ is torsionless, i.e., $k_R(E(R)) = 0$. Recently, Jans [7] has shown that $R$ is left QF-3' if and only if $F(k_R)$ is closed under taking extensions and $T(k_R)$ is closed under taking submodules (cf. also Kato [8] and Tsukerman [14]). From this point of view we now make the following definition.

**Definition.** An $R$-module $Q$ is called QF-3' if $Q$ satisfies each one of the conditions of Theorem 1.4.

It follows from this definition that every injective $R$-module is QF-3'. The following example pointed out by Tsukerman without proof shows that there exist non-injective QF-3' $R$-modules.

**Example 2.1.** Every direct sum of injective $R$-modules is QF-3'.

To see this, let $Q = \sum_{\lambda \in \Lambda} Q_\lambda$ be a direct sum of injective $R$-modules. Then $F(k_{Q_\lambda}) \subseteq F(k_Q)$ for all $\lambda$ and hence $E(Q) \subseteq E(Q_\lambda)$. Since $Q \subseteq E(Q) \subseteq \prod_{\lambda \in \Lambda} E(Q_\lambda)$, and thus $Q$ is QF-3'.

**Proposition 2.2.**

1. Every direct product of QF-3' $R$-modules is QF-3'.
2. Every direct sum of QF-3' $R$-modules is QF-3'.

**Proof.** (1) Let $Q = \prod_{\lambda \in \Lambda} Q_\lambda$ be a direct product of QF-3' $R$-modules. Then $F(k_{Q_\lambda}) \subseteq F(k_Q)$ for all $\lambda$ and hence $E(Q) \subseteq E(Q_\lambda)$. Since $Q \subseteq E(Q) \subseteq \prod_{\lambda \in \Lambda} E(Q_\lambda)$, and thus $Q$ is QF-3'. The proof of (2) is similar to that of (1) and so it will be omitted.

It should be noted that, as we shall show later, direct summands of a QF-3' $R$-module need not be QF-3' in general.

**Proposition 2.3.** Every essential extension of a QF-3' $R$-module is QF-3'.

**Proof.** Suppose that $Q$ is QF-3' and $Q'$ is an essential extension of $Q$. Then we can assume that $Q \subseteq Q' \subseteq E(Q)$ and hence we have $k_{E(Q)} \leq k_{Q'} \leq k_Q$. By Theorem 1.4, $k_Q = k_{Q'}$, and thus $Q'$ is QF-3' again by Theorem 1.4.

It follows from this that every rational extension of a QF-3' $R$-module is also QF-3'. This appeared in [13] for left QF-3' rings.

Let $Q$ be an $R$-module. As is easily seen, $Q$ is faithful if and only if $k_Q(R) = 0$ and this is so if and only if $k_Q \leq k_R$. On the other hand, $Q$ is torsionless if and only if $k_R(Q) = 0$, or equivalently, $k_R \leq k_Q$. Therefore if $Q$ is both
faithful and torsionless, then we have \( k_\mathcal{Q} = k_R \). Applying Theorem 1.4, we have

**Theorem 2.4.** For a ring \( R \), the following conditions are equivalent:

1. \( R \) is a left QF-3' ring.
2. The \( R \)-module \( R \) is QF-3'.
3. There exists a QF-3' \( R \)-module \( Q \) which is both faithful and torsionless.

**Proposition 2.5.** Let \( Q \) be an \( R \)-module.

1. If \( Q \) is a QF-3' \( R \)-module with non-zero socle, then the injective hull of every simple submodule of \( Q \) is isomorphic to a submodule of \( Q \).
2. If the injective hull of every cyclic submodule of \( Q \) is isomorphic to a submodule of \( Q \), then \( Q \) is QF-3'.

Proof. (1) Let \( S \) be a simple submodule of \( Q \). Take \( \alpha(\pm 0) \in E(S) \). Then there exists \( ax(\pm 0) \in R_S \cap S \). \( S \) is in \( F(k_\mathcal{Q}) \) and so \( E(S) \) is in \( F(k_\mathcal{Q}) \) by Theorem 1.4. We can find an \( R \)-homomorphism \( f: E(S) \to Q \) such that \( f(ax) \neq 0 \). Hence we have \( f(S) \neq 0 \) and \( f \) must be a monomorphism.

(2) Take \( \alpha(\pm 0) \in E(Q) \). There exists \( ax(\pm 0) \in R_S \cap Q \). By assumption, \( E(Rax) \subseteq Q \), and the inclusion mapping \( Rax \to E(Rax) \) can be extended to an \( R \)-homomorphism \( f: E(Q) \to Q \) such that \( f(x) \neq 0 \), which shows that \( Q \) is QF-3'.

Clearly, for a direct sum \( Q \) of injective \( R \)-modules, the injective hull of every cyclic submodule is isomorphic to a submodule of \( Q \) and so (2) above gives another proof of Example 2.1.

As an immediate consequence of this proposition, we have at once

**Corollary 2.6.** For an \( R \)-module \( Q \) with non-zero socle, the following conditions are equivalent:

1. \( Q \) is indecomposable and QF-3'.
2. \( Q = E(S) \) for every simple submodule \( S \) of \( Q \).
3. \( Q = E(Q') \) for every non-zero submodule \( Q' \) of \( Q \).

**Proposition 2.7.** Let \( Q \) be an \( R \)-module. If every cyclic submodule of \( Q \) is QF-3', then \( Q \) is itself QF-3'.

Proof. Take \( \alpha(\pm 0) \in E(Q) \) and claim that there exists an \( R \)-homomorphism \( f^*: E(Q) \to Q \) with \( f^*(x) \neq 0 \). Choose an element \( a \) in \( R \) such that \( ax(\pm 0) \) is in \( Q \). Then we have \( Rax \subseteq E(Rax) \subseteq E(Q) \) and \( E(Q) = E(Rax) \oplus Q' \), for some submodule \( Q' \) of \( E(Q) \). By assumption, \( Rax \) is QF-3' and so there exists an \( R \)-homomorphism \( f: E(Rax) \to Rax \) such that \( f(ax) \neq 0 \). Then it is easy to see that the composition \( f^*: E(Q) \to Q \) of \( f \) and the projection mapping \( E(Q) \to E(Rax) \) has the desired property.

As a direct consequence of this, we see that if every cyclic \( R \)-module is QF-3', then every \( R \)-module is also QF-3'. This was proved by Tsukerman
Recall that an $R$-module $Q$ is a cogenerator for $R$-mod if $F(k_Q) = R$-mod. Therefore, a cogenerator for $R$-mod is necessarily QF-3'. We now consider the question of when a QF-3' $R$-module becomes a cogenerator for $R$-mod. To do this we shall prove

**Proposition 2.8.** For an $R$-module $Q$ ($\neq 0$), the following conditions are equivalent:

1. $Q$ contains a copy of every simple $R$-module.
2. Every simple $R$-module belongs to $F(k_Q)$.
3. For every simple $R$-module $S$, $\text{Hom}_R(S, Q) \neq 0$.
4. For every maximal left ideal $m$ of $R$, $r_Q(m) \neq 0$.
5. For every proper left ideal $m$ of $R$, $r_Q(m) \neq 0$.
6. For every non-zero finitely generated $R$-module $M$, $\text{Hom}_R(M, Q) \neq 0$.
7. For every non-zero cyclic $R$-module $M$, $\text{Hom}_R(M, Q) \neq 0$.
8. $E(Q)$ is a cogenerator for $R$-mod.
9. Every non-zero injective $R$-module $M$ with $k_M(Q) = 0$ is a cogenerator for $R$-mod.

Proof. We shall show only (7)$\Rightarrow$(8)$\Rightarrow$(9)$\Rightarrow$(1).

(7)$\Rightarrow$(8). Let $M$ be an $R$-module. Take $x(\neq 0)$ in $M$. Then by assumption there exists a non-zero $R$-homomorphism $f: Rx \to Q$. Since $E(Q)$ is injective, it can be extended to an $R$-homomorphism $f': M \to E(Q)$ and $f'(x) = f(x) \neq 0$. This shows that $E(Q)$ is a cogenerator for $R$-mod.

(8)$\Rightarrow$(9). Since $k_M(Q) = 0$, $Q \subseteq \prod M$, a direct product of copies of $M$, and hence $E(Q) \subseteq \prod M$. Then we have $R$-mod = $\mathbf{F}(k_{E(Q)}) \subseteq \mathbf{F}(k_{k_M})$. It follows that $\prod M$ is a cogenerator for $R$-mod and so is $M$ by [12, Lemma 1].

(9)$\Rightarrow$(1). Since $E(Q)$ is a cogenerator for $R$-mod, for every simple $R$-module $S$, there exists a non-zero $R$-homomorphism $f: S \to E(Q)$. $S$ is simple, so $f$ must be a monomorphism. Since $f(S) \cap Q \neq 0$, $f(S) \cap Q = f(S)$ and hence $f(S)$ is contained in $Q$.

An $R$-module satisfying (1) and (8) was called lower distinguished by Azumaya [1] and a quasi-cogenerator by Morita [10] respectively.

Generalizing results due to Kato [8], Jans [6] and Sugano [12], we have

**Theorem 2.9.** The following conditions on an $R$-module $Q$ are equivalent:

1. $Q$ is a cogenerator for $R$-mod.
2. $Q$ is QF-3' and contains a copy of every simple $R$-module.
3. $\sum_{\alpha \in A} \oplus E(S_{\alpha}) = F(k_Q)$, where $\{S_{\alpha}\}_{\alpha \in A}$ be a complete set of representatives for the isomorphism classes of simple $R$-modules.
4. There exists a cogenerator for $R$-mod contained in $F(k_Q)$.
5. Every $R$-module $M$ with $k_M(Q) = 0$ is a cogenerator for $R$-mod.
(6) \( Q \) is faithful QF-3' and \( F(k_Q) \) is closed under taking homomorphic images.

Proof. (1)\( \Rightarrow \) (2)\( \Rightarrow \) (3) follow from Proposition 1.1 and Theorem 1.4 and (3)\( \Rightarrow \) (4) and (5)\( \Rightarrow \) (6) are easy.

(4)\( \Rightarrow \) (5). Let \( N \) be a cogenerator for \( R\)-mod contained in \( F(k_Q) \) and let \( M \) be an \( R \)-module with \( k_M(Q) = 0 \). Then we have \( k_M \leq k_Q \leq k_N \) and \( F(k_N) = R\)-mod. Hence \( F(k_M) = R\)-mod as desired.

(6)\( \Rightarrow \) (1). By assumption, there exists a class \( T \) of \( R \)-modules such that \( (T(k_Q), F(k_Q), T) \) forms a 3-fold torsion theory for \( R\)-mod in the sense of [9]. It follows from Lemma 2.1 of [9] that \( k_Q(M) = k_Q(R) \cdot M \) for each \( R \)-module \( M \). Hence it results that \( F(k_Q) = R\)-mod since \( Q \) is faithful.

3. Non-singular QF-3' \( R \)-modules

In case the singular submodule \( Z(Q) = 0 \), we can give a simple criterion for \( Q \) being QF-3'.

**Theorem 3.1.** Let \( Q \) be an \( R \)-module with \( Z(Q) = 0 \). Then \( Q \) is QF-3' if and only if \( T(k_Q) \) is closed under taking submodules.

This was also obtained by the same method in Bican [2] and we will omit the proof.

As is well-known, the functor \( Z \) of \( R\)-mod which assigns to each \( R \)-module \( M \) its singular submodule \( Z(M) \) is a left exact preradical of \( R\)-mod. It is to be noted that, for this preradical, \( F(Z) \) is nothing but the torsion-free class of the so-called Goldie torsion theory. We shall now give other characterizations of non-singular QF-3' \( R \)-modules by means of the functor \( Z \). To do this, we first prove the following which appeared in Colby and Rutter [4] for the case \( Q = R \).

**Proposition 3.2.** The following conditions on an \( R \)-module \( Q \) are equivalent:

1. \( Z(Q) = 0 \).
2. \( F(k_Q) \subseteq F(Z) \).
3. \( T(Z) \subseteq T(k_Q) \).

Proof. (1)\( \Rightarrow \) (2). Since \( Z \leq k_Q \), we have \( F(k_Q) \subseteq F(Z) \).

(2)\( \Rightarrow \) (3). Let \( M \) be in \( T(Z) \). Take \( f \) in \( \text{Hom}_R(M, Q) \) and \( x \) in \( M \). Then, since \( \text{Ann}_R(x) \) is essential in \( R \), so is \( \text{Ann}_R(f(x)) \) and hence \( f(x) \) is in \( Z(Q) \). But by assumption (2) \( Z(Q) = 0 \) and this implies that \( M \) is contained in \( T(k_Q) \).

(3)\( \Rightarrow \) (1). Since \( Z \) is an idempotent preradical, \( Z(Q) \) is in \( T(Z) \) and hence is in \( T(k_Q) \). This shows that \( \text{Hom}_R(Z(Q), Q) = 0 \) and \( Z(Q) = 0 \).
Lemma 3.3. Let $Q$ be a faithful $R$-module. Then we have

1. $T(k_{E(Q)}) \subseteq T(Z)$, and
2. $F(Z) \subseteq F(k_{E(Q)})$.

Proof. (1) For every $R$-module $M$ in $T(k_{E(Q)})$ and every element $x$ in $M$, we shall claim that $\text{Ann}_R(x)$ is essential in $R$. Suppose that $m$ is a non-zero left ideal in $R$ such that $\text{Ann}_R(x) \cap m = 0$. Define $f: mx \to R$ such that $f(ax) = a$ for $a \in m$. Clearly this is a well defined $R$-homomorphism. Let $a$ be a non-zero element of $m$. Then there exists an $R$-homomorphism $g: R \to E(Q)$ such that $g(a) \neq 0$ since $E(Q)$ is faithful. The composition map $g \circ f: mx \to E(Q)$ can be extended to an $R$-homomorphism $h: M \to E(Q)$ and $h(ax) = g(f(ax)) = g(a) \neq 0$. Thus we have $\text{Hom}_R(M, E(Q)) \neq 0$, but this is a contradiction. Similarly we can show that (2) holds.

It follows from Lemma 3.3 that, if $Q$ is faithful and non-singular, then $E(Q)$ is a cogenerator for $F(Z)$. However, we can show that this is also true for more general $QF$-3' $R$-modules.

Theorem 3.4. For a faithful $R$-module $Q$, the following conditions are equivalent:

1. $Q$ is $QF$-3' and $Z(Q) = 0$.
2. $T(k_Q) = T(Z)$.
3. $F(k_Q) = F(Z)$.
4. $k_Q = Z$.
5. $Q$ is a cogenerator for $F(Z)$.

Proof. (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3) follow from Proposition 3.2 and Lemma 3.3. (2)$\Rightarrow$(1). By Proposition 3.2, $Z(Q) = 0$. Since $T(Z)$ is closed under taking submodules, so is $T(k_Q)$. Therefore, $Q$ is $QF$-3' by Theorem 3.1. (3)$\Rightarrow$(1). By Proposition 3.2, $Z(Q) = 0$. Since $F(Z)$ is closed under taking injective hulls, so is $F(k_Q)$. Therefore, $Q$ is $QF$-3' by Theorem 1.4. (4)$\Rightarrow$(1) follows from Theorem 1.4 since $Z$ is left exact. So we assume (2) and also (3). By Proposition 3.2, $Z(Q) = 0$ and we have $Z \subseteq k_Q$. $F(k_Q) = F(Z)$ is closed under taking extensions and so by Proposition 1.2 $k_Q$ is idempotent. For each $R$-module $M$, $k_Q(M) \subseteq T(k_Q) = T(Z)$ and $k_Q(M) = Z(k_Q(M)) \subseteq Z(M)$. Therefore we have $k_Q \subseteq Z$.

(3)$\Rightarrow$(5). The fact that $Q$ is a cogenerator for $F(Z)$ means that $Z(Q) = 0$ and $F(Z) \subseteq F(k_Q)$, or equivalently $F(Z) = F(k_Q)$ by Proposition 3.2. This completes the proof of the theorem.

In [4], it was given a similar characterization, except for (4) and (5), of non-singular left $QF$-3 rings in case these are semi-primary, and (4) may be viewed as a generalization of a result of [15].

In Proposition 2.3 we have shown that every essential extension of a $QF$-3'}
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Let \( Q \) be a faithful QF-3' \( R \)-module and let \( Q' \) be a non-singular \( R \)-module such that \( Q \subset Q' \). Then \( Q' \) is also QF-3'.

Proof. By Proposition 3.2 and Theorem 3.4, \( F(Z) = F(k_\rho) \subset F(k_{\rho'}) \subset F(Z) \). Hence we have \( F(k_{\rho'}) = F(Z) \) and \( Q' \) is QF-3'.

As another corollary to this theorem, we have

**Corollary 3.6.** For a ring \( R \) with \( Z(R) = 0 \) and its maximal ring of left quotients \( Q \), the following conditions are equivalent:

1. Every non-singular \( R \)-module is torsionless, i.e., \( R \) is a cogenerator for \( F(Z) \).
2. \( R \) is a left QF-3' ring.
3. \( RQ \) is torsionless.

Recently, Cateforis [3] has given a necessary and sufficient condition for a non-singular \( R \)-module to be a cogenerator for \( F(Z) \). The following theorem is motivated by his Theorem 1.1, and provides alternative characterizations of non-singular QF-3' \( R \)-modules to that given in Theorem 3.4.

**Theorem 3.7.** For a non-singular \( R \)-module \( Q \), the following conditions are equivalent:

1. \( Q \) is faithful and QF-3'.
2. \( Q \) contains non-zero injective submodules and the sum \( Q^* \) of all such injective submodules is faithful.
3. There exists a faithful submodule \( Q_0 \) of \( Q \) such that \( Q_0 \) contains the injective hull of every one of its finitely generated submodules.

Before proving the theorem, we shall quote Lemma 0.2 of [3] and give its proof for the sake of completeness.

**Lemma 3.8.** If \( A \) is an injective \( R \)-module and \( B \) is a non-singular \( R \)-module, then, for every \( R \)-homomorphism \( f: A \to B \), both \( \ker(f) \) and \( \text{Im}(f) \) are injective.

Proof. Since \( A \) is injective, we can assume that \( \ker(f) \subset E(\ker(f)) \subset A \). Take \( x(\neq 0) \) in \( E(\ker(f)) \) and \( a(\neq 0) \) in \( R \). If \( ax = 0 \), then \( a \) is in \( Ra \cap \text{Ann}_R(f(x)) \). If \( ax \neq 0 \), then we can find \( bax(\neq 0) \) in \( Rax \cap \ker(f) \) for some \( b \) in \( R \). Since \( f(bax) = 0 \) and \( ba \neq 0 \), \( Ra \cap \text{Ann}_R(f(x)) \neq 0 \). At any rate, we have \( Ra \cap \text{Ann}_R(f(x)) \neq 0 \) and hence \( \text{Ann}_R(f(x)) \) is essential in \( R \). \( f(x) \) is then in \( Z(B) = 0 \). Therefore, \( x \) is in \( \ker(f) \) which shows that \( \ker(f) = E(\ker(f)) \).

Proof of Theorem 3.7. \( (1) \Rightarrow (2) \). By assumption, \( \text{Hom}_R(E(Q), Q) \neq 0 \)
and so by Lemma 3.8 $Q$ contains certainly non-zero injective submodules. Moreover $k_{Q^*}(E(Q)) = k_Q(E(Q))$ again by Lemma 3.8. Hence we have $k_{Q^*}(Q) = 0$ which implies that $k_{Q^*} \leq k_Q$ and $Q^*$ is faithful. (Moreover in this case $k_Q = k_{Q^*}$ holds.)

(2) $\Rightarrow$ (3). For every finite family $\{M_1, M_2, \ldots, M_n\}$ of non-zero injective submodules of $Q$, $\sum_{i=1}^n M_i$ is a homomorphic image of an injective $R$-module $\sum_{i=1}^n \oplus M_i$ and so by Lemma 3.8 it is also injective. It follows from this that $Q^*$ contains the injective hull of every one of its finitely generated submodules.

(3) $\Rightarrow$ (1). By Proposition 2.5 $Q_0$ is QF-3'. $Q_0$ is faithful and $Q$ is non-singular, so by Corollary 3.5 $Q$ is also QF-3'. (Here we shall point out that $k_Q = k_{Q_0}$ holds. To see this it is sufficient to show that $k_{Q_0}(Q) = 0$. Take $x_t(\pm 0)$ in $E(Q)$. Then $\text{Ann}_R(x) = 0$ so we can find $d(\pm 0)$ in $R$ such that $Ra \cap \text{Ann}_R(x) = 0$. Since $ax$ is a non-zero element of $E(Q)$, there exists some $bax(\pm 0)$ in $Rax \cap Q$. $ba$ is a non-zero element in $R$ and $Q_0$ is faithful and so for some $x_t$ in $Q_0$ we have $bax_t \neq 0$. Then the mapping $f: Rbax \rightarrow Rbax$, given by $f(rbax) = rbax_t$ for $r$ in $R$, is a well-defined $R$-homomorphism. By assumption, $E(Rbax_t) \subseteq Q_0$ and so $f$ has an extension $f^*: E(Q) \rightarrow Q_0$ and $f^*(x_t) \neq 0$. Thus $k_{Q_0}(E(Q)) = 0$ and $k_{Q_0}(Q) = 0$.)

To illustrate the theorem, we shall give some examples.

**Example 3.9.** Let $R$ be the ring of $2 \times 2$ upper triangular matrices over a field $K$. Then it is a faithful non-singular left module over itself. It has only one non-zero injective left ideal, namely

$$
\begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix},
$$

and this is also a faithful $R$-module. Hence $R$ is a QF-3' $R$-module with

$$
R^* = \begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix}.
$$

There is no faithful left ideal of $R$ properly contained in $R^*$, so we have $R_0 = R^*$. Moreover $R = R^* \oplus R'$, where

$$
R' = \begin{pmatrix}
K & 0 \\
0 & 0
\end{pmatrix}
$$

and is not QF-3'.

**Example 3.10.** Let $R$ be as above and $Q$ the ring of all $2 \times 2$ matrices over $K$. Then $Q$ is also a faithful non-singular $R$-module and is QF-3' since $Q = E(RR)$. In this case, $Q = Q^*$ and we may take for $Q_0$, for example, as

$$
\begin{pmatrix}
K & 0 \\
K & 0
\end{pmatrix}, \begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix}, \text{ or } Q = \begin{pmatrix}
K & K \\
K & K
\end{pmatrix}.$$
Hence the submodule $Q_o$ in the theorem is not uniquely determined within isomorphisms.

**REMARK.** Let $Q$ be a faithful, non-singular QF-3' $R$-module. Then there exist faithful submodules $Q^*$ and $Q_o$ of $Q$ with properties mentioned in Theorem 3.7. As was pointed out in the proof of the theorem, $k_{Q^*} = k_{Q_o} = k_Q$ hold and hence by Theorem 1.4 both $Q^*$ and $Q_o$ are also QF-3'. These, as well as $Q$ and $E(Q)$, are faithful, non-singular QF-3' $R$-modules. Clearly $Q^*$ includes $Q_o$ and moreover it is a unique maximal one of those submodules of $Q$ which contain the injective hull of every one of its finitely generated submodules. Since each injective submodule of $Q$ is that of $Q^*$, we can conclude that $Q^*$ coincides with the sum of all non-zero injective submodules of $Q^*$, i.e., $(Q^*)^* = Q^*$.

Let us suppose furthermore that every direct sum of non-singular injective $R$-modules is injective. For example, we may take a finite dimensional ring $R$ in the sense that it contains no infinite direct sum of submodules. Then $Q^*$ is itself injective and hence $Q$ can be decomposed into a direct sum of submodules $Q^*$ and $Q': Q = Q^* \oplus Q'$. Since $Q^*$ is a unique maximal non-zero injective submodule of $Q$, if $Q' \neq 0$, then $Q'$ does not contain any non-zero injective submodule of $Q$. Therefore by Lemma 3.8 $\text{Hom}_R(E(Q'), Q') = 0$. This shows that $Q'$ can not be QF-3'.

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**References**


